Exercises of Numerical PDEs
Sheet 2
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Exercise 1. Let Ω be a bounded open set in $\mathbb{R}^n$, and consider the boundary value problem

$$-\Delta u + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + c(x)u = f(x), \quad in \quad \Omega, \tag{1}$$

$$u = 0, \quad on \quad \partial\Omega \tag{2}$$

where $c \in C(\bar{\Omega})$, $c \geq 0$ on $\bar{\Omega}$, $b = (b_1, b_2, ..., b_n)$ is a constant vector, and $f \in L_2(\Omega)$.

1. Explain what it means for u to be a classical solution of the problem $(1) - (2)$.
2. State the weak formulation of this boundary value problem, and explain what it means for u to be a weak solution of $(1) - (2)$.

Let

$$a(u,v) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x) dx + \sum_{i=1}^{n} b_i \int_{\Omega} \frac{\partial u}{\partial x_i}(x) v(x) dx + \int_{\Omega} c(x) u(x) v(x) dx,$$

and

$$l(v) = \int_{\Omega} f(x) v(x) dx.$$

1. Show that $a(\cdot,\cdot)$ is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$, and $l(\cdot)$ is a linear form on $H_0^1(\Omega)$.
2. Show also using the Poincaré- Friedrichs inequality, that
   - $\exists c_0 > 0 \; \forall v \in H_0^1(\Omega) \; a(v,v) \geq c_0 ||v||^2_{H^1(\Omega)}$.
   - $\exists c_1 > 0 \; \forall v, w \in H_0^1(\Omega) \; |a(v,w)| \leq c_1 ||v||_{H^1(\Omega)} ||w||_{H^1(\Omega)}$.
   - $\exists c_2 > 0 \; \forall v \in H_0^1(\Omega) \; |l(v)| \leq c_2 ||v||_{H^1(\Omega)}$.

Hence deduce, using the Lax-Milgram theorem, that the boundary value problem $(1)-(2)$ has a unique weak solution $u \in H_0^1(\Omega)$.
(Points: 0.5+1+1+3).

Exercise 2. Consider the two-points boundary value problem for the second-order ordinary differential equation

$$-u'' + u' + x^2 u = 1 - |x|, \quad x \in (-1, 1), \tag{3}$$

$$u(-1) = u(1) = 0. \tag{4}$$

- Show that this boundary value problem has a unique weak solution, u, in $H_0^1(-1,1)$. Show also that the function $x \mapsto 1 - |x|$ belongs to $H^1(-1,1)$, but not to $C^1[-1,1]$. Hence deduce, using the differential equation, that u belongs to $H^3(-1,1)$, but not to $C^3[-1,1]$.

Now suppose that N is a positive even integer, $h = 2/N$, and let $x_i = -1+ih$, $i = 0, ..., N$. Consider the following finite difference scheme for the numerical solution of the above problem:

$$-\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + \frac{U_{i+1} - U_{i-1}}{2h} + x_i^2 U_i = 1 - |x_i|, \quad 1 \leq i \leq N-1,$$

$$U_0 = U_N = 0.$$

1. Rewrite the difference scheme as a system of linear equations in matrix form with the vector of unknowns $U = (U_1, ..., U_{N-1})^T$, and comment on the structure of the matrix.

2. Define the global error, e, of this finite difference scheme by $e_i = u(x_i) - U_i$, $i = 0, ..., N$. Show that

$$-\frac{e_{i+1} - 2e_i + e_{i-1}}{h^2} + \frac{e_{i+1} - e_{i-1}}{2h} + x_i^2 e_i = \Phi_i, \quad 1 \leq i \leq N-1,$$

$$e_0 = e_N = 0.$$

where Φ is the truncation error.

3. Show that

$$||e||_{1,h} \leq C_1 ||\Phi||_h,$$

where C_1 is a positive constant, and the mesh-dependent norms $||\cdot||_h$ and $||\cdot||_{1,h}$ are as in the Lecture Notes.

4. Express Φ_i in terms of $u(x_{i-1}), u(x_i), u(x_{i+1}), u''(x_i)$ and $u'(x_i)$. Show that

$$||\Phi||_h \leq C_2 h ||u||_{H^3(-1,1)},$$

where C_2 is another positive constant. Hence deduce that

$$||u - U||_{1,h} \leq C_1 C_2 h ||u||_{H^3(-1,1)}.$$

(Points: 2+1+1.5+2+2)