Exercises of Numerical PDEs
Sheet 3
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Exercise 1. Let $\Omega = (0,1) \times (0,1)$. Consider the elliptic boundary value problem

$$-\Delta u + c(x,y)u = f(x,y), \quad in \quad \Omega, \tag{1}$$

$$u = g(x,y), \quad on \quad \Gamma = \partial\Omega, \tag{2}$$

where $c \in C(\bar{\Omega})$, $c(x,y) \geq 0$ on $\bar{\Omega}$, $f \in C(\bar{\Omega})$ and $g \in C(\Gamma)$.
Construct a five point difference scheme for the approximate solution of this problem on a uniform mesh $\bar{\Omega}_h = \Omega_h \cup \Gamma_h$ of stepsize $h = 1/N$. Rewrite the difference scheme as a system of linear equations and comment on the structure of the matrix of the system.
Explain what is meant by the global error e, and the truncation error, Φ, of this finite difference scheme. Show that

$$-(D_x^+ D_x^- e_{ij} + D_y^+ D_y^- e_{ij}) + c(x_i, y_j)e_{ij} = \Phi_{ij}, \quad in \quad \Omega_h,$$

$$e_{ij} = 0, \quad on \quad \Gamma_h.$$

Express Φ_{ij} in terms of the values of u, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ at the mesh points. Hence deduce that if the solution of the problem (1) – (2) belongs to $C^4(\bar{\Omega})$, then

$$|\Phi_{ij}| \leq \frac{h^2}{12}\left(||\frac{\partial^4 u}{\partial x^4}||_{C(\bar{\Omega})} + ||\frac{\partial^4 u}{\partial y^4}||_{C(\bar{\Omega})}\right).$$

Show also that

$$||u - U||_{1,h} = ||e||_{1,h} \leq \frac{5}{4}||\Phi||_h,$$

and that

$$||u - U||_{1,h} \leq \frac{5}{48}h^2\left(||\frac{\partial^4 u}{\partial x^4}||_{C(\bar{\Omega})} + ||\frac{\partial^4 u}{\partial y^4}||_{C(\bar{\Omega})}\right).$$

Now, suppose that on

$$\Gamma_N := \{(x,y) : x = 1, 0 < y < 1\},$$

the Dirichlet boundary condition $u = g(x,y)$ has been replaced by the Neumann boundary condition $\frac{\partial u}{\partial n} = G(x,y)$, where n is the unit outward normal to Γ_N and $G \in C(\Gamma_N)$; i.e. consider the following mixed (Dirichlet-Neumann) boundary value problem:

$$-\Delta u + c(x,y)u = f(x,y), \quad in \quad \Omega,$$

$$u = g(x,y), \quad on \quad \Gamma_D = \Gamma / \Gamma_N,$$

$$\frac{\partial u}{\partial n} = G(x,y), \quad on \quad \Gamma_N.$$

Construct a finite difference scheme for the approximate solution of this problem on a uniform mesh $\bar{\Omega}_h = \Omega_h \cup \Gamma_h$ of stepsize $h = 1/N$. Rewrite the difference scheme as a system of linear equations and comment on the structure of the matrix of the system.
(Points: 14).