Übungen zur Optimale Steuerung (A. Borzì), Blatt 4
SoSe 2019 1 St., Mi 13-14, SE 30
(Calendar of Variation ODE, Optimal Control ODE, Optimal Control PDE)

(1) Prove the following:
Let $Q \in \mathbb{R}^{n \times n}$ and $A \in \mathbb{R}^{m \times n}$ be given, and it holds that $\text{rank}(A) = m$, and $Q$ is positive definite in the subspace $\ker(A)$. Then the following matrix is invertible:

$$
\begin{pmatrix}
Q & A^T \\
A & 0
\end{pmatrix}.
$$

(2) Let $X$ be a normed space and $f : X \to \mathbb{R}$ be a function. We say that $f$ is lower semi-continuous at $x_0$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$
f(x) - f(x_0) > -\epsilon,
$$
whenever $\|x - x_0\| < \delta$.

Prove that $f$ is lower semi-continuous at $x_0$ if and only if for every sequence $x_n$ with $\lim_{n \to \infty} x_n = x_0$, it follows that $\liminf_{n \to \infty} f(x_n) \geq f(x_0)$.

(3) Let $(f_j), (g_j)$ be any two strongly convergent sequences in an arbitrary infinite dimensional Hilbert space $H$ with scalar product $(\cdot, \cdot)_H$, and $(h_j), (k_j)$ be any two weakly convergent sequences in $H$. Prove or find a counterexample:

a) The sequence $(f_j, g_j)_H$ is always convergent.

b) The sequence $(f_j, h_j)_H$ is always convergent.

c) The sequence $(h_j, k_j)_H$ is always convergent.

(4) Find a function in $C[0, 1]$ that is not contained in $H^1(0, 1)$. Find a sequence $(v_n)$ in $C_0^\infty(0, 1)$ which converges to the constant function $y = 1$ in the $L^2(0, 1)$ sense.

(5) Consider the functional

$$
J(y) = \int_0^1 (1 + x) (y'(x))^2 \, dx,
$$
where $y \in C^2[0, 1]$, and $y(0) = 0$ and $y(1) = 1$. Of all functions of the form

$$
y(x) = x + c_1 x (1 - x) + c_2 x^2 (1 - x),
$$
where $c_1$ and $c_2$ are constants, find the one that minimizes $J$. 