Algebraic Multigrid for Optimality Systems

Alfio Borzì and Giuseppe Borzì

Objective: The fast solution of elliptic optimality systems on general 3D domains.

Applications: Distributed optimal control of elliptic systems

$$\min_{u \in U} J(y(u), u)$$

$$-\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(d_k \frac{\partial y}{\partial x_k}\right) + \sum_{k=1}^{d} c_k \frac{\partial y}{\partial x_k} \;=\; u + g$$

The optimality system has the following structure $\hat{\mathcal{A}}_k \mathbf{w}_k + \mathcal{B}_k \mathbf{w}_k = \mathcal{F}_k$ where

$$\hat{\mathcal{A}}_k = \begin{pmatrix} A_k & 0 \\ 0 & A_k^* \end{pmatrix}, \quad \mathcal{B}_k = \begin{pmatrix} 0 & -\frac{1}{\nu} B_k \\ B_k & 0 \end{pmatrix}, \text{ and } \mathcal{F}_k = \begin{pmatrix} \tilde{g}_k \\ \tilde{z}_k \end{pmatrix}.$$

Idea: Define AMG components based on A_k. Store only A_k, B_k, and $\mathbf{w}_k$.

Achievement: Robust AMG w.r.t changes of ν. Optimal memory and computational complexities.

In $\Omega = (0,1)^3$ with discontinuous anisotropic diffusion coefficients.

$$d_1(\mathbf{x}) = \begin{cases} 1 & y \ge x, \\ 10^2 & y < x, \end{cases} \quad d_2(\mathbf{x}) = \begin{cases} 10^2 & y \ge x, \\ 1 & y < x, \end{cases} \quad d_3(\mathbf{x}) = \begin{cases} 10^1 & y + z \ge 1, \\ 10^{-1} & y + z < 1. \end{cases}$$

Objective function: $z(x, y, z) = \sin(2\pi x)\cos(2\pi y)\sin(2\pi z)$.

Tracking properties; $N_i = 115200$, complexity $c_i/c_r = 3.25/3.31$.

ν	$\lvert u - z\rvert_0$	ρ	$No.iter.$
10^{-4}	3.56(-1)	0.30	21
10^{-6}	3.23(-1)	0.16	17
10^{-8}	4.57(-2)	0.13	16

Convergence properties ($\nu = 10^{-6}$)

	Standard coarsening			
N_i	ρ	$No.iter.$	c_i/c_r	$\lvert\lvert u - z\rvert\rvert$
486720	0.28	22	3.33/3.39	3.23(-1)
960400	0.30	25	3.35/3.42	3.22(-1)
	Aggressive coarsening			
N_i	ρ	$No.iter.$	c_i/c_r	$\lvert\lvert u - z\rvert\rvert$
486720	0.66	73	2.25/2.23	3.23(-1)
960400	0.67	80	2.26/2.25	3.22(-1)

Convergence analysis of AMG for optimality systems.

Under appropriate conditions there exists a positive constants $\delta < 1$ such that

$$(\hat{\mathcal{A}}_1 \mathcal{E}_1 \mathbf{w}, \mathcal{E}_1 \mathbf{w})_1 \le \delta^2 (\hat{\mathcal{A}}_1 \mathbf{w}, \mathbf{w})_1,$$

where $\mathcal{E}_k$ is the AMG error operator.