Parabolic Multigrid and Optimal Control Problems

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Objective: Solution of time-dependent reaction-diffusion problems.

Applications: Optimal control of systems governed by reaction-diffusion equations

$$\min_{u \in L^2(Q)} \frac{1}{2}||y - z||^2_{L^2(Q)} + \frac{\nu}{2}||u||^2_{L^2(Q)}$$

$$-\partial_t y + \Delta y + \delta e^y = u \quad \text{in } Q = \Omega \times (0, T)$$

The optimality system consists of two parabolic equations with opposite orientation:

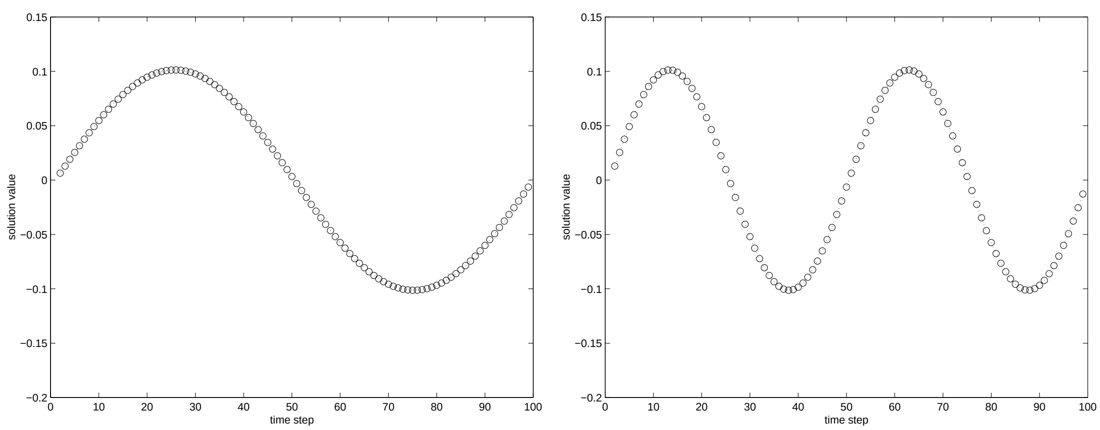
$$\begin{aligned} -\partial_t y + \Delta y + \delta e^y &= u, \\ \partial_t p + \Delta p + \delta e^y p &= -(y - z), \\ \nu u - p &= 0, \end{aligned}$$

with initial condition $y(\mathbf{x}, 0) = y_0(\mathbf{x})$ and terminal condition $p(\mathbf{x}, T) = 0$.

Approach: Develop space-time multigrid methods for systems of coupled parabolic equations with opposite orientation.

Achievement: Fast and robust multigrid solution.

A challenging test: $z(x, y, t) = \frac{1}{\pi^2}\sin(\pi x)\sin(\pi y)\sin(n\pi t)$. Choose $\delta = 10.0$ and $\nu = 10^{-5}$. The uncontrolled system ($u = 0$) blows up at finite time.



The computed solutions at $(x, y) = (0.5, 0.5)$ for all time steps (dashed line) almost coincide with the desired value (circle line). The case $n = 2$ (left) and $n = 4$ (right).

Convergence of space-time multigrid for linear optimality systems proved in the framework of twolevel Fourier analysis.