Trends in Mathematics

Alfio Borzì

Institut für Mathematik, Universität Würzburg
Lehrstuhl Mathematik IX - Wissenschaftliches Rechnen
and Würzburg - Wroclaw Center for Stochastic Computing
(WWCSC)

The Team Scientific Computing - WiReMIX

- Prof. Dr. Alfio Borzì
- Prof. Dr. Roland Griesmaier
- Petra Markert-Autsch
- Dr. Stephan Schmidt
- Dr. Souvik Roy
- Andreas Schindele
- Suttida Wongkaew
- Martin Sprengel
- Beatrice Gaviraghi
- Gabriele Ciaramella
- Juri Merger
- Duncan Kioi Gathungu
- Christian Schmiedecke
- Tanvir Rahman

www9.mathematik.uni-wuerzburg.de

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What is Mathematics?

Mathematics is the most practical science and the most abstract art

The Artist, the object, and the aesthetic value

The Artist is a person that feels, sees, ears, smells, tastes, touches an object and creates a work of aesthetic value.

Das Objekt ist ein Grenzwert im mathematischen Sinn. Man nähert sich der Objektivität ständig an, ohne das Objekt selbst je zu erreichen. Das Objekt, das man zu erreichen glaubt, ist stets das von der Intelligenz des Subjektes repräsentierte und interpretierte Objekt. (J.Piaget)

Aesthetic value is the value that an art work possesses in virtue of its capacity to elicit pleasure (positive value) or displeasure (negative value) when experienced.

The Mathematician, its senses and objects

A Mathematician is an artist with its own sense for ‘objects’.

A Mathematician is attracted by patterns observed with the five senses and the intellect: regularities, repetitions, symmetries, similarities, etc.. These constitute the object.

A Mathematician observes with the five senses and the intellect as any other person. However, he/she sees with the tools of mathematical analysis.

The act of creating a work of aesthetic value is the statement of a theorem and, more generally, of a theoretical framework.
A Mathematical work is usually considered beautiful as it recognizes new structures. It may be considered beautiful or ugly because of the way of how it is stated (e.g., the proof).

Mathematics is the most practical science

"Ohne Mathematik tappt man doch immer im Dunkeln."

Werner von Siemens.

Today, mathematics increasingly provide the modern foundation for much of science, engineering, economics, and business. The increasingly important challenges of deriving knowledge from huge amounts of data, whether numerical or experimental, of simulating complex phenomena accurately, and of dealing with uncertainty intelligently are some of the areas where mathematical scientists have important contributions to make going forward.

Mathematics is interdisciplinary

Mathematics work is becoming an increasingly integral and essential component of a growing array of areas of investigation in biology, medicine, social sciences, business, advanced design, climate, finance, advanced materials, and many more.

Education in the Mathematics

Mathematical sciences work involves the integration of mathematics, statistics, and computation, with sciences in the broadest sense and the interplay of these areas with areas of potential application.

Because of these interdisciplinary opportunities, mathematical education is never complete and, at the moment, it does not reflect these new trends in science.

Mathematical education in schools is often misleading: it poses emphasis on solving problems by using given formulas. This is wrong as much as teaching painting by paint-by-numbers.

Mathematics, like painting or poetry, is creative work. That makes it very difficult to teach. It takes time and effort to produce a work of art, and it takes a skilled teacher to recognize one. Unfortunately, schools' teachers are usually not fully educated mathematicians.

Classification and myth of mathematics

There is a traditional division of mathematics into pure mathematics and applied mathematics.

This division is misleading.

It reflects a conservative science politics, sometime cultural limitation and a superficial knowledge of mathematics history. However, it is common and standard.

There never have been and never will be any "applied sciences", there are only applications of sciences. (L. Pasteur)

It is often stated that 'Mathematics is the language of science'. However, language is the ability to acquire and use complex systems of communication; but mathematics does not communicate!

Some people also believe that 'Mathematics does not discover anything'. However, discovery is the act of detecting something new: Mathematics discovers new structures, new relationships and similarities, providing new reasoning to explain the knowledge gathered through observations.

Some new trends in Mathematics

- The topology of three-dimensional spaces
- Uncertainty quantification and stochastic models
- Networks and collective phenomena
- Computational biology
- Inverse problems and reconstruction techniques
- The interplay of mathematics and theoretical physics

The topology of a three-dimensional space

The three-dimensional sphere sits in a four-dimensional coordinate space as the collection of 4D vectors of unit length

$$x^2 + y^2 + z^2 + w^2 = 1$$

It has the property that any closed path on the sphere can be continuously deformed through closed paths, always staying on the sphere, so as to shrink it down to a point. Poincaré asked if the sphere was the only geometrical entity of finite extent, up to topological equivalence, with this property.

In 2002, Grigory Perelman succeeded in answering this question in the affirmative. He invoked deep ideas from analysis and geometry and focused on an idea of Hamilton on Ricci flow evolution equations and the development of singularities.

In mathematics, various geometric problems belong to this class, as do equations for the evolution of many different types of systems in science and engineering. Understanding singularity development for these equations would have a huge impact, because the behaviour of solutions near singularities can be so important. As a by-product the intertwining of the areas of differential geometry, topology, and PDEs will emerge.

Uncertainty quantification and stochastic models

It is one of the most important objectives of computer simulation to accurately represent real processes. However, our capabilities are limited by uncertainties in the models, their coefficients, and the input data. Moreover, different models may be required to represent different scales of the same system.

The purpose of uncertainty quantification is to develop techniques that augment present simulation methods with deterministic models to take into account of our statistical rather than deterministic knowledge of reality. With these methodologies the result of a simulation will be complete from a statistical point of view, thus enormously increasing its validity.

Another way to take into account uncertainty in deterministic models is to extend them by introducing stochastic terms.

It is reasonable to think that uncertainty quantification and stochastic theory will merge in a unique discipline.

Networks and collective phenomena

The emergence of online social networks is changing behaviour in many contexts, allowing decentralized collective phenomena and interaction among large groups. More generally, networks pervade our world from the behaviour of groups of animals or robots to communication and energy distribution.

Starting from a few interaction rules among agents, dynamical systems with very special structures result that allow to predict or explain unexpected phenomena and the emergence of collective configurations.

The study of networks is putting together different mathematical, physical, and social disciplines as dynamical systems, algebra, graph theory, statistics, statistical mechanics, optimization, chaos theory, synergetics, psychology, economy, political sciences, etc..

Computational biology

Biology is becoming one of the most important application fields of mathematics. At the level of micro biology, we certainly aim at the mathematical modelling of cells as evolutionary systems. In the same way, we need to develop accurate evolutionary models of complex systems as a group of cancer cells, an ensemble of bacillus, etc.. The problem of modelling temporal evolution of biological systems appears at all scales up to the study of people's population growth on the earth.

In micro biology there is the problem of pre- and post-processing of super-resolution microscopy images, including data mining.

Another fundamental mathematical problem in biology is the protein-folding problem. In fact, knowing the geometrical structure of a protein is an essential step in understanding its biological function. The protein-folding problem is to predict the three-dimensional structure of a protein from its primary sequence of amino-acids. This unsolved problem is usually formulated as a global optimization problem.

Inverse problems and reconstruction techniques

Inverse problems are those for which one creates an understanding of the internal structure of a system from external observations. The system itself is hidden, a black box that cannot be probed directly. Inverse problems arise in many different ways in medical imaging as well as in oil exploration.

It is reasonable to think that inverse problems will be combined with modern techniques of statistical inference. In fact, the number of parameters about which we seek inference is becoming the same as, or much larger than, the number of observations at our disposal (e.g., gene expression monitoring, functional MRI).

On the other hand, increasing the number of measurements results in a slowing down of the inversion process. For this reason, it is important to reconstruct a 'complete' information based only on partial measurements. This need is motivating the development of compressed sensing techniques.

The interplay of mathematics and theoretical physics

In my opinion, it is impossible to detach mathematics from physics. Already the works of Archimede show how these two disciplines enrich each other. In the mid-nineteenth century these fields were essentially one and the same. The formulation of Hilbert spaces and quantum mechanics as well as the formulation of Riemann geometry and general relativity are further recent examples.

Today theoretical physics is facing the quest of unifying quantum field theories with quantum gravity. One recent development, the Maldacena gauge gravity duality (AdS/CFT) theory, connecting general relativity with quantum field theories, is posing new challenges to mathematics that will require decades of investigation.

The AdS/CFT duality and further developments are showing new connections between gravity theory, hyperbolic geometry, equations of hydrodynamics, supersymmetry, Maxwell-, Dirac-, and Klein-Gordon equations, string theory, etc..

Unsolved fundamental problems

There are unsolved fundamental problems in algebra, additive and algebraic number theories, analysis, combinatorics, algebraic, discrete, and Euclidean geometries, dynamical systems, partial differential equations, and graph-, group-, model-, number-, set- and Ramsey-theories, as well as miscellaneous unsolved problems.

In the quest of solving these problems, new trends in Mathematics appear.

References

1. Herbert Robbins and Richard Courant, What is Mathematics? An Elementary Approach to Ideas and Methods, Oxford University Press, 1996.
2. National Research Council, The Mathematical Sciences in 2025. Washington, D.C., The National Academies Press, 2013.
3. Paul Lockhart, A Mathematician's Lament, Bellevue Literary Press, 2009.
4. Reviel Netz and William Noel, The Archimedes Codex, Weidenfeld & Nicolson, 2007.
5. Joel E. Cohen, Mathematics is Biology's Next Microscope, Only Better; Biology is Mathematics' Next Physics, Only Better, PLoS Biology 2 (12): e439, 2004.
6. Stanislaw M. Ulam, Adventures of a Mathematician, University of California Press, 1991.
7. Wikipedia, The Free Encyclopedia.
8. All my readings and thoughts.