Julius-Maximilians-
UNIVERSITÄT
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Lehrstuhl für
MATHEMATIK IX

Übungen zur Optimale Steuerung (A. Borzì), Blatt 9

SoSe 2019 1 St., Mi 13-14, SE 30

(Calculus of Variation ODE, Optimal Control ODE, Optimal Control PDE)

(1) Use the Lagrange function to derive the first-order optimality system for the following optimal control problem

$$\begin{aligned} &\min J(y,u) := \frac{1}{2}\int_a^b |y(x) - y_d(x)|^2 \, dx + \frac{\nu}{2}\int_a^b |u(x)|^2 \, dx + (y(b))^2 \\ &\text{s.t.} \quad y'(x) = f(x)\, y(x) + u(x), \qquad y(a) = y_a, \\ &u \in L^2(a,b), \end{aligned}$$

where $\nu > 0$ and $f \in C^1[a,b]$, and $y_d \in L^2(a,b)$ is a given target trajectory.

(2) Derive the first-order optimality system for the following optimal control problem by Fréchet differentiation of the corresponding reduced cost functional

$$\begin{aligned} &\min J(y,u) := \frac{1}{2}\int_a^b |y(x) - y_d(x)|^2 \, dx + \frac{\nu}{2}\int_a^b |u(x)|^2 \, dx \\ &\text{s.t.} \quad -y''(x) = u(x), \qquad y(a) = y_a,\, y(b) = y_b, \\ &u \in L^2(a,b), \end{aligned}$$

where $\nu > 0$ and $y_d \in L^2(a,b)$ is a given target configuration.

(3) Consider the following optimal control problem

$$\begin{aligned} &\min J(y,u) := \frac{1}{2}\int_0^1 \left(|y(x)|^2 + |u(x)|^2\right) dx \\ &\text{s.t.} \quad y'(x) = u(x), \qquad y(0) = 0, \\ &u \in U_{ad} := \{u \in L^2(0,1) \,|\, u(x) \in [-1,1] \text{ a.e. in } (0,1)\}. \end{aligned}$$

Recall the Legendre condition and use the optimality system to show that the Hamiltonian has a maximum at the optimal trajectory. Write the optimal control as a function of the adjoint variable.

(4) Consider the optimal control problem given in (3). Show that the corresponding Hamiltonian has two relative minima that correspond to the controls $u(x) = 1$ and $u(x) = -1$, $x \in [0, 1]$, respectively. Show that corresponding to these controls the Hamiltonian system can be solved explicitly, and compute the value of the cost functional. Compare this result with the unique solution of the optimal control problem.

(5) The Hamiltonian system of the optimal control problem given in (3) admits infinite many solutions where the state function is not differentiable (only continuous). Construct one of these solutions assuming that there exists a point $\xi \in (0, 1)$ such that the adjoint function p is positive for $x < \xi$ and negative for $x > \xi$.