Julius-Maximilians-
UNIVERSITÄT
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Lehrstuhl für
MATHEMATIK IX

Übungen zur Optimale Steuerung (A. Borzì), Blatt 2

SoSe 2019 1 St., Mi 13-14, SE 30

(Calculus of Variation ODE, Optimal Control ODE, Optimal Control PDE)

(1) To approximate a function $g(x) = \sin(\pi x)$ over the interval $[0, \pi]$ by a polynomial p of degree 2, we minimize the criterion

$$f(a) = \frac{1}{2} \int_0^1 (g(x) - p(x))^2 \, \mathrm{d}x,$$

where $p(x) = a_0 + a_1 x + a_2 x^2$. Find the equations satisfied by the optimal coefficients $a = (a_0, a_1, a_2)$, and solve them.

(2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{2}\, x^T Q x + b^T x + \gamma$$

where $Q \in \mathcal{S}_n$ (the vector space of all symmetric $(n \times n)$-matrices), $b \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$. Prove the following statements:

(a) f is convex if and only if Q is positive semi-definite.

(b) f is uniformly convex if and only if Q is positive definite.

(c) Determine the gradient of f.

(3) Let $X \subseteq \mathbb{R}^n$ be an open set, $f : X \to \mathbb{R}$ be continuously differentiable, $x \in X$ and $d \in \mathbb{R}^n \setminus \{0\}$ given. Show that

$$\lim_{t \to 0^+} \frac{f(x + t\, d) - f(x)}{t} = \nabla f(x) \cdot d.$$

Apply this result to $f(x) = \|Ax - b\|^2_{\mathbb{R}^m}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

(4) Solve the following optimization problem with constraints in $\mathbb{R}^2$

$$\begin{aligned} \min f(x) &:= x_1^2 - x_2^2 \\ s.t. \quad & x_1 + 2x_2 + 1 = 0 \\ & x_1 - x_2 \leq 3. \end{aligned}$$

Write the Lagrange function and derive the first- and second-order optimality conditions.

(5) Let $Q \in \mathcal{S}_n$, $q \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m \times n}$ $(rank(A) = m)$, $b \in \mathbb{R}^m$. Consider the following optimization problem in $\mathbb{R}^n$.

$$\begin{aligned} &\min f(x) := x^T Q x + q^T x \\ &s.t. \quad Ax = b \end{aligned}$$

Write the KKT optimality system and discuss its solution.