MASTER THESIS

The spherically symmetric Vlasov-Poisson system

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Introduction

For centuries, mankind has been fascinated by the beauty and complexity of our universe. As part of the scientific revolution, the exploration of our solar system, stars and outer space has started, and the universe has been examined from a scientific perspective. Nowadays, there are endless examples of how scientists of various disciplines explore the depth of our cosmos. Along with this development, mathematics has been forming the basis, and precise mathematical models have become essential. A well-known topic arising in astrophysics is the N-body problem. Using this model, one can describe our solar system, which comprises the eight planets, dwarf planets and other small solar system bodies orbiting the sun, interacting by Newton's law of gravity. When it comes to galaxies, N is of the order 10^{10}-10^{12} and a different, statistical approach is favourable. A galaxy is described by a distribution function $f = f(t, x, v)$, where $t \in \mathbb{R}$ denotes the time and the phase space element $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ denotes the position and velocity of a star. Integrating over some phase space region $dx\,dv$ yields the numbers of stars contained in this region at the time t. In this thesis we make the reasonable assumption that stars only interact by their self-generated gravitational field $F = F(t, x)$, neglecting the existence of relativistic or thermodynamic effects, the expansion of the universe, electromagnetic fields of galaxies or other local phenomena like supernovas. Hence, any trajectory of a star with unit mass only obeys Newton's law of motion

$$\dot{x} = v, \quad \dot{v} = F(t, x).$$

Collisions of stars are to be neglected and no stars are born or die in our model. Hence, the distribution function f is constant along orbits of stars, and satisfies a first-order conservation law on phase space, which yields the Vlasov equation

$$\partial_t f + v \cdot \partial_x f + F \cdot \partial_v f = 0.$$

The gravitational potential $U = U(t, x)$ induced by the stars is given by the Poisson equation

$$\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t, x) = 0,$$

and is coupled with the Vlasov equation via

$$F = -\partial_x U, \quad \rho(t, x) = \int f(t, x, v)\,\mathrm{d}v.$$

The Vlasov-Poisson system is well understood for smooth initial data. In order to understand a differential equation it is important to consider its steady states, especially to analyse the stability under certain perturbations. There is a whole class of different types of steady states. Mathematicians and physicists are most interested in those, which are somehow profit-yielding to analyse, and which are more or less close to real galaxies. This immediately leads us to the class of compactly supported steady states with finite mass. One family of such steady states was constructed by Kurth, cf. [1]. One specific feature is the singularity of the phase space distribution at the boundary and the discontinuity of the corresponding mass density. Hence, one cannot talk about Kurth-type steady states as solutions of the Vlasov-Poisson system in the classical sense; one needs to find a more general solution concept. It is the motivation of Chapter 2 to do this and to give a global existence result.

Despite the astrophysical motivation, a more general solution concept is also interesting in a theoretical mathematical aspect. For all differential equations, one can easily introduce a weak formulation of the problem. Our more general formulation is special for kinetic equations and can be seen somewhere between the weak and classical formulation of the Vlasov-Poisson system.

In Chapter 3, we investigate flat galaxies, which are modelled by density functions $f = f(t, x, v)\delta(x_3)\delta(v_3)$, which are delta distributed in the x_3 and v_3-axis. Hence, all stars have to be in the $(x_1, x_2, 0)$ plane with velocity $v = (v_1, v_2, 0)$. However, they still create a 3-dimensional gravitational potential collectively, causing the so-called flat Vlasov-Poisson system to differ significantly from the 2-dimensional Vlasov-Poisson system. In particular, we want to model certain flat steady states numerically, which are functions of the energy and the angular momentum. We will elaborate on our numerical algorithm and discuss the numerical observations.

Notations

The natural numbers are here defined without zero, i.e., $\mathbb{N} = \mathbb{N} \setminus \{0\}$, and we write $\mathbb{N}_0$ to add zero. For $x, y \in \mathbb{R}^n$ we denote by

$$x \cdot y := \sum_{k=1}^{n} x_k y_k, \quad |x| := \sqrt{x \cdot x}$$

the Euclidean scalar product and norm. The term "$R > 0$" always induces that $R \in \mathbb{R}$. For any $R > 0$ and $x \in \mathbb{R}^n$, the n-dimensional open ball is denoted by

$$B_R^n(x) := B_R(x) := \{y \in \mathbb{R}^n : |x - y| < R\}, \quad B_R := B_R(0).$$

If I is an interval, we always demand that $\mathring{I}$ is non-empty, where $\mathring{I}$ denotes the interior of I. For any differentiable function $f = f(t, x, v), t \in \mathbb{R}, x \in \mathbb{R}^3, v \in \mathbb{R}^3$, the partial derivatives are denoted by

$$\partial_t f := \frac{\partial}{\partial t} f, \quad \partial_x f := \nabla_x f, \quad \partial_v f := \nabla_v f.$$

For $t \in \mathbb{R}$, we denote by $f(t)$ the function $f(t) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with $(x, v) \mapsto f(t, x, v)$. In estimates the letter C is used as any positive constant, where the dependency is obvious from the context. C may also change from line to line. To emphasise any dependency on for example a function f or number n, we write, often just one time:

$$C[f], \quad C[n] \quad \text{or} \quad C[f, n].$$

If the constant is fixed during this thesis, we will use subscripts. For $k \in \mathbb{N}$, the space of k times continuously differentiable functions is denoted by $C^k(\mathbb{R}^n)$, where a subscript "c" indicates compactly supported functions. As usual, the support of a function is denoted by $\operatorname{supp} f$. For any $p \in [1, \infty]$, we denote by $L^p(\mathbb{R}^n)$ the Lebesgue space and by $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^n)}$ the corresponding norm. For any set $M \subset \mathbb{R}^n$, $\mathbb{1}_M$ denotes its indicator function,

$$\mathbb{1}_M(x) = 1 \quad \text{if } x \in M, \quad \mathbb{1}_M(x) = 0 \quad \text{if } x \notin M.$$

For any measurable set M, the Lebesgue measure is denoted by $\operatorname{vol}(M)$. In Chapter 2, the variables $x, \tilde{x}$ and $v, \tilde{v}$ are always in $\mathbb{R}^3$ and $z := (x, v), \tilde{z} := (\tilde{x}, \tilde{v}) \in \mathbb{R}^6$. Any integral $\int$ without domain of integration always extends over $\mathbb{R}^3$ or $\mathbb{R}^6$, depending on what makes sense in the context.

By $\operatorname{Lip}(\mathbb{R}^n)$ we denote the space of locally Lipschitz-continuous functions, i.e, for all $R > 0$ there exists some $L > 0$ such that $|f(y) - f(\tilde{y})| \leq L|y - \tilde{y}|$ for all $|y - \tilde{y}| < R$. From now on, Lipschitz-continuous always means locally Lipschitz-continuous.

The following definition of spherical symmetry is fundamental. Some functions $\rho = \rho(t, x)$ or some functions on phase space $f = f(t, x, v)$ are **spherically symmetric** if

$$\rho(t, x) = \rho(t, Ax), \quad f(t, x, v) = f(t, Ax, Av), \quad A \in \mathrm{SO}(3);$$

note that the latter symmetry differs from the symmetry on $\mathbb{R}^6$. The spaces

$$\begin{aligned}
\mathbb{B}(\mathbb{R}^3) &:= \{\rho : \mathbb{R}^3 \to \mathbb{R} : \rho \text{ is measureable and bounded}\}, \\
\mathbb{B}_c(\mathbb{R}^3) &:= \{\rho \in \mathbb{B}(\mathbb{R}^3) : \rho \text{ has compact support}\}, \\
\mathbb{B}^s(\mathbb{R}^3) &:= \{\rho \in \mathbb{B}(\mathbb{R}^3) : \rho \text{ is spherically symmetric}\}, \\
\mathbb{B}^s_c(\mathbb{R}^3) &:= \mathbb{B}_c \cap \mathbb{B}^s(\mathbb{R}^3)
\end{aligned}$$

are of great interest in this thesis as well; for phase space functions $\mathring{f} = \mathring{f}(x, v)$, they are defined analogously with the corresponding definition of spherical symmetry given above. If a variable of time is involved, we will just write

$$\rho \in \mathbb{B}(I \times \mathbb{R}^3), \ldots, \mathbb{B}^s_c(I \times \mathbb{R}^3),$$

which means that $\rho(t) \in \mathbb{B}(\mathbb{R}^3), \ldots, \mathbb{B}^s_c(\mathbb{R}^3)$ for all $t \in I$. Note that the support of some $\rho \in \mathbb{B}_c(I \times \mathbb{R}^3)$ may depend on $t \in I$. Finally, for $\rho \in \mathbb{B}^s(\mathbb{R}^3)$ its **radial function** exists and is denoted by $\bar{\rho}$, i.e,

$$\bar{\rho} : [0, \infty[\to \mathbb{R}, \quad \rho(x) = \bar{\rho}(|x|), \quad x \in \mathbb{R}^3.$$

2 The spherically symmetric Vlasov-Poisson system with a new solution concept

2.1 Classical solutions and results

We want to introduce the classical gravitational Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{2.1}$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0, \tag{2.2}$$

$$\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v, \tag{2.3}$$

where the initial distribution function $\mathring{f} = f(0,x,v)$ is given. For any interval I, where $\mathring{I}$ is non-empty, we say that $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a classical solution of this system if the following holds:

i) $f \in C^1(I \times \mathbb{R}^3 \times \mathbb{R}^3)$, $U, \rho \in C^1(I \times \mathbb{R}^3)$ and U is twice continuously differentiable with respect to x.

ii) f, ρ, U solve (2.1)-(2.3) on $I \times \mathbb{R}^3 \times \mathbb{R}^3$ and $I \times \mathbb{R}^3$ respectively.

iii) For all compact subintervals $J \subset I$, the force field $-\partial_x U$ is bounded on $J \times \mathbb{R}^3$.

For smooth initial data, the following classical global existence result is known:

Theorem 2.1.
For every non-negative initial data $\mathring{f} \in C_c^1(\mathbb{R}^6)$, there exists a unique classical solution f with $f(0) = \mathring{f}$ on $\mathbb{R}^6$ which is global in time, i.e. $I = [0, \infty[$.

This result was established over a long period of time and we will briefly say a few words about the milestones, on which this thesis is based. In 1952 the astronomer Rudolf Kurth [2] established a local existence and uniqueness result. Jürgen Batt [3] found in 1977 a criterion under which a solution is global in time and established a global existence result for spherically symmetric data. In 1989 two different proofs of Theorem 2.1 were independently found by Klaus Pfaffelmoser [4] and by Pierre-Louis Lions and Benoit Perthame [5], based on the above mentioned results.

In order to understand the Vlasov-Poisson system, we want to analyse both the Vlasov equation (2.1) and the Poisson equation (2.2).

2.2 The Poisson equation

For some function $\rho : \mathbb{R}^3 \to \mathbb{R}$, we say that $U \in C^2(\mathbb{R}^3)$ is the solution of the Poisson equation if

$$\Delta U(x) = 4\pi\rho(x), \quad x \in \mathbb{R}^3,$$
$$\lim_{|x|\to\infty} U(x) = 0.$$

We collect some well known results about potential theory.

Lemma 2.2.
Let $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$ and define $U(x) := U_\rho(x) := -\int_{\mathbb{R}^3} \rho(y)/|x-y|\,\mathrm{d}y$. Then

a) $U \in C^1(\mathbb{R}^3)$ *and*

$$\partial_x U(x) = \int \frac{x-y}{|x-y|^3}\rho(y)\,\mathrm{d}y, \qquad \lim_{|x|\to\infty} U(x) = 0.$$

b) For all $p \in [1,3[$, there exists constants $C_p > 0$ such that

$$\|\partial_x U\|_\infty \leq C_p \|\rho\|_p^{p/3} \|\rho\|_\infty^{1-p/3}.$$

Especially, we have for $p = 1$ that $C_1 = 3(2\pi)^{2/3}$.

c) If $\rho \in C_c^1(\mathbb{R}^3)$, then $U_\rho \in C^2(\mathbb{R}^3)$ solves the Poisson equation on $\mathbb{R}^3$.

Proof. We refer to [6] for a proof. □

Later, we will consider spherically symmetric initial data. As we will see in Section 2.4, this property is preserved by the Vlasov-Poisson system. Thus, the Poisson equation with spherically symmetric mass density ρ is of great interest.

Lemma 2.3.
Let $\rho \in C_c^1(\mathbb{R}^3)$ be spherically symmetric, $U := U_\rho$ and let $\bar{\rho} : [0,\infty[\to \mathbb{R}$ be the radial function of ρ, i.e., $\bar{\rho}(|\cdot|) = \rho$ on $\mathbb{R}^3$. Then

a) U is spherically symmetric and for its radial function $\bar{U}$ and $r > 0$ it holds that

$$\frac{1}{r^2}(r^2\bar{U}'(r))' = 4\pi\bar{\rho}(r), \quad \lim_{r\to\infty} \bar{U}(r) = 0, \tag{2.4}$$

b)

$$\begin{aligned}
\bar{U}(r) &= -\frac{4\pi}{r}\int_0^r \eta^2\bar{\rho}(\eta)\,\mathrm{d}\eta - 4\pi\int_r^\infty \eta\bar{\rho}(\eta)\,\mathrm{d}\eta,\\
\bar{U}'(r) &= \frac{4\pi}{r^2}\int_0^r \eta^2\bar{\rho}(\eta)\,\mathrm{d}\eta, \quad r > 0,\\
\partial_x U(x) &= \frac{x}{r}\,\bar{U}'(r), \quad |x| = r > 0,\\
\bar{U}'(0) &= \lim_{r\to 0}\bar{U}'(r) = 0 \quad \textit{and} \quad \bar{U}(0) = \lim_{r\to 0}\bar{U}(r) \ \textit{exists in}\ \mathbb{R}.
\end{aligned} \tag{2.5}$$

Proof. Since any $A \in \mathrm{SO}(3)$ is length-preserving, we have

$$U(Ax) = \int \frac{\rho(y)}{|Ax - y|}\,\mathrm{d}y = \int \frac{\rho(Ay)}{|A(x-y)|}\,\mathrm{d}y = \int \frac{\rho(y)}{|x-y|}\,\mathrm{d}y = U(x).$$

Using Lemma 2.2 c), we know that $\Delta U = 4\pi\rho$ on $\mathbb{R}^3$. The Laplace operator in spherical coordinates reads

$$\Delta\cdot = \frac{1}{r^2}\partial_r(r^2\partial_r\cdot) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\cdot) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\cdot\,.$$

Using the symmetry of U, we obtain with $r = |x|$ that

$$\frac{1}{r^2}(r^2\bar{U}'(r))' = \Delta U(x) = 4\pi\rho(x) = 4\pi\bar{\rho}(r).$$

To verify part b), we simply show that $\bar{U}$ solves the radial Poisson equation (2.4). The uniqueness of the Poisson equation then yields the uniqueness of $\bar{U}$. Using the fundamental theorem of calculus and the compact support of ρ, we get

$$\begin{aligned}\bar{U}'(r) &= \frac{4\pi}{r^2}\int_0^r \eta^2\bar{\rho}(\eta)\,\mathrm{d}\eta - \frac{4\pi}{r}r^2\bar{\rho}(r) - 0 + 4\pi r\bar{\rho}(r)\\ &= \frac{4\pi}{r^2}\int_0^r \eta^2\bar{\rho}(\eta)\,\mathrm{d}\eta,\end{aligned}$$

and hence

$$(r^2\bar{U}'(r))' = 4\pi r^2\bar{\rho}(r),$$

i.e., $\bar{U}$ is a solution of the radial Poisson equation (2.4). □

We want to extend part b) to $\rho \in \mathbb{B}_c^s(\mathbb{R}^3)$ by a density argument. We can choose some $(\rho_n) \subset C_c^1(\mathbb{R}^3)$ spherically symmetric with $\rho_n \to \rho$ in $L^2(\mathbb{R}^3)$ and $\|\rho_n\|_\infty \le C$, where C only depends on $\|\rho\|_\infty$. By Lemma 2.2, U is in $C^1(\mathbb{R}^3)$ and

$$\|\partial_x U_n - \partial_x U\|_\infty \le C_2\|\rho_n - \rho\|_2^{2/3}\|\rho_n - \rho\|_\infty^{1/3} \to 0.$$

Given that (2.5) holds for $\rho \in \mathbb{B}_c^s(\mathbb{R}^3)$, we have the very important fact that $\bar{U}' \in \mathrm{Lip}([0,\infty[)$ and hence $F := -\partial_x U \in \mathrm{Lip}(\mathbb{R}^3)$, cf. (2.20). This so-called force field F couples the Poisson equation with the Vlasov equation (2.1), which is analysed in the following section:

2.3 The characteristic system and solutions in the characteristic sense

Imagine a galaxy, where $X(t) \in \mathbb{R}^3$ is the position of a star at some time $t \ge 0$. According to Newton's laws of motion, the trajectory of the star and the force field are related by

$$\dot{X}(t) = V(t), \quad \dot{V}(t) = F(t, X(t)). \tag{2.6}$$

A proven method of solving many kinetic equations, like the transport equation, the Vlasov-Maxwell system or the Vlasov-Poisson system, is the method of characteristics. Simply speaking, we want to solve (2.6) for "all" particles at "every" starting point $(X(t=0), V(t=0)) = (x, v)$. If we know the position and velocity of all particles for all times, we can merge them together to a solution f of the original kinetic equation. Newton's equations (2.6) will be named the *characteristic system*, and a trajectory will be named *characteristic*. Strictly speaking, we have the following definition:

Definition and Remark 2.4.
Let $F : I \times \mathbb{R}^3 \to \mathbb{R}^3$ be continuous, Lipschitz-continuous with respect to x and bounded on $J \times \mathbb{R}^3$ for every compact subinterval $J \subset I$. Then the following holds:

a) For every $t \in I$ and $z := (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, there exists a unique solution

$$I \ni s \mapsto (X, V)(s, t, x, v) \in \mathbb{R}^6$$

*of the **characteristic system***

$$\dot{X} = V, \quad \dot{V} = F(s, X), \quad (X, V)(t, t, x, v) = (x, v). \tag{2.7}$$

*We call $s \mapsto (X, V)(s, t, x, v) = (X, V)(s)$ the **characteristics** of the system and we will sometimes suppress the (t, x, v) argument. The **characteristic flow** $Z := (X, V)$ is continuous on $I \times I \times \mathbb{R}^6$ and Lipschitz-continuous with respect to z.*

b) For every $s, t \in I$, the mapping $Z(s, t, \cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is a homeomorphism with inverse $Z^{-1}(s, t, \cdot) = Z(t, s, \cdot)$, and for all $\tau \in I$, $Z(s, t, z) = Z(s, \tau, Z(\tau, t, z))$ holds.

c) For every $s, t \in I$, $Z(s, t, \cdot)$ is measure preserving, i.e.,

$$|\det \partial_z Z(s, t, \cdot)| = 1, \qquad \text{almost everywhere on } \mathbb{R}^6.$$

d) For any $\Phi \in \mathbb{B}_c(\mathbb{R}^6)$ and any measurable subset $D \subset \mathbb{R}^6$, the integral transformation rule holds:

$$\int_D \Phi(Z(s, t, z))\,\mathrm{d}z = \int_{Z(s,t,D)} \Phi(z)\,\mathrm{d}z, \qquad s, t \in I.$$

It is worth comparing this remark to the analogous result in the classical setup, cf. [6]. There, one additionally demands that F is continuously differentiable with respect to x. In return, $Z(s, t, \cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is a C^1-diffeomorphism and is measure preserving for all $z \in \mathbb{R}^6$. However, since we want to investigate characteristics from some discontinuous distribution function f, we have to work with a discontinuous mass density ρ, which generally does not induce a continuously differentiable force field F. On the other hand, a Lipschitz-continuous force field seems to be just enough to have well-defined characteristics, which is, as we will see throughout this thesis, all we need.

Proof. In the following we will write $G(s,z) := (v, F(s,x))$ for the right side of (2.7).
a) We fix some $(t,z) \in I \times \mathbb{R}^6$ and write $Z(s) := Z(s,t,z)$, $s \in I$. Note that G is Lipschitz-continuous with respect to z. By Picard-Lindelöf, there exists some $\delta > 0$ and a unique solution $(X,V) : [t-\delta, t+\delta] \to \mathbb{R}^3 \times \mathbb{R}^3$ of (2.7) with $(X,V)(t) = (x,v)$. In order to extend the solution to $I \times \mathbb{R}^3$, we need to bound (X,V). Integrating the characteristic equations yields

$$|V(s)| \le |V(t)| + \left| \int_t^s F(\tau, X(\tau))\, \mathrm{d}\tau \right| \le |v| + \delta \|F\|_\infty \qquad \text{and}$$
$$|X(s)| \le |X(t)| + \left| \int_t^s V(\tau)\, \mathrm{d}\tau \right| \le |x| + |v|\delta + \delta^2 \|F\|_\infty, \qquad s \in [t-\delta, t+\delta].$$

Since $F_{|J \times \mathbb{R}^3}$ is bounded for every compact subinterval $J \subset I$, the solution can be extended to $I \times \mathbb{R}^6$. Let us consider the characteristic flow Z. To show the Lipschitz-continuity, we fix some s and t to obtain with the Lipschitz-continuity of F

$$|Z(s,t,z) - Z(s,t,\tilde{z})| \le |z - \tilde{z}| + C[F] \int_s^t |Z(\tau,t,z) - Z(\tau,t,\tilde{z})|\, \mathrm{d}\tau,$$

and hence, by Gronwall,

$$|Z(s,t,z) - Z(s,t,\tilde{z})| \le \mathrm{e}^{C|t-s|} |z - \tilde{z}| = C|z - \tilde{z}|.$$

b) For a start, we want to show that $Z(s,t,z) = Z(s,\tau,Z(\tau,t,z))$ for all $s,\tau \in I$. For this purpose we fix some $\tau \in I$ and use the initial condition to see that we have equality at the time $s = \tau$:

$$Z(\tau,\tau,Z(\tau,t,z)) = Z(\tau,t,z) = Z(s,t,z), \qquad t \in I, z \in \mathbb{R}^6.$$

Since both $s \mapsto Z(s,t,z)$ and $s \mapsto Z(s,\tau,Z(\tau,t,z))$ solve (2.7), equality follows by uniqueness. In particular, we have $Z(s,t,Z(t,s,z)) = Z(t,t,z) = z$ and $Z^{-1}(s,t,\cdot) = Z(t,s,\cdot)$.
c) The idea to verify c) is the following: We need to construct some smooth Z_ε, which approximates our flow Z and where the classical result holds. Since our assertion only needs to hold almost everywhere and the Integral transformation does work for Z due to Lemma 2.5, it seems reasonable to show equality with a Du-Bois-Reymond argument.

To this end let $J \in C_c^\infty(\mathbb{R}^3)$ be a mollifier with $J \ge 0$, $\operatorname{supp} J \subset B_1^3$ and $\int_{\mathbb{R}^3} J = 1$. For any small $\varepsilon > 0$, we define

$$J_\varepsilon(x) := \frac{1}{\varepsilon^3} J\Big(\frac{x}{\varepsilon}\Big), \quad x \in \mathbb{R}^3$$

and define the smooth forces

$$F_\varepsilon(t,x) := \big(J_\varepsilon * F(t)\big)(x), \quad (t,x) \in I \times \mathbb{R}^3,$$

where $*$ denotes the convolution on $\mathbb{R}^3$. Obviously, for $R > 0$ and $x, \tilde{x} \in B_R^3$ we find that

$$|F_\varepsilon(t,x) - F_\varepsilon(t,\tilde{x})| \le \int J_\varepsilon(y)|F(t,x-y) - F(t,\tilde{x}-y)|\,\mathrm{d}y \le C[R]|x-\tilde{x}|, \tag{2.8}$$

$$|F_\varepsilon(t,x) - F(t,x)| \le \int_{B_\varepsilon(x)} J_\varepsilon(x-y)|F(t,y) - F(t,x)|\,\mathrm{d}y \le C[R]\varepsilon. \tag{2.9}$$

By $s \mapsto Z(s,t,z)$ and $s \mapsto Z_\varepsilon(s,t,z)$ we denote the solutions of

$$\begin{aligned} &\dot{X} = V, \qquad \dot{V} = F(s,X) \qquad \text{with } Z(t,t,z) = z \\ \text{and} \quad &\dot{X}_\varepsilon = V_\varepsilon, \qquad \dot{V}_\varepsilon = F_\varepsilon(s,X) \qquad \text{with } Z_\varepsilon(t,t,z) = z \end{aligned}$$

respectively. With (2.8), (2.9) and a Gronwall argument similar to the one appearing in the proof of a), we obtain

$$Z_\varepsilon \to Z, \quad \text{as} \quad \varepsilon \to 0, \quad \text{uniformly on } J \times J \times B_R^6, \tag{2.10}$$

for any $R > 0$ and any compact interval $J \subset I$. Next, we denote $H(s) := \det \partial_z Z_\varepsilon(s,t,z)$ for $s,t \in I$, $z \in \mathbb{R}^6$. As we can now interchange the order of taking partial derivatives of $Z_\varepsilon \in C^1(I \times I \times \mathbb{R}^6; \mathbb{R}^6)$, we have by Jacobi's formula

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} H(s) &= H(s)\,\mathrm{trace}\Big((\partial_z Z_\varepsilon(s,t,z))^{-1}\partial_z\partial_s Z_\varepsilon(s,t,z)\Big) \\ &= H(s)\,\mathrm{trace}\Big((\partial_z Z_\varepsilon(s,t,z))^{-1}\partial_z G_\varepsilon(s, Z_\varepsilon(s,t,z))\partial_z Z_\varepsilon(s,t,z)\Big) \\ &= H(s)\,\mathrm{div}_z G_\varepsilon\big(s, Z_\varepsilon(s,t,z)\big) = 0. \end{aligned}$$

In the last step we used that G_ε is a divergence-free vector field. Hence, $\det \partial_z Z_\varepsilon(\cdot,t,z)$ is constant on I. With $Z_\varepsilon(t,t,z) = z$, it follows that

$$\det \partial_z Z_\varepsilon(s,t,z) = 1, \quad s \in I, \tag{2.11}$$

which is just the classical result. Next, we pick any test function $\varphi \in C_c^\infty(\mathbb{R}^6)$ and observe, using Lemma 2.5 with $Z(s,t,\cdot) = Z^{-1}(t,s,\cdot)$ and (2.11), that

$$\begin{aligned} \int \varphi(z)|\det \partial_z Z(s,t,z)|\,\mathrm{d}z &= \int \varphi(Z(t,s,z))\,\mathrm{d}z = \lim_{\varepsilon\to 0}\int \varphi(Z_\varepsilon(t,s,z))\,\mathrm{d}z \\ &= \lim_{\varepsilon\to 0}\int \varphi(z)|\partial_z Z_\varepsilon(s,t,z)|\,\mathrm{d}z = \int \varphi(z)\,\mathrm{d}z. \end{aligned}$$

Note that we can go to the limit simply utilising Lebesgue's dominated convergence theorem since $\operatorname{supp}\varphi(Z_\varepsilon(s,t,\cdot)) = Z_\varepsilon(t,s,\operatorname{supp}\varphi)$ is uniformly compact and $Z_\varepsilon \to Z$ pointwise. Hence, $|\det \partial_z Z(s,t,\cdot)| = 1$ almost everywhere, where the null set may depend on s,t. This proves c), and combining c) with Lemma 2.5 immediately yields d). □

The following integral transformation rule was used in the latter proof:

Lemma 2.5. *(Transformation rule for Lipschitz functions)*
Let $n \in \mathbb{N}$, $D \subset \mathbb{R}^n$ be a measurable set, $f \in \mathbb{B}_c(\mathbb{R}^n)$ and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be (locally) Lipschitz-continuous and bijective. Then Φ is differentiable almost everywhere, and both of the following integrals exist:

$$\int_D f(\Phi(y))\,\mathrm{d}y = \int_{\Phi(D)} f(y)|\det \nabla\Phi(y)|\,\mathrm{d}y.$$

Proof. We refer to [7], *263F Corollary. □

It is the main task of this chapter to expand the solution concept. To this end, we make the following important consideration: Let $f \in C^1(I \times \mathbb{R}^6)$ and let $s \mapsto (X, V)(s)$ be any characteristics. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} f(s, X(s), V(s)) &= (\partial_t f + \partial_x f \cdot \dot{X} + \dot{V} \cdot \partial_v f)(s, X(s), V(s)) \\ &= (\partial_t f + \partial_x f \cdot V + F \cdot \partial_v f)(s, X(s), V(s)).\end{aligned}$$

The first term describes the change of f along characteristics, and is 0 if f is constant along characteristics. The last term is 0 if f solves the classical Vlasov equation (2.1) with $F = -\partial_x U$. By Remark 2.4 b), we know that through each point (t, x, v) there passes a characteristic curve. Hence, we get the following equivalence, which will be the motivation for our more general solution concept:

Lemma 2.6.
Let $f \in C^1(I \times \mathbb{R}^6)$ and let $F \in C(I \times \mathbb{R}^3; \mathbb{R}^3)$ be continuously differentiable with respect to x. Then $f(s, Z(s))$ is constant for every characteristic $s \mapsto Z(s)$ of (2.7) if and only if $\partial_t f + v \cdot \partial_x f + F \cdot \partial_v f = 0$ on $I \times \mathbb{R}^6$.

By reducing the regularity of f, " $\iff$ " becomes obviously wrong, since the partial derivatives of f do not exist. The left side, however, remains well-defined even for discontinuous functions f. This leads to the following generalisation of the solution concept:

Definition 2.7.
*A measurable function $f : I \times \mathbb{R}^6 \to \mathbb{R}$ solves the Vlasov-Poisson system in the **characteristic sense** if the following holds:*

*(i) The **mass density** ρ and the **force field** F*

$$\rho_f(t, x) := \rho(t, x) := \int f(t, x, v)\,\mathrm{d}v,\ F_f(t, x) := F(t, x) := -\int \frac{x - y}{|x - y|^3}\rho_f(t, y)\,\mathrm{d}y,$$

exist for all $(t, x) \in I \times \mathbb{R}^3$.

(ii) For all $(t, z) \in I \times \mathbb{R}^6$, there exists a unique solution $s \mapsto Z_f(s, t, z) = Z(s, t, z)$ of

$$\dot{X}_f = V_f, \quad \dot{V}_f = F_f(s, X), \quad Z_f(t, t, z) = z.$$

with the properties of Remark 2.4.

(iii) *f is constant along its* ***induced*** *characteristics, i.e., for all $(t,z) \in I \times \mathbb{R}^6$, the mapping $s \mapsto f(s, Z_f(s,t,z))$ is constant on I.*

The terminology "Z_f is induced by f" comes from the fact, that the force field F_f is exactly the Newtonian gravitational force induced by the mass density ρ_f of the distribution function f. Note that Definition 2.7 is a successful generalisation in the following sense:

- Being a classical solution implies being a characteristic solution due to Lemma 2.2 a).
- Being a smooth characteristic solution implies being a classical solution due to Lemma 2.6.
- Steady states of the well-known type $\Phi(E,L)$ for reasonable Φ satisfy (i)-(iii) and are therefore solutions in the characteristic sense. For details on Φ, E, L, we refer to Section 3.3 or Chapter 2 in [6]. In particular, Kurth steady states solve the Vlasov-Poisson system in the characteristic sense.

The following lemma, especially its proof, demonstrates how the Vlasov-Poisson system can be entirely described by its initial datum and its characteristic system.

Lemma 2.8.
Let $\mathring{f} \in \mathbb{B}_c(\mathbb{R}^6)$ and let Z be any characteristic flow like in Remark 2.4. Define $f(t,z) := \mathring{f}(Z(0,t,z))$ for all $(t,z) \in I \times \mathbb{R}^6$ with $0 \in I$. Then the following holds:

a) If $Z = Z_f$ in the sense of Definition 2.7 $(i),(ii)$, then f is constant along its induced characteristics with $f(0) = \mathring{f}$ on $\mathbb{R}^6$.

b) For every $t \in I$ and $p \in [1,\infty]$, it holds that $\operatorname{supp} f(t) = Z_f(t,0,\operatorname{supp} \mathring{f})$ and $\|f(t)\|_p = \|\mathring{f}\|_p$. The latter property will be called the ***conservation of L^p-norms****.*

c) $f \in C(I; L^1(\mathbb{R}^6))$, i.e., the mapping $t \mapsto f(t)$ is continuous with respect to the $L^1(\mathbb{R}^6)$-norm.

Proof. *a)* Let $s \mapsto W_f(s)$ be any characteristic induced by f. By Remark 2.4 b), there exists some $(t_0, z_0) \in I \times \mathbb{R}^6$ such that $W_f(s) = Z_f(s, t_0, z_0)$ for all $s \in I$. Since

$$Z_f(0, s, Z_f(s, t_0, z_0)) = Z_f(0, t_0, z_0),$$

we have

$$f(s, W_f(s)) = \mathring{f}(Z_f(0, s, W_f(s))) = \mathring{f}(Z_f(0, s, Z_f(s, t_0, z_0))) = \mathring{f}(Z_f(0, t_0, z_0)),$$

which implies that $f(\cdot, W_f(\cdot))$ is constant on I.
b) For $t \in I$, using $Z^{-1}(t,0,\cdot) = Z(0,t,\cdot)$, we have

$$\begin{aligned}\operatorname{supp} f(t) = \operatorname{supp} \mathring{f}(Z(0,t,\cdot)) &= \left\{z \in \mathbb{R}^6 : Z(0,t,z) \in \operatorname{supp} \mathring{f}\right\} \\ &= \left\{Z(t,0,Z(0,t,z)) : Z(0,t,z) \in \operatorname{supp} \mathring{f}\right\} \\ &= \left\{Z(t,0,z) : z \in \operatorname{supp} \mathring{f}\right\} = Z(t,0,\operatorname{supp} \mathring{f}).\end{aligned}$$

By Remark 2.4 b) and d), we have for all $p \in [1, \infty[$

$$\|f(t)\|_p^p = \int |\mathring{f}(Z(0,t,z))|^p \,\mathrm{d}z = \int_{Z(0,t,\mathbb{R}^6)} |\mathring{f}(z)|^p \,\mathrm{d}z = \|\mathring{f}\|_p^p,$$
$$\|f(t)\|_\infty = \sup_{z\in\mathbb{R}^6} |\mathring{f}(Z(0,t,z))| = \sup_{z\in\mathbb{R}^6} |\mathring{f}(z)| = \|\mathring{f}\|_\infty.$$

c) For any $\varepsilon > 0$, we can find some $g \in C_c^\infty(\mathbb{R}^6)$ such that

$$\|\mathring{f} - g\|_1 < \varepsilon. \tag{2.12}$$

For any $t, t' \in I$, using the compact support and (2.12) with Remark 2.4 d), we find that

$$\begin{aligned}\|f(t) - f(t')\|_1 &\leq \int |\mathring{f}(Z(0,t,z)) - g(Z(0,t,z))| \,\mathrm{d}z + \int |\mathring{f}(Z(0,t',z)) - g(Z(0,t',z))| \,\mathrm{d}z \\ &\quad + \int |g(Z(0,t,z)) - g(Z(0,t',z))| \,\mathrm{d}z \\ &\leq 2\varepsilon + \int_{B_R^6} |g(Z(0,t,z)) - g(Z(0,t',z))| \,\mathrm{d}z \\ &\leq 2\varepsilon + \|\partial_z g\|_\infty \frac{4\pi}{3} R^3 \sup_{z\in B_R^6} |Z(0,t,z) - Z(0,t',z)|.\end{aligned}$$

Since Z is continuous, the last term tends to zero as $t \to t'$ and the assertion follows. □

2.4 Local existence

Before diving into the local existence result, let us comment on why a characteristic solution was defined for all measurable functions, whereas the following local existence result additionally needs boundedness and spherical symmetry for the initial data. In Definition 2.7 a characteristic solution requires unique characteristics, which suggests a Lipschitz-continuous right side F. As we will see in the proof, this is accomplished by the spherical symmetry of $\mathring{f}$. The crucial point is, that spherical symmetry is preserved by the Vlasov-Poisson system, ensuring the right side F to stay Lipschitz-continuous. However, with regard to steady states of Kurth type [1], we allow in Definition 2.7 distribution functions to become singular, as long as the force fields and characteristics are well-defined. Note that it is easy to check that Kurth type steady states are indeed solutions in the characteristic sense, even though our existence result does not apply, since they are not bounded.

Theorem 2.9.
Let $\mathring{f} \in \mathbb{B}_c^s(\mathbb{R}^6)$ be non-negative. Then there exists some $T > 0$ and a unique characteristic solution $f \in \mathbb{B}_c^s([0,T[\times\mathbb{R}^6)$ with $f(0) = \mathring{f}$ on $\mathbb{R}^6$.

Due to Lemma 2.6 and 2.8 and the definition of characteristic solutions, the Vlasov-Poisson system can be fully treated by considering the characteristic flow. Thereby, we need to find for any given initial datum $\mathring{f} \in \mathbb{B}_c^s(\mathbb{R}^6)$ some $T > 0$ and for all $(t, z) \in [0, T[\times\mathbb{R}^6$ a unique solution $[0, T[\ni s \mapsto Z(s, t, z)$ of

$$\dot{X} = V, \quad \dot{V} = -\iint \frac{X(s,t,z) - \tilde{x}}{|X(s,t,z) - \tilde{x}|^3} \mathring{f}(Z(0, s, \tilde{x}, \tilde{v}))\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v}, \quad Z(t, t, z) = z, \tag{2.13}$$

which enjoys the properties of Remark 2.4. In order to attack this differential equation for Z, it is the first step of our proof to consider the following iterative scheme:

STEP 1 We define the 0th iterate by

$$f_0(t, z) := \mathring{f}(z), \quad (t, z) \in [0, \infty[\times\mathbb{R}^6.$$

If the nth iterate is defined, we define the mass density and force field by

$$\rho_n(t, x) := \int f_n(t, x, v)\,\mathrm{d}v, \quad F_n(t, x) := -\int \frac{x - y}{|x - y|^3} \rho_n(t, y)\,\mathrm{d}y, \quad (t, x) \in [0, \infty[\times\mathbb{R}^3.$$

Next, we denote by $s \mapsto Z_n(s, t, z)$ the unique solution of

$$\dot{X} = V, \quad \dot{V} = F_n(s, X), \quad Z_n(t, t, z) = z.$$

For all $n \in \mathbb{N}_0, t \geq 0, z \in \mathbb{R}^6$, we can now define

$$f_{n+1}(t, z) := \mathring{f}(Z_n(0, t, z)).$$

For all $n \in \mathbb{N}_0, t \geq 0$, the iterates are well-defined and have the following properties:

a) $f_n(t) \in \mathbb{B}_c^s(\mathbb{R}^6)$, $f_n \in C([0, \infty[; L^1(\mathbb{R}^6))$, $\rho_n(t) \in \mathbb{B}_c^s(\mathbb{R}^3)$, $F_n \in C([0, \infty[\times\mathbb{R}^3; \mathbb{R}^3)$, $F_n(t)$ is Lipschitz-continuous and, most importantly, again spherically symmetric. Furthermore, all characteristic flows Z_n have the properties of Remark 2.4.

b) Defining $\mathring{R}, \mathring{P} > 0$ such that $\operatorname{supp} \mathring{f} \subset B^3_{\mathring{R}} \times B^3_{\mathring{P}}$, we have

$$f_n(t, x, v) = 0 \quad \text{for } |v| \geq P_n(t) \text{ or } |x| \geq \mathring{R} + \int_0^t P_n(s)\,\mathrm{d}s,$$

where $P_n(t) := \sup\{|V_{n-1}(s, 0, z)| : z \in \operatorname{supp} \mathring{f}, 0 \leq s \leq t\}$ and $P_0(t) := \mathring{P}$.

c)

$$\|\rho_n(t)\|_\infty \leq \frac{4\pi}{3} \|\mathring{f}\|_\infty P_n(t)^3. \tag{2.14}$$

d)

$$\|F_n(t)\|_\infty \leq C_{\mathring{f}} P_n(t)^2, \quad \text{where } C_{\mathring{f}} \text{ is given in (2.18).}$$

With the exception of the spherical symmetry, assertion a) follows by induction with Lemma 2.2, Lemma 2.8 and rewriting

$$F_n(t,x) = -\frac{x}{r^3}\int_{|y|\le r}\int f_n(t,y,v)\,\mathrm{d}v\,\mathrm{d}y.$$

We want to show spherical symmetry by induction: Note that $f_0(t)$ is spherically symmetric by definition. Now assume that $f_n(t)$ is spherically symmetric for any $n \in \mathbb{N}_0$. Hence, $\rho_n(t)$ and $F_n(t)$ are spherically symmetric. We need to make sure that this property is transferred to Z_n in such way that the next iterate $f_{n+1}(t)$ is again spherically symmetric. Therefore, let $A \in \mathrm{SO}(3)$ and let $B \in \mathbb{R}^{6\times 6}$ such that $Bz = B(x,v) = (Ax, Av)$. Recall that

$$f_{n+1}(t, Bz) = \mathring{f}(Z_n(0,t,Bz)).$$

We want to show that the characteristics are invariant under simultaneous rotation in x and v in the sense that

$$Z_n(0,t,Bz) = BZ_n(0,t,z). \tag{2.15}$$

To this end we use the spherical symmetry of $f_n(t)$ and Lemma 2.3 to rewrite the characteristic system:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}X_n(s,t,z) = -\frac{X_n(s,t,z)}{r}\frac{4\pi}{r^2}\int_0^r \eta^2\bar{\rho}_n(t,\eta)\,\mathrm{d}\eta, \tag{2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}X_n(s,t,z) = V_n(s,t,z), \tag{2.17}$$

where $r := |X_n(s,t,z)|$. By no means, $z \mapsto Z_n(0,t,z)$ is linear, as equation (2.15) may mislead us. The same holds for (2.16), where the nonlinearity in X_n is beautifully hidden in r. Nevertheless, since A is some rotation and vanishes under the norm, (2.16) is "linear" in the following sense: If

$$s \mapsto X_n(s,t,z), \quad X_n(t,t,z) = x$$

is a solution of (2.16), then

$$s \mapsto AX_n(s,t,z), \quad AX_n(t,t,z) = Ax$$

is a solution of (2.16), too. Involving the linear equation (2.17) to our consideration, we have $Z_n(s,t,Bz) = BZ_n(s,t,z)$ by uniqueness, which finally proves (2.15) and hence

$$f_{n+1}(t,Bz) = \mathring{f}(BZ_n(0,t,z)) = f_{n+1}(t,z).$$

We prove b) by induction: $n = 0$ is trivial. Now pick any $n \in \mathbb{N}$. Using Lemma 2.8 b), i.e. $\operatorname{supp} f_n(t) = Z_{n-1}(t,0,\operatorname{supp}\mathring{f})$, we can rewrite

$$\begin{aligned} P_n(t) &= \sup\big\{|V_{n-1}(s,0,z)| : z \in \operatorname{supp}\mathring{f}, 0 \le s \le t\big\} \\ &= \sup\big\{|v| : z \in Z_{n-1}(s,0,\operatorname{supp}\mathring{f}), 0 \le s \le t\big\} \\ &= \sup\big\{|v| : z \in \operatorname{supp} f_n(s), 0 \le s \le t\big\}. \end{aligned}$$

If $|v| \geq P_n(t)$, then $f_n(t,x,v) = 0$ for all $x \in \mathbb{R}^3$. If on the other hand

$$|x| \geq \mathring{R} + \int_0^t P_n(s)\,\mathrm{d}s,$$

we have

$$\begin{aligned} |X_{n-1}(0,t,z)| &= |x + \int_0^t V_{n-1}(0,s,z)\,\mathrm{d}s| \geq |x| - \int_0^t |V_{n-1}(0,s,z)|\,\mathrm{d}s \\ &\geq |x| - \int_0^t P_n(s)\,\mathrm{d}s \geq \mathring{R}. \end{aligned}$$

By definition of $\mathring{R}$, we have $\mathring{f}(x,\cdot) = 0$ for $x \geq \mathring{R}$, and hence

$$f_n(t,x,v) = \mathring{f}(X_{n-1}(0,t,z), V_{n-1}(0,t,z)) = 0, \quad v \in \mathbb{R}^3.$$

Assertion c) follows immediately with b) and Remark 2.4 d):

$$\begin{aligned} \|\rho_n(t,\cdot)\|_\infty &= \Big\| \int \mathring{f}(Z_{n-1}(0,t,\cdot,v)) \mathbb{1}_{\{|v|\leq P_n(t)\}}(v)\,\mathrm{d}v \Big\|_\infty \\ &\leq \mathrm{vol}\{|v| \leq P_n(t)\} \|\mathring{f}\|_\infty = \frac{4\pi}{3} P_n(t)^3 \|\mathring{f}\|_\infty, \quad t \geq 0. \end{aligned}$$

In the last step we only applied the formula of the volume of a ball with radius $P_n(t)$. To verify d), we simply use Lemma 2.2 and the conservation of L^p-norms to obtain

$$\begin{aligned} \|F_n(t)\|_\infty &\leq (3(2\pi)^{2/3}) \|\rho_n(t)\|_1^{1/3} \|\rho_n(t)\|_\infty^{2/3} \\ &\leq (3(2\pi)^{2/3}) \big(4\pi/3\big)^{2/3} \|\mathring{f}\|_1^{1/3} \|\mathring{f}\|_\infty^{2/3} P_n(t)^2, \quad t \geq 0. \end{aligned}$$

Hence,

$$C_{\mathring{f}} := 4 \cdot 3^{1/3} \cdot \pi^{4/3} \|\mathring{f}\|_1^{1/3} \|\mathring{f}\|_\infty^{2/3} \tag{2.18}$$

only depends on $\mathring{f}$ and the properties of the iterates are proved.

Step 2 In this step we want to control the support of $f_n(t)$ and $\rho_n(t)$ uniformly in n. Using b) from Step 1, we discover that it is sufficient to bound $P_n(t)$, i.e., we can find some $\delta > 0$ and some function $Q : [0,\delta[\to [0,\infty[$ such that

$$P_n(t) \leq Q(t), \quad n \in \mathbb{N}_0, t \in [0,\delta[.$$

For this purpose remember $\mathring{P}$ and $C_{\mathring{f}}$ from Step 1. We define $\delta := (\mathring{P} C_{\mathring{f}})^{-1}$ and

$$Q : [0,\delta[\to [0,\infty[, \quad t \mapsto \frac{\mathring{P}}{1 - \mathring{P} C_{\mathring{f}} t} = \frac{C_{\mathring{f}}}{\delta - t}. \tag{2.19}$$

It is easy to check that Q is the maximal solution of the integral equation

$$Q(t) = \mathring{P} + C_{\mathring{f}} \int_0^t Q^2(s)\,\mathrm{d}s.$$

We want to prove that Q is the desired bound of P_n on $[0, \delta[$ by induction:
For $n = 0$, we obviously have $P_0(t) = \mathring{P} \le Q(t)$. From here on, assume that the assertion holds for $n \in \mathbb{N}_0$, i.e., $P_n(t) \le Q(t)$ for $t \in [0, \delta[$. In order to bound $P_{n+1}(t)$, we need to consider $|V_n(s, 0, z)|$ for $s \in [0, t]$, $z \in \operatorname{supp} \mathring{f}$. With d) from Step 1 and the induction hypothesis we obtain

$$\begin{aligned} |V_n(s,0,z)| &= \Big|V_n(0,0,z) + \int_0^s \dot{V}_n(\tau,0,z)\,\mathrm{d}\tau\Big| \le |v| + \int_0^s \|F_n(\tau)\|_\infty \,\mathrm{d}\tau \\ &\le \mathring{P} + C_{\mathring{f}} \int_0^s P_n^2(\tau)\,\mathrm{d}\tau \le \mathring{P} + C_{\mathring{f}} \int_0^t Q^2(\tau)\,\mathrm{d}\tau = Q(t) \end{aligned}$$

for all $z = (x, v) \in \operatorname{supp} \mathring{f}$, $0 \le s \le t < \delta$. Thereby, we have $P_{n+1}(t) \le Q(t)$ for $t \in [0, \delta[$, and the proof is completed. Accordingly, for every positive $\delta_0 < \delta$, there exists a radius $\mathcal{R} > 0$ such that

$$\operatorname{supp} f_n(t) \subset B^6_{\mathcal{R}} \subset \mathbb{R}^6 \quad \text{and} \quad \operatorname{supp} \rho_n(t) \subset B^3_{\mathcal{R}} \subset \mathbb{R}^3$$

for all $t \in [0, \delta_0], n \in \mathbb{N}_0$. With this uniform radius $\mathcal{R}$ and the length of the time interval δ in mind, we are ready for the next step.

Step 3 In order to expect convergence of the iterates, we need to control the difference of two consecutive iterates. To be successful, estimates may only depend on $\mathring{f}$ and δ_0 but not on $n \in \mathbb{N}$ or $t \in [0, \delta_0]$. In this step we want to find a uniform Lipschitz-bound of the force fields in the sense that

$$|F_n(t,x) - F_n(t,y)| \le C[\delta_0, \mathring{f}, R]|x - y|, \quad R > 0 \text{ and } x, y \in B^3_R. \tag{2.20}$$

In the classical setup, this can be done by finding a uniform bound on $\partial_x^2 U_n$. In [6], the entire regularity of the characteristics and the mass density was exploited to obtain this estimate, using a Gronwall argument. However, we have to find another way, since $\partial_x Z$ or $\partial_x \rho$ do not exist in our case. Once again, we rely on the spherical symmetry of the iterates and use the special formula of our force field

$$F_n(t,x) = \frac{x}{r}\,\bar{F}_n(t,r), \quad \bar{F}_n(t,r) = -\frac{4\pi}{r^2}\int_0^r \eta^2 \bar{\rho}_n(t,\eta)\,\mathrm{d}\eta.$$

Recall that C or $C[\delta_0, \mathring{f}, R]$ may change from line to line and that any dependencies are often denoted just once. First of all, we want to show that $\bar{F}_n(t)$ is Lipschitz-continuous uniformly in n and t. For any $R > 0$ and $t \in [0, \delta_0]$, $0 < u < r \le R$, we obtain with (2.14) and Q from (2.19) that

$$\begin{aligned} |\bar{F}_n(r) - \bar{F}_n(u)| &\le C\Big|\frac{1}{r^2}\int_u^r \eta^2 \bar{\rho}_n(t,\eta)\,\mathrm{d}\eta\Big| + C\Big|\Big(\frac{1}{r^2} - \frac{1}{u^2}\Big)\int_0^u \eta^2 \bar{\rho}_n(t,\eta)\,\mathrm{d}\eta\Big| \\ &\le C[\delta_0, \mathring{f}]\Big(\frac{1}{r^2}|r^3 - u^3| + |r-u|\frac{u^3(u+r)}{u^2 r^2}\Big) \le C[R]|r-u|. \end{aligned} \tag{2.21}$$

Deriving (2.20) from (2.21) is straight forward: First, we fix some $x, y \in B^3_R$ and assume that $x, y \neq 0$; the case $x \neq 0$ and $y = 0$ is trivial. If $|x| = |y| = r$, then

$$|F_n(t,x) - F_n(t,y)| = \Big|\frac{x}{r}\,\bar{F}_n(t,r) - \frac{y}{r}\,\bar{F}_n(t,r)\Big| \le |x-y|\,\Big|\frac{\bar{F}_n(t,r)}{r}\Big| \le C|x-y|.$$

If $x = \lambda y$ for some $\lambda > 0$ and writing $u = |y|$, we use (2.21) to obtain

$$|F_n(t,x) - F_n(t,y)| = \Big|\frac{\lambda y}{\lambda u}\,\bar{F}_n(t,\lambda u) - \frac{y}{u}\,\bar{F}_n(t,u)\Big| \le C\big||\lambda y| - |y|\big| \le C|x-y|.$$

The general case $x, y \neq 0$ follows immediately by inserting the term $F_n(t, |x|/|y|\,y)$, which completes Step 3. Due to the fact that $F_n(t,x) = x\|\mathring{f}\|/|x|^3$ for all $x \in \mathbb{R}^3 \backslash B^3_{\mathcal{R}}$ and $t \in [0,\delta_0], n \in \mathbb{N}_0$, the local Lipschitz-continuity immediately implies global Lipschitz-continuity.

Step 4 This is the main component to prove the local existence. In the classical setup, one can show that f_n is a Cauchy sequence, then prove the necessary regularity of the induced potential U to obtain uniform convergence of the characteristic flow $Z_n \to Z$. This approach does not seem to work in our setup for several reasons; the difference of the regularity of $f \in C^1_c$ in the classical and $f \in \mathbb{B}^s_c$ in our setup is too much. On the other hand, the regularity of our characteristic flow is much closer to the classical characteristic flow. As we have seen at the beginning of this proof, the Vlasov-Poisson system can be completely treated by investigating its characteristic flow and it seems reasonable to focus on this. The idea to show convergence is the following: Taking advantage of the spherical symmetry, we start by estimating the difference of two iterates of the force by the difference of two iterates of the corresponding characteristic flow. On the other hand, the characteristic flow can easily be estimated by the force field. Exploiting this Gronwall loop will show that F_n is a Cauchy sequence. Accordingly, we obtain a limiting field F, which hopefully has the necessary regularity to obtain a limiting characteristic flow Z with the properties of Remark 2.4.

Firstly, we fix any $t \in [0,\delta_0]$, $z \in \mathbb{R}^6$ and write $Z_n(s,t,z) = Z_n(s)$. By (2.20), we have

$$|\dot{V}_{n+1}(s) - \dot{V}_n(s)| \le C|X_{n+1}(s) - X_n(s)| + |F_{n+1}(s, X_n(s)) - F_n(s, X_n(s))|, \qquad (2.22)$$

and hence, we have to consider the second term. For $|x| = r \le \mathcal{R}, s \in [0,\delta_0]$, simply observe that

$$F^*_n := |F_{n+1}(s,x) - F_n(s,x)| = \frac{4\pi}{r^2}\Big|\int_0^r \eta^2\big(\bar{\rho}_{n+1}(s,\eta) - \bar{\rho}_n(s,\eta)\big)\,\mathrm{d}\eta\Big| \le Cr. \qquad (2.23)$$

On the other hand, since

$$\begin{aligned}\{Z_n(0,s,z) : |y| \le r\} &= \{\tilde{z} \in \mathbb{R}^6 : \exists\, z \in B^3_r \times \mathbb{R}^3 \text{ such that } Z_n(s,0,\tilde{z}) = z\} \\ &= \{\tilde{z} \in \mathbb{R}^6 : |X_n(s,0,\tilde{z})| \le r\},\end{aligned}$$

and denoting $z = (y,v)$, we can rewrite the force

$$\begin{aligned} -\bar{F}_n(s,r) &= \frac{1}{r^2}\int_{|y|\le r} \rho_n(s,y)\,\mathrm{d}y = \frac{1}{r^2}\int_{\{z\in\mathbb{R}^6 : |y|\le r\}} \mathring{f}(Z_{n-1}(0,s,z))\,\mathrm{d}z \\ &= \frac{1}{r^2}\int_{\{Z_{n-1}(0,s,z) : |y|\le r\}} \mathring{f}(z)\,\mathrm{d}z = \int_{\{z\in\mathbb{R}^6 : |X_{n-1}(s,0,z)|\le r\}} \mathring{f}(z)\,\mathrm{d}z \end{aligned} \qquad (2.24)$$

to obtain

$$
\begin{aligned}
F_n^* &\le \frac{1}{r^2}\Big| \int_{\{z\in\mathbb{R}^6:|X_n(s,0,z)|\le r\}} \mathring{f}(z)\,\mathrm{d}z - \int_{\{z\in\mathbb{R}^6:|X_{n-1}(s,0,z)|\le r\}} \mathring{f}(z)\,\mathrm{d}z\Big| \\
&\le \frac{1}{r^2}\|\mathring{f}\|_\infty \operatorname{vol}(D_n),
\end{aligned}
\tag{2.25}
$$

where we defined

$$
\begin{aligned}
D_n := \{z \in \operatorname{supp}\mathring{f} :&|X_n(s,0,z)| \le r < |X_{n-1}(s,0,z)| \\
&\vee |X_{n-1}(s,0,z)| \le r < |X_n(s,0,z)|\}.
\end{aligned}
$$

Next, defining

$$
d_n := \sup_{z\,\in\,\operatorname{supp}\mathring{f}} |X_n(s,0,z) - X_{n-1}(s,0,z)|,
$$

we easily observe that

$$
\begin{aligned}
\operatorname{vol}(D_n) \le\; &\operatorname{vol}\{z \in \operatorname{supp}\mathring{f} : |X_n(s,0,z)| \le r < d_n + |X_n(s,0,z)| \\
&\qquad\qquad \vee |X_{n-1}(s,0,z)| \le r < d_n + |X_{n-1}(s,0,z)|\} \\
\le\; &\operatorname{vol}\{z \in \operatorname{supp}\mathring{f} : |X_n(s,0,z)| \le r < d_n + |X_n(s,0,z)|\} \\
&+ \operatorname{vol}\{z \in \operatorname{supp}\mathring{f} : |X_{n-1}(s,0,z)| \le r < d_n + |X_{n-1}(s,0,z)|\} \\
=:\; &\operatorname{vol}(D_n^1) + \operatorname{vol}(D_n^2).
\end{aligned}
$$

Our aim is to get rid of the X_n-terms and obtain an estimate of the form

$$
F_n^* \le \frac{C}{r^2}\big(\operatorname{vol}(D_n^1) + \operatorname{vol}(D_n^2)\big) \le C d_n.
$$

To do so, we will use the fact that our characteristic flow is measure preserving to eliminate the X_n-terms in the following way:

$$
\begin{aligned}
\operatorname{vol}(D_n^1) &= \operatorname{vol}(Z_n(s,0,D_n^1)) \\
&= \operatorname{vol}\{Z_n(s,0,z) : z \in \operatorname{supp}\mathring{f} \;\wedge\; |X_n(s,0,z)| \le r < d_n + |X_n(s,0,z)|\} \\
&= \operatorname{vol}\{(y,v) \in Z_n(s,0,\operatorname{supp}\mathring{f}) : |y| \le r < d_n + |y|\} \\
&\le \operatorname{vol}\Big(\{y \in B^3_{\mathcal{R}} : r - d_n < |y| < r\} \times B^3_{\mathcal{R}}\Big) \\
&\le C[\mathcal{R}]\left(r^3 - (r-d_n)^3\right) \le C(d_n^3 + 3d_n^2 r + 3d_n r^2),
\end{aligned}
$$

Obviously, the same result holds for D_n^2; we just have to insert Z_{n-1} instead of Z_n and hence

$$
\operatorname{vol}(D_n) \le C(d_n^3 + 3d_n^2 r + 3d_n r^2). \tag{2.26}
$$

We remove the powers of d_n as follows: If $r \le d_n$, we make use of (2.23) to obtain

$$
F_n^* \le Cr \le Cd_n.
$$

If $r > d_n$, (2.25) with (2.26) yields

$$F_n^* \le \frac{C}{r^2}(r^2 d_n + 3r^2 d_n + 3r^2 d_n) \le C d_n.$$

After combining both results, we see our first success:

$$\begin{aligned}\|F_{n+1}(s) - F_n(s)\|_\infty &\le C \sup_{z \in \operatorname{supp} \mathring{f}} |X_n(s,0,z) - X_{n-1}(s,0,z)| \\ &\le C \sup_{z \in \operatorname{supp} \mathring{f}} |Z_n(s,0,z) - Z_{n-1}(s,0,z)|. \end{aligned} \tag{2.27}$$

The other way around, i.e, estimating the difference of the flows by the difference of the forces, can be easily obtained with the characteristic system and (2.22):

$$\begin{aligned}&|Z_{n+1}(s,0,z) - Z_n(s,0,z)| = \left| \int_0^s \left(\dot{Z}_{n+1}(\tau,0,z) - Z_n(\tau,0,z)\right) \mathrm{d}\tau \right| \\ &\le C \int_0^s |Z_{n+1}(\tau,0,z) - Z_n(\tau,0,z)| \,\mathrm{d}\tau + C \int_0^s \|F_{n+1}(\tau) - F_n(\tau)\|_\infty \,\mathrm{d}\tau,\end{aligned}$$

and by Gronwall

$$|Z_{n+1}(s,0,z) - Z_n(s,0,z)| \le C \int_0^s \|F_{n+1}(\tau) - F_n(\tau)\|_\infty \,\mathrm{d}\tau. \tag{2.28}$$

After combining (2.27) and (2.28), we can conclude the convergence as follows. First,

$$\|F_{n+1}(\tau) - F_n(\tau)\|_\infty \,\mathrm{d}\tau \le C \int_0^s \|F_n(\tau) - F_{n-1}(\tau)\|_\infty \,\mathrm{d}\tau.$$

Applying this estimate n-times, then using $\|F_1(\tau) - F_0(\tau)\|_\infty \le C[\mathring{f}, \delta_0]$, yields

$$\|F_{n+1}(t) - F_n(t)\|_\infty \le C^n \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} \|F_1(\tau_1) - F_0(\tau_1)\|_\infty \,\mathrm{d}\tau_1 \,\mathrm{d}\tau_2 \ldots \mathrm{d}\tau_n \le C \frac{C^n t^n}{n!}.$$

Using the power series of the exponential function and the Cauchy criterion finally implies

$$\|F_{n+k}(t) - F_n(t)\|_\infty \le \sum_{j=n}^{j=n+k-1} \|F_{j+1}(t) - F_j(t)\|_\infty \le C \sum_{j=n}^{\infty} \frac{(Ct)^j}{j!} \longrightarrow 0,$$

as $n \to \infty$, for all $t \in [0, \delta_0]$. Consequently, $F_n \to F$ uniformly on $[0, \delta_0] \times \mathbb{R}^3$. Moreover, (2.20) implies that F is Lipschitz-continuous with respect to x, which yields well-defined characteristics $s \mapsto Z(s,t,z)$ with

$$\dot{X} = V, \quad \dot{V} = F(s, X), \quad Z(t,t,z) = z,$$

and hence $Z_n \to Z$ uniformly on $[0, \delta_0] \times [0, \delta_0] \times \mathbb{R}^6$. Note that Z has the properties of Remark 2.4. Next, we need to make sure that

$$f : [0, \delta_0] \times \mathbb{R}^6 \to \mathbb{R}, \quad f(t,z) := \mathring{f}(Z(0,t,z)) \tag{2.29}$$

is the desired local solution in the characteristic sense, i.e, we have to check (*i*)-(*iii*) from Definition 2.7. Obviously, f is measureable and $\int f(t,x,v)\,\mathrm{d}v \in \mathbb{R}$ for all $(t,x) \in [0,\delta_0] \times \mathbb{R}^3$. Since $X_n(t,0,\cdot) \to X(t,0,\cdot)$ pointwise on $\mathbb{R}^6$, we obtain by Lebesgue's dominated convergence theorem and (2.24) that

$$\begin{aligned}\bar{F}(t,r) &= \lim_{n\to\infty} \bar{F}_n(t,r) = -\lim_{n\to\infty} \int_{\{z\in\mathbb{R}^6:|X_{n-1}(t,0,z)|\le r\}} \mathring{f}(z)\,\mathrm{d}z \\ &= -\int_{\{z\in\mathbb{R}^6:|X(t,0,z)|\le r\}} \mathring{f}(z)\,\mathrm{d}z = -\frac{1}{r^2}\int_{\{z\in\mathbb{R}^6:|y|\le r\}} \mathring{f}(Z(0,t,z))\,\mathrm{d}z.\end{aligned}$$

This implies

$$F(t,x) = -\iint \frac{x-y}{|x-y|^3} f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v,$$

which is the necessary relation between F, f and Z from (*i*) and (*ii*). Lastly, f is constant along its induced characteristics due to Lemma 2.8, which proves (*iii*).

Since $\delta_0 \in]0,\delta[$ was arbitrary, our constructed solution (2.29) and all induced quantities can at least be extended to $[0,\delta[$. Next, we want to show uniqueness for the characteristic solution we have just constructed.

2.5 Uniqueness in the space of spherically symmetric functions

We pick two characteristic solutions $f, g \in \mathbb{B}^s_c([0,T[\times\mathbb{R}^6)$ with $\mathring{f} = f(0) = g(0) \in \mathbb{B}^s_c(\mathbb{R}^6)$ and want to show that $f = g$ on $[0,T[\times\mathbb{R}^6$. Therefore, we fix any (t,z) and use the fact that there passes a characteristic curve through each point due to Definition 2.7 (*ii*). We can define $\xi := Z_f(0,t,z)$ to get $z = Z_f(t,0,\xi)$, where Z_f is the characteristic according to f. Since f is constant along its characteristics, we have

$$f(t,z) = f(t, Z_f(t,0,\xi)) = f(0, Z_f(0,0,\xi)) = \mathring{f}(Z_f(0,0,\xi)) = \mathring{f}(\xi) = \mathring{f}(Z_f(0,t,z)).$$

We can do the same with g to obtain

$$g(t,z) = \mathring{f}(Z_g(0,t,z)).$$

Once again, we pick some $0 < \delta_0 < T$ and only need to show uniqueness on the compact time intervall $[0,\delta_0]$. First, we need to make sure that both mass densities ρ_f, ρ_g and both force fields F_f, F_g have the necessary properties to go through Step 4 of our local existence proof:

Since $f, g \in \mathbb{B}^s_c([0,\delta_0] \times \mathbb{R}^6)$, we find for every $t \in [0,\delta_0]$ some $\mathcal{R}(t) > 0$ such that $\operatorname{supp} f(t) \cup \operatorname{supp} g(t) \subset B^6_{\mathcal{R}(t)}$. Indeed, this function $\mathcal{R} : [0,\delta_0] \to \mathbb{R}$ can be designed to be continuous by Lemma 2.8 b), simply using $\operatorname{supp} f(t) = Z_f(t,0,\operatorname{supp}\mathring{f})$ and $\operatorname{supp} g(t) = Z_g(t,0,\operatorname{supp}\mathring{f})$ and the continuity of the characteristics. Note that Lemma 2.8 is applicable since Z_f and Z_g have the properties of Remark 2.4 by the definiton of a characteristic solution. The uniform bound $R := \max\{\mathcal{R}(t) : t \in [0,\delta_0]\}$ implies

$\|\rho_f(t)\|_\infty + \|\rho_g(t)\|_\infty \leq C[R,\delta_0]\|\mathring{f}\|_\infty$ like in Step 1 and hence the global Lipschitz-continuity of F_f and F_g like in (2.20) from Step 3 due to spherical symmetry.

Next, we want to show that $Z_f(0,t,z) = Z_g(0,t,z)$ by going through the estimates of Step 4, where we replace F_n, F_{n-1} with F_f, F_g and Z_n, Z_{n-1} with Z_f, Z_g: First, let us define the well-known quantities

$$\begin{aligned} D_{f,g} &:= \{z \in \operatorname{supp} \mathring{f} : |X_f(s,0,z)| \leq r < |X_g(s,0,z)| \\ &\qquad\qquad \vee |X_g(s,0,z)| \leq r < |X_f(s,0,z)|\}, \\ d_{f,g} &:= \sup_{z \in \operatorname{supp} \mathring{f}} |X_f(s,0,z) - X_g(s,0,z)|. \end{aligned}$$

Since $F_f(s,\cdot)$ and $F_g(s,\cdot)$ are spherically symmetric for all times $s \in [0,T[$, we find analogously to Step 4 that

$$|F_f(s,x) - F_g(s,x)| \leq Cr, \quad |F_f(s,x) - F_g(s,x)| \leq \frac{1}{r^2}\|\mathring{f}\|_\infty \mathrm{vol}\,(D_{f,g}) \tag{2.30}$$

for $(s,x) \in [0,\delta_0] \times \mathbb{R}^3$, $r = |x|$. Furthermore,

$$\mathrm{vol}\,(D_{f,g}) \leq C(d_{f,g}^3 + 3d_{f,g}^2 r + 3d_{f,g}r^2),$$

which implies with (2.30) that

$$\|F_f(s) - F_g(s)\|_\infty \leq C \sup_{z \in \operatorname{supp} \mathring{f}} |Z_f(s,0,z) - Z_g(s,0,z)|.$$

Analogously to (2.28) we conclude with Gronwall that

$$|Z_f(s,0,z) - Z_g(s,0,z)| \leq C \int_0^s \|F_f(\tau) - F_g(\tau)\|_\infty \,\mathrm{d}\tau$$

and obtain $Z_f(s,0,z) = Z_g(s,0,z)$ for $(s,z) \in [0,\delta_0] \times \mathbb{R}^6$. By Definition 2.7 (ii) both Z_f and Z_g have the properties of Remark 2.4. Hence by the formula for the inverse of the characteristic flow, we find that

$$Z_f(0,s,\cdot) = {Z_f}^{-1}(s,0,\cdot) = {Z_g}^{-1}(s,0,\cdot) = Z_g(0,s,\cdot), \quad s \in [0,T[,$$

which finally implies $f(t,z) = \mathring{f}(Z_f(0,t,z)) = \mathring{f}(Z_g(0,t,z)) = g(t,z)$ for $(t,z) \in [0,\delta_0] \times \mathbb{R}^6$. Since $0 < \delta_0 < T$ was arbitrary, uniqueness on $[0,T[$ follows.

2.6 The continuation criterion

For the desired global existence result we need to investigate the conditions under which a solution can be extended. Vividly speaking, this is the case as long as the mass density or the velocities of our particles remain bounded. To be specific, we want to prove the following *continuation criterion*:

Theorem 2.10.
Consider $T \in]0,\infty]$ and let $f : [0,T[\times\mathbb{R}^6 \to \mathbb{R}$ be the maximal solution in the characteristic sense, i.e., if $\tilde{f} : [0,\tilde{T}[\times\mathbb{R}^6 \to \mathbb{R}$ is another solution with $f(0) = \tilde{f}(0)$, then $\tilde{T} \leq T$. If

$$P^* := \sup\left\{|v| : (x,v) \in \operatorname{supp} f(t), 0 \leq t < T\right\} < \infty, \quad \textit{or}$$
$$\rho^* := \sup\left\{\rho(t,x) : x \in \mathbb{R}^3, 0 \leq t < T\right\} < \infty,$$

then $T = \infty$; the solution is global in time.

Proof. Let $f : [0,T[\times\mathbb{R}^6 \to \mathbb{R}$ be the maximal solution in the characteristic sense and assume $T < \infty$. We have to check the two cases $P^* < \infty$ and $\rho^* < \infty$. In Step 2 we computed δ, which was the length of the time interval of our constructed solution.
The idea of the proof is the following: As a start, we pick some $t_0 \in]0,T[$ sufficiently close to T. Then, we consider the initial value problem with $f(t_0)$ as initial datum, and go through Step 1-4. If we manage to extend the solution beyond T and use the uniqueness, we obtain the contradiction to the assumption that f is a maximal solution.

1) Let $P^* < \infty$:
Since the L^p-norms are preserved due to Lemma 2.8 b), we have

$$\|f(t_0)\|_\infty = \|\mathring{f}\|_\infty, \quad \|f(t_0)\|_1 = \|\mathring{f}\|_1, \quad 0 \leq t_0 < T,$$

and consequently $C_{f(t_0)} := 4 \cdot 3^{1/3} \cdot \pi^{4/3}\|f(t_0)\|_1^{1/3}\|f(t_0)\|_\infty^{2/3} = C_{\mathring{f}}$, which was defined in Step 1. For any $t_0 \in]0,T[$, we define $\delta^* := (P^* C_{f(t_0)})^{-1}$. The key idea is to take advantage of the fact that $C_{f(t_0)}$ and δ^* are independent of t_0. Thus, we can choose t_0 sufficiently close to T such that $t_0 + \delta^* > T$. Analogously to Step 2, we consider the maximal solution of the integral equation

$$\tilde{Q}(t) = P^* + C_{\mathring{f}} \int_{t_0}^t \tilde{Q}^2(s)\,\mathrm{d}s.$$

We know that $\tilde{Q}$ exists on $[t_0, t_0 + \delta^*[$. Due to our local existence result we have $f(t_0) \in \mathbb{B}_c^s(\mathbb{R}^6)$. Hence, we can consider $f(t_0)$ as the new initial datum and go through Step 1-4: Step 1 is exactly the same, note that

$$f(t_0,x,v) = 0 \quad \text{for all } |v| \geq P^*, x \in \mathbb{R}^3$$

by definition of P^*. In Step 2 we obtained the length of our time interval, and hence the radius $\mathcal{R}$, which bounded the support of $f_n(t)$ and $\rho_n(t)$. We can bound the crucial quantity

$$\tilde{P}_n(t) := \sup\left\{|V_{n-1}(s,t_0,z)| : z \in \operatorname{supp} f(t_0), t_0 \leq s \leq t\right\}, \quad t \in [t_0, t_0+\delta^*[\,,$$

once again by proving $\tilde{P}_n(t) \leq \tilde{Q}(t)$ by induction. The case $n = 0$ is trivial, and the induction step remains almost the same:

$$\begin{aligned}|V_n(s,t_0,z)| &\leq |V_n(t_0,t_0,z)| + \left|\int_{t_0}^s \dot{V}_n(\tau,t_0,z)\,\mathrm{d}\tau\right| \leq |v| + \int_{t_0}^s \|F_n(\tau)\|_\infty\,\mathrm{d}\tau \\ &\leq P^* + C_{f(t_0)}\int_{t_0}^s \tilde{P}_n^2(\tau)\,\mathrm{d}\tau \leq P^* + C_{\mathring{f}}\int_{t_0}^t \tilde{Q}^2(\tau)\,\mathrm{d}\tau = \tilde{Q}(t).\end{aligned}$$

The same procedures from Step 3 can now be conducted again to obtain a uniform Lipschitz-bound of the force fields. Now we have all necessary estimates and can go through Step 4. Thereafter, we obtain a solution $\tilde{f} \in \mathbb{B}_c^s([t_0, t_0+\delta^*[\times\mathbb{R}^6)$ with $\tilde{f}(t_0) = f(t_0)$. Since both $\tilde{f}$ and f exist on $[t_0, T[\times\mathbb{R}^6$, we have $\tilde{f} = f$ on $[t_0, T[\times\mathbb{R}^6$ by uniqueness. In particular, we extended f to $[0, t_0+\delta^*[\times\mathbb{R}^6$ which contradicts the maximality of T.

2) Let $\rho^* < \infty$:
This can be traced back to the first case $P^* < \infty$ as follows: By Lemma 2.2 b), we have

$$\|F(t)\|_\infty \leq C\|\rho(t)\|_1^{1/3}\|\rho(t)\|_\infty^{2/3},$$

and hence

$$\sup_{t\in[0,T[} \|F(t)\|_\infty < C[\mathring{f}, \rho^*] < \infty.$$

Using $\operatorname{supp} f(t) = Z(t, 0, \operatorname{supp}\mathring{f})$, we have

$$\begin{aligned}P^* &= \sup\left\{|v| : (x,v) \in \operatorname{supp} f(t), 0 \leq t < T\right\} \\ &= \sup\left\{|v| : (x,v) \in Z(t,0,\operatorname{supp}\mathring{f}), 0 \leq t < T\right\} \\ &= \sup\left\{|V(t,0,z)| : z \in \operatorname{supp}\mathring{f}, 0 \leq t < T\right\}\end{aligned}$$

and since we can estimate

$$|V(t,0,z)| \leq |v| + \int_0^t \|F(\tau)\|_\infty\,\mathrm{d}\tau < C[\mathcal{R}, \mathring{f}, \rho^*, T] < \infty,$$

$P^* < \infty$ follows and the proof is completed.

□

2.7 Global existence

Theorem 2.11.
For every non-negative $\mathring{f} \in \mathbb{B}_c^s(\mathbb{R}^6)$ there exists a unique global solution $f : [0,\infty[\times\mathbb{R}^6 \to \mathbb{R}$ in the characteristic sense. Additionally, there exists some $P_0 > 0$ such that

$$f(t,x,v) = 0, \quad |v| \geq P_0,\ (t,x) \in [0,\infty[\times\mathbb{R}^3.$$

P_0 depends only on $\|\mathring{f}\|_1, \|\mathring{f}\|_\infty$ and $\mathring{P}$.

Proof. Let $\mathring{f} \in \mathbb{B}_c^s(\mathbb{R}^6)$ and $f \in \mathbb{B}_c^s([0,T[\times\mathbb{R}^6)$ be the corresponding local solution in the characteristic sense. According to the continuation criterion, it is sufficient to limit

$$P(t) := \sup\big\{|v| : z \in \operatorname{supp} f(s), 0 \le s \le t\big\}, \quad 0 \le t < T.$$

Lemma 2.2 b) and the conservation of the L^p-norms imply that

$$\|F(t)\|_\infty \le C\|f(t)\|_1^{1/3}\|f(t)\|_\infty^{2/3} P^2(t) \le C_{\mathring{f}} P^2(t).$$

Due to the spherical symmetry of f, Lemma 2.3 provides the estimate

$$|F(t,x)| \le \frac{\|\mathring{f}\|_1}{r^2}, \quad (t,x) \in [0,T[\times\mathbb{R}^3,\ |x| = r.$$

Assembling these estimates yields

$$|F(t,x)| \le C^* \min\left\{\frac{1}{r^2}, P^2(t)\right\}. \tag{2.31}$$

with the constant C^* only depending on $\mathring{f}$. We fix any $t \in [0,T[, z \in \operatorname{supp}\mathring{f}$ and write for the characteristics associated with f

$$(X,V)(s) = (X,V)(s,0,z), \quad 0 \le s \le t < T.$$

For any $i \in \{1,2,3\}$ arbitrary but fixed and all $r \in \mathbb{R}$, we define

$$\xi := X_i \in C^2([0,t]) \quad \text{and} \quad g(r) := C^* \min\left\{\frac{1}{r^2}, P^2(t)\right\}.$$

Our first goal is to find an estimate for $|\dot\xi(t) - \dot\xi(0)|$. Since $\ddot X(s) = F(s, X(s))$, we have

$$|\ddot\xi(s)| \le g(\xi(s)), \quad 0 \le s \le t.$$

1) We consider the case $\dot\xi(s) \ne 0$ for all $s \in]0,t[$:
Since $\dot\xi$ is in $C^1([0,t])$, we know that $\dot\xi$ does not change sign:

$$|\dot\xi(t) - \dot\xi(0)|^2 \le |\dot\xi(t) - \dot\xi(0)||\dot\xi(t) + \dot\xi(0)| = |\dot\xi(t)^2 - \dot\xi(0)^2|.$$

Using the fundamental theorem of calculus and after some straight forward computations, we find that

$$\begin{aligned}
|\dot\xi(t)^2 - \dot\xi(0)^2| &= \Big|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\dot\xi(s)^2)\,\mathrm{d}s\Big| = 2\Big|\int_0^t \ddot\xi(s)\dot\xi(s)\,\mathrm{d}s\Big| \le 2\int_0^t g(\xi(s))|\dot\xi(s)|\,\mathrm{d}s \\
&= 2\int_{\xi(0)}^{\xi(t)} g(s)\,\mathrm{d}s \le \int_{-\infty}^{\infty} 2C^* \min\left\{\frac{1}{s^2}, P(t)^2\right\}\mathrm{d}s \\
&= 2C^*\Big(\int_{-\infty}^{-1/P(t)} \frac{1}{s^2}\,\mathrm{d}s + \int_{-1/P(t)}^{+1/P(t)} P(t)^2\,\mathrm{d}s + \int_{+1/P(t)}^{\infty} \frac{1}{s^2}\,\mathrm{d}s\Big) \\
&= 8C^* P(t).
\end{aligned}$$

2) In the other case, there exists some $s \in]0,t[$ such that $\dot\xi(s) = 0$.
We define

$$s_- := \inf\big\{s \in]0,t[: \dot\xi(s) = 0\big\},$$
$$s_+ := \sup\big\{s \in]0,t[: \dot\xi(s) = 0\big\}.$$

Obviously,

$$0 \le s_- \le s_+ \le t, \quad \dot\xi(s_-) = 0 = \dot\xi(s_+).$$

We also know that there are no more zeros of $\dot\xi$ on $]0,s_-[$ and $]0,s_+[$. Hence, we can apply case 1) on these intervals and obtain

$$|\dot\xi(t) - \dot\xi(0)| \le |\dot\xi(t) - \dot\xi(s_+)| + 0 + |\dot\xi(s_-) - \dot\xi(0)| \le 2\sqrt{8C^*P(t)}.$$

Since P is nondecreasing and $\dot\xi = V_i$ by definition, we get in both cases

$$|V(s) - V(0)| \le \sqrt{3} \max_{i\in\{1,2,3\}} |V_i(s) - V_i(0)| \le 2\sqrt{3}\sqrt{8C^*P(t)}, \quad s \in [0,t].$$

This yields the desired bound of $P(t)$:

$$\begin{aligned}
P(t) &= \sup\big\{|v| : (x,v) \in \operatorname{supp} f(s), 0 \le s \le t\big\}\\
&= \sup\big\{|V(s,0,x,v)| : (x,v) \in \operatorname{supp} \mathring{f}, 0 \le s \le t\big\}\\
&\le \sup\big\{\sqrt{3} \max_{i\in\{1,2,3\}} |V_i(s,0,x,v) - V_i(0,0,x,v)| : (x,v) \in \operatorname{supp} \mathring{f}, 0 \le s \le t\big\} + P(0)\\
&\le 2\sqrt{3}\sqrt{8C^*}P(t)^{1/2} + P(0),
\end{aligned}$$

and hence

$$P(t) \le 4\sqrt{6}\sqrt{24C^{*2} + C^*\mathring{P}} + 48C^* + \mathring{P}, \quad t \in [0,T[\,,$$

which completes the proof due to the continuation criterion. Note that both $P(0) = \mathring{P}$ and C^* depend only on $\|\mathring{f}\|_1$ and $\|\mathring{f}\|_\infty$. □

2.8 Conservation of the energy and the Casimir functional

In the following we consider both local and global solutions, i.e. let $T > 0$ or $T = \infty$. For any compactly supported characteristic solution $f : [0,T[\times\mathbb{R}^6 \to \mathbb{R}$ we introduce its **kinetic** and **potential energy**

$$E_{\text{kin}}(f(t)) := \frac{1}{2}\iint |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v, \qquad E_{\text{pot}}(f(t)) := \frac{1}{2}\iint U(t,x)f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$$

for $t \in [0,T[$, where the potential induced by f is defined as

$$U_f(t,x) := U(t,x) := -\iint \frac{f(t,y,v)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}v, \quad (t,x) \in [0,T[\times\mathbb{R}^3.$$

For any continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(0) = 0$ we define the **Casimir functional** as

$$\mathcal{C}(f(t)) := \iint \Phi\big(f(t,x,v)\big)\, \mathrm{d}v\, \mathrm{d}x, \quad t \in [0,T[.$$

Note that both $|E_{\text{kin}}(f(t))|$ and $|E_{\text{pot}}(f(t))|$ are finite, and $U_f(t) \in C^1(\mathbb{R}^3)$ with $-\partial_x U_f = F_f$ due to the fact that $\rho_f(t) \in L^1 \cap L^\infty(\mathbb{R}^3)$ for all times.

Theorem 2.12.
The total energy and the Casimir functional are preserved in time: For all $t \in [0,T[$, we have

$$E_{\text{tot}} := E_{\text{kin}}(f(t)) + E_{\text{pot}}(f(t)) = E_{\text{kin}}(\mathring{f}) + E_{\text{pot}}(\mathring{f}), \qquad \mathcal{C}(f(t)) = \mathcal{C}(\mathring{f}).$$

Proof. In order to prove this, it is essential that our characteristic flow is at least almost everywhere measure preserving, which holds by Definition 2.7 (*ii*). Hence, the conservation of the Casimir functional is a direct consequence of the Transformation rule for Lipschitz functions:

$$\mathcal{C}(f(t)) = \int \Phi\big(f(t,z)\big)\, \mathrm{d}z = \int \Phi\big(\mathring{f}(Z(0,t,z))\big)\, \mathrm{d}z = \int \Phi\big(\mathring{f}(z)\big)\, \mathrm{d}z = \mathcal{C}(\mathring{f}).$$

There are several ideas in the literature to prove the conservation of energy. In the classical Vlasov-Poisson setup, one can exploit the fact that f, ρ, U solve (2.1), (2.2) by considering $\frac{d}{dt}\big(E_{\text{kin}}(f(t)) + E_{\text{pot}}(f(t))\big)$.
However, our f is just a solution in the characteristic sense, and in general the Vlasov and the Poisson equation do not hold since $\partial_t f, \partial_x f, \partial_v f$ or ΔU do not exist. Thus, we have to find another approach that only uses $s \mapsto f(s, (Z(s,t,z))$ being constant and our characteristic flow $Z = (X, V)$ having the good properties of Remark 2.4. We will exploit these properties by inserting a time derivative with the fundamental theorem of calculus as follows:
Remember the notation $z = (x, v)$ and $\mathrm{d}z = \mathrm{d}x\, \mathrm{d}v$. Starting directly with the total energy, using the measure preserving transformation $z \mapsto Z(t, 0, z)$ and the fundamental

theorem of calculus, we have

$$
\begin{aligned}
2E_{\text{tot}} =& \int |v|^2 f(t,z)\,\mathrm{d}z + \iint \frac{f(t,z)f(t,\tilde{z})}{|x-\tilde{x}|}\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
=& \int |V(t,0,z)|^2 f\big(t,Z(t,0,z)\big)\,\mathrm{d}z - \iint \frac{f\big(t,Z(t,0,z)\big)f\big(t,Z(t,0,\tilde{z})\big)}{|X(t,0,z)-X(t,0,\tilde{z})|}\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
=& \iint_0^t \frac{\mathrm{d}}{\mathrm{d}s}\Big(|V(s,0,z)|^2 f\big(s,Z(s,0,z)\big)\Big)\,\mathrm{d}s\,\mathrm{d}z + \int |v|^2 \mathring{f}(z)\,\mathrm{d}z \\
& - \iiint_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{f\big(s,Z(s,0,z)\big)f\big(s,Z(s,0,\tilde{z})\big)}{|X(s,0,z)-X(s,0,\tilde{z})|}\right)\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}\tilde{z} - \iint \frac{\mathring{f}(z)\mathring{f}(\tilde{z})}{|x-\tilde{x}|}\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
=& 2\big(E_{\text{kin}}(\mathring{f}) + E_{\text{pot}}(\mathring{f})\big) + 2\iint_0^t V(s,0,z)\cdot F(s,X(s,0,z))f\big(s,Z(s,0,z)\big)\,\mathrm{d}s\,\mathrm{d}z \\
& + \iiint_0^t \frac{X(s,0,z)-X(s,0,\tilde{z})}{|X(s,0,z)-X(s,0,\tilde{z})|^3} \\
& \times \big(V(s,0,z)-V(s,0,\tilde{z})\big) f\big(s,Z(s,0,z)\big) f\big(s,Z(s,0,\tilde{z})\big)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}\tilde{z}.
\end{aligned}
$$

In the last step we used that f is a characteristic solution:

$$
\frac{\mathrm{d}}{\mathrm{d}s} f(s,Z(s,0,z)) = 0, \quad s \in [0,T[.
$$

Using Fubini and transforming back with $z \mapsto Z(0,s,z)$, $\tilde{z} \mapsto Z(0,s,\tilde{z})$, we obtain

$$
\begin{aligned}
E_{\text{tot}} =& E_{\text{kin}}(\mathring{f}) + E_{\text{pot}}(\mathring{f}) + \int_0^t \int v\cdot F(s,x) f(s,z)\,\mathrm{d}z\,\mathrm{d}s \\
& + \int_0^t \iint v\cdot \frac{x-\tilde{x}}{|x-\tilde{x}|^3} f(s,\tilde{z})\, f(s,z)\,\mathrm{d}\tilde{z}\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}
$$

Since f is a characteristic solution, the formula

$$
F(t,x) = -\int \frac{x-\tilde{x}}{|x-\tilde{x}|^3} f(t,\tilde{z})\,\mathrm{d}\tilde{z}
$$

holds for all $(t,x) \in [0,T[\times\mathbb{R}^3$. Therefore, the last two terms cancel each other and the assertion $E_{\text{kin}}(f(t)) + E_{\text{pot}}(f(t)) = E_{\text{kin}}(\mathring{f}) + E_{\text{pot}}(\mathring{f})$ follows. □

Obviously, this proof works in the classical setup as well since the regularity of f immediately implies the necessary properties of the characteristic flow and the force field.

Outlook

Let us give a brief overview of interesting problems that can be attacked based on the concept of characteristic solutions and the given existence result. One exciting question

is how any irregularities of a characteristic solution spread in phase space. For example if $f \in \mathbb{B}_c([0,T[\times\mathbb{R}^6)$ is a characteristic solution which is smooth on its support but discontinuous at the boundary of its support, what can we say about the regularity of $f(t)$ for $t > 0$?
Another open question is whether we can weaken the requirements for $\mathring{f}$ in our local existence result. It seems hard to get rid of the spherical symmetry assumption since this ensures the Lipschitz-continuity of the force field which seems to be necessary for well-defined characteristics. In view of the Kurth-type solutions, it is perhaps sufficient to demand $0 \leq \mathring{f} \in L^1(\mathbb{R}^6)$, $\int \mathring{f}\,\mathrm{d}v \in L^\infty(\mathbb{R}^3)$ and $\mathring{f}$ being spherically symmetric and compactly supported for a pointwise defined representative.

3 The flat Vlasov-Poisson system

In this chapter, we want to analyse the flat axially symmetric Vlasov-Poisson system numerically. For this purpose we need certain analytical results of the potential and the force field. Before introducing the flat Vlasov-Poisson system, we want to point out how it differs from the 2-dimensional Vlasov-Poisson system, where the distribution function f is defined on $\mathbb{R}^2 \times \mathbb{R}^2$ and the Vlasov equation is coupled to the 2-dimensional Poisson equation. For some reasonable f, the induced gravitational potential takes the form

$$U(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) f(y,v) \,\mathrm{d}v \,\mathrm{d}y, \quad x \in \mathbb{R}^2,$$

which is just of mathematical interest and has no interpretation in the astrophysical context known to the author. When it comes to flat galaxies, we need the 3-dimensional Vlasov-Poisson system with flat phase space functions.

Flat phase space functions can be expressed by delta distributions and phase space functions on $\mathbb{R}^3 \times \mathbb{R}^3$ via

$$\mathring{f}(x_1, x_2, x_3, v_1, v_2, v_3) = \underline{\mathring{f}}(x_1, x_2, v_1, v_2)\delta(x_3)\delta(v_3). \tag{3.1}$$

Take into consideration that they do not fulfil $\mathring{f} \in C_c^1(\mathbb{R}^6)$ or $\mathring{f} \in \mathbb{B}(\mathbb{R}^6)$ from Chapter 2. In the following, underlining a function $\underline{f}, \underline{\rho}, \underline{F}$ emphasises that it is defined on the **flat phase space** $\mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbb{R}^2$ respectively. This indicates whether x and v are elements of $\mathbb{R}^2$ or $\mathbb{R}^3$. Once we insert (3.1) into the classical formula of the force, where formally

$$\rho(x) = \int_{\mathbb{R}^3} f(x_1, x_2, x_3, v_1, v_2, v_3) \,\mathrm{d}v = \int_{\mathbb{R}^3} \underline{f}(x_1, x_2, v_1, v_2)\delta(x_3)\delta(v_3) \,\mathrm{d}v = \underline{\rho}(x_1, x_2)\delta(x_3)$$

holds, we obtain the **flat force**

$$\underline{F}(x_1, x_2) = \int_{\mathbb{R}^2} \frac{(x_1, x_2) - (y_1, y_2)}{|(x_1, x_2) - (y_1, y_2)|^3} \underline{\rho}(y_1, y_2) \,\mathrm{d}y.$$

This is the motivation to consider the **flat Vlasov-Poisson system**

$$\partial_t \underline{f}(t,x,v) + v \cdot \partial_x \underline{f}(t,x,v) + \underline{F}(t,x) \cdot \partial_v \underline{f}(t,x,v) = 0, \tag{3.2}$$

$$\underline{F}(t,x) = -\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^3} \,\underline{\rho}(t,y) \,\mathrm{d}y, \tag{3.3}$$

$$\underline{\rho}(t,x) = \int_{\mathbb{R}^2} \underline{f}(t,x,v) \,\mathrm{d}v. \tag{3.4}$$

We are only interested in the initial value problem, where $\underline{\mathring{f}} = \underline{f}_{|t=0}$ is given on $\mathbb{R}^2 \times \mathbb{R}^2$. We call $(\underline{f}, \underline{\rho}, \underline{F})$ a solution of the flat Vlasov-Poisson system or a **flat solution** if the following holds:

i) $\underline{f} \in C^1(I \times \mathbb{R}^2 \times \mathbb{R}^2)$ and $\underline{F}, \underline{\rho} \in C^1(I \times \mathbb{R}^2)$,

ii) $\underline{f}, \underline{\rho}, \underline{F}$ solve (3.2)-(3.4) on $I \times \mathbb{R}^2 \times \mathbb{R}^2$ and $I \times \mathbb{R}^2$ respectively,

iii) for all compact subintervals $J \subset I$, the force field $\underline{F}$ is bounded on $J \times \mathbb{R}^2$.

The flat force $\underline{F}$ and the classical force F from Chapter 2 look similar but looks can be deceiving: For $n \in \{2, 3\}$, $R > 0$, $x \in B_R^n$ and some discontinuous mass density in $L^1 \cap L^\infty(\mathbb{R}^n)$, for example $\rho := \mathbb{1}_{B_R^n}$, we have

$$|F(x)| = \Big| \int_{B_R^n} \frac{x-y}{|x-y|^3} \, \mathrm{d}y \Big| \begin{cases} = \infty, & n = 2 \\ < \infty, & n = 3 \end{cases} . \tag{3.5}$$

Hence, useful properties like Lemma 2.2 are lost as we go from the classical to the flat case and it is fair to say that the flat Vlasov-Poisson system is more complicated. In the dissertation of Svetlana Dietz [8], one can find a local existence result for initial data $\mathring{\underline{f}} \in C_c^{1;\alpha}(\mathbb{R}^2 \times \mathbb{R}^2)$, i.e, $\mathring{\underline{f}}$ and its first derivatives are Hölder-continuous to the exponent α. However, we will not pursue this any further and assume from here on that our initial datum shall always enjoy this regularity. The topic we are interested in is the axially symmetric case:

3.1 The flat, axially symmetric Vlasov-Poisson system

We call some function f on the flat phase space $\mathbb{R}^2 \times \mathbb{R}^2$ **axially symmetric** if

$$\underline{f}(x, v) = \underline{f}(Ax, Av), \quad x, v \in \mathbb{R}^2, \quad A \in \mathrm{SO}(2).$$

For any axially symmetric initial datum $\mathring{\underline{f}}$, we consider the unique local solution $\underline{f}$. Since $\underline{f}(\cdot, A\cdot, A\cdot)$ is also a solution of the flat Vlasov-Poisson system with the same initial datum, $\underline{f} = \underline{f}(\cdot, A\cdot, A\cdot)$ by uniqueness; thus axial symmetry is preserved. Before we continue to compare the spherically and axially symmetric cases, we want to recall and expand our notations: If anything is overlined or underlined, it is one dimensional or two dimensional respectively. To differ functions in $\mathbb{R}^2$ or $\mathbb{R}^3$ depending only on the radius, we write

$$\begin{aligned} \mathcal{U}(r) &= \underline{U}(x), \quad \Sigma(r) = \underline{\rho}(x), \qquad r = |x| = |(x_1, x_2)|, \\ \bar{U}(r) &= U(x), \quad \bar{\rho}(r) = \rho(x), \qquad r = |x| = |(x_1, x_2, x_3)|. \end{aligned}$$

Next, we want to investigate the flat axially symmetric potential and the corresponding force and compare the results to the spherically symmetric case.

3.2 Properties of the potential and the force field

Definition and Remark 3.1. ***(Complete elliptic integrals)***
We denote by $k \mapsto \mathbb{K}(k)$ and $k \mapsto \mathbb{E}(k)$ the complete elliptic integrals of the first and of

the second kind:

$$\mathbb{K}(k) := \int_0^{\pi/2} \frac{\mathrm{d}\vartheta}{\sqrt{1-k^2(\sin\vartheta)^2}}, \quad k \in [0,1[\,, \tag{3.6}$$

$$\mathbb{E}(k) := \int_0^{\pi/2} \sqrt{1-k^2(\sin\vartheta)^2}\,\mathrm{d}\vartheta, \quad k \in [0,1]. \tag{3.7}$$

These two functions have the following properties:

a) $\mathbb{K}(0) = \pi/2$, $\mathbb{K}(k) \to \infty$, *as* $k \to 1$, *and* $\mathbb{K}$ *is strictly increasing on* $[0,1[$.

b) $\mathbb{E}(0) = \pi/2$, $\mathbb{E}(1) = 1$ *and* $\mathbb{E}$ *is strictly decreasing on* $[0,1]$.

c) $\mathbb{K}$ *and* $\mathbb{E}$ *are continuously differentiable with*

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathbb{K}(k) = \frac{1}{k}\Big(\frac{\mathbb{E}(k)}{1-k^2} - \mathbb{K}(k)\Big), \tag{3.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathbb{E}(k) = \frac{1}{k}\big(\mathbb{E}(k) - \mathbb{K}(k)\big). \tag{3.9}$$

d) $\mathbb{E} \in L^1 \cap L^\infty([0,1])$ *and* $\mathbb{K} \in L^1 \cap L^p([0,1[\,)$ *for all* $p \in [1,\infty[$ *and*

$$\mathbb{K}(k) = \frac{1}{1+k}\mathbb{K}\Big(\frac{2\sqrt{k}}{1+k}\Big), \quad k \in [0,1[\,. \tag{3.10}$$

Proof. This is standard theory for elliptic integrals and we refer to [9], [10] for a proof. □

Lemma 3.2.
Let $\underline{\rho} \in C_c^{1,\alpha}(\mathbb{R}^2)$ *be axially symmetric with* $\operatorname{supp}\underline{\rho} \subset \underline{B}_R^2$ *and let* $\Sigma(|\cdot|) = \underline{\rho}$ *denote its radial function. Then we have the following formulas for the potential and the force field:*

$$\mathcal{U}(r) = -4\int_0^R \frac{s\Sigma(s)}{r+s}\mathbb{K}\Big(\frac{2\sqrt{rs}}{r+s}\Big)\,\mathrm{d}s \tag{3.11}$$

$$= -\frac{4}{r}\int_0^r s\Sigma(s)\mathbb{K}\Big(\frac{s}{r}\Big)\,\mathrm{d}s - 4\int_r^R \Sigma(s)\mathbb{K}\Big(\frac{r}{s}\Big)\,\mathrm{d}s, \tag{3.12}$$

$$\mathcal{U}'(r) = \frac{4}{r^2}\int_0^r s\Sigma(s)\frac{\mathbb{E}(s/r)}{1-s^2/r^2}\,\mathrm{d}s - \frac{4}{r}\int_r^R \Sigma(s)\left(\frac{\mathbb{E}(r/s)}{1-r^2/s^2} - \mathbb{K}\Big(\frac{r}{s}\Big)\right)\mathrm{d}s, \tag{3.13}$$

$$\partial_x \underline{U}(x) = x/r\,\mathcal{U}'(r), \qquad \textit{for } r = |x|, \quad x \in \mathbb{R}^2. \tag{3.14}$$

Before giving a proof, we want to recall Lemma 2.2 and compare the formulas of the forces, as this might give us insight into the upcoming problems. First, note that the axially symmetric potential $\mathcal{U}$, written as (3.12), and the spherically symmetric potential $\bar{U}$ look similar: Just multiply with π and replace $\mathbb{K}(\cdot/r)$ and $\mathbb{K}(r/\cdot)$ by the identity. Unfortunately, this small difference destroys the similarity for the corresponding forces

$\bar{U}'$ and $\mathcal{U}'$, where an entire $\int_r^R$-term and the singularities $(1-s^2/r^2)^{-1}$ and $(1-r^2/s^2)^{-1}$ additionally appear. This has several physical and mathematical consequences:

Starting with the mathematical aspect, we have that $\mathcal{U}'(R)$ can be infinite if Σ is not Hölder-continuous in R. Of course, (3.5) was an indication for this to happen but there was hope that the forces gain some regularity by the symmetry assumption, as it is the case with spherical symmetry.

The physical aspect concerns the influence area of the force. In the spherically symmetric setting, one test particle located at $r > 0$ only experiences the mass, which is closer to the origin than itself. Mathematically speaking, $\bar{U}'(r)$ is independent of $\bar{\rho}$ on $[r, \infty[$. On the other hand, the gravitational force is always attractive to the center of mass, i.e., $\bar{U}'(r) \geq 0$, where $\bar{U}'(r) = 0$ if and only if $\bar{\rho} = 0$ on $[0, r[$ almost everywhere. Both of these properties vanish in the axially symmetric setting. Indeed, in Figure 3.6 we numerically constructed a steady state $(\underline{f}, \Sigma, \mathcal{U})$ such that $\mathcal{U}' < 0$ on some subinterval.

Proof. Using the symmetry of $\underline{U}$, we can assume that $(x_1, x_2) = (r, 0)$. Next, we introduce polar coordinates and write $y = (s \sin\alpha, s\cos\alpha)$ to obtain

$$|x-y|^2 = (r - s\cos\alpha)^2 + s^2\sin^2\alpha = r^2 + s^2 - 2rs\cos\alpha = (r+s)^2 - 2rs(1+\cos\alpha).$$

We use the well known trigonometric identity

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

and define $k := 2\sqrt{rs}/(r+s)$ to conclude

$$|x-y| = \left((r+s)^2 - 4rs\cos^2\frac{\alpha}{2}\right)^{1/2} = (r+s)\left(1 - k^2\cos^2\frac{\alpha}{2}\right)^{1/2}.$$

Using this equation for our axially symmetric potential yields

$$\underline{U}(x_1, x_2) = -\int_{\mathbb{R}^2} \frac{\underline{\rho}(y)}{|x-y|}\,\mathrm{d}y = -2\int_0^\infty \int_0^\pi \frac{s\Sigma(s)}{r+s} \frac{\mathrm{d}\alpha\,\mathrm{d}s}{\sqrt{1-k^2\cos^2(\alpha/2)}},$$

where " $\int_0^{2\pi} = 2\int_0^\pi$ " due to the symmetry at $\alpha = \pi$. Upon substituting

$$t = \cos(\alpha/2), \quad \mathrm{d}t = -\frac{1}{2}\sqrt{1-t^2}\,\mathrm{d}\alpha,$$

and observing that

$$\mathbb{K}(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-k^2t^2}\sqrt{1-t^2}},$$

we can prove equation (3.11):

$$\begin{aligned}\underline{U}(x_1, x_2) &= -2\int_0^\infty \int_0^1 \frac{s\Sigma(s)}{r+s} \frac{1}{\sqrt{1-k^2t^2}} \frac{-2}{\sqrt{1-t^2}}\,\mathrm{d}t\,\mathrm{d}s \\ &= -4\int_0^\infty \frac{s\Sigma(s)}{r+s}\mathbb{K}(k)\,\mathrm{d}s.\end{aligned}$$

As pointed out in [11], (3.12) can be easily derived by splitting the domain of integration into $[0,r]$ and $[r,\infty]$ and using (3.10) with $k = s/r$ and $k = r/s$ respectively; recall that $\Sigma = 0$ on $[R,\infty[$. Using (3.8) and the Hölder-continuity of Σ, (3.13) follows by differentiating (3.12).

□

3.3 Flat axially symmetric steady states

For given $x = (x_1, x_2)$, $v = (v_1, v_2)$ let us introduce the following coordinates:

$$r := |x|, \quad w := \frac{x \cdot v}{r}, \quad L_z := x_1 v_2 - x_2 v_1,$$
$$u := |v|, \quad \alpha := \sphericalangle(x, v), \quad L := |(x,0) \times (v,0)|^2 = L_z^2 = r^2 u^2 - (x \cdot v)^2 = (ru \sin\alpha)^2.$$

Hence, for any particle located at x with velocity v, w is the radial velocity, L is the modulus of the angular momentum squared and α is the angle between x and v. In the following exploration we assume that all appearing functions are sufficiently regular. It is well known that both the axially and spherically symmetric Vlasov-Poisson systems can be formulated in the (r, u, α) and (r, w, L) coordinates, i.e., with abuse of notation,

$$f(t, x, v) = f(t, r, u, \alpha) = f(t, r, w, L), \quad \text{for } x, v \in \mathbb{R}^n \text{ with } n = 2, 3.$$

Let us explain the benefits of solving the axially symmetric Vlasov-Poisson system numerically in the (r, w, L) coordinates as opposed to the cartesian ones. In the former case, the Vlasov equation takes the form

$$\partial_t f + w\partial_r f + \Big(\frac{L}{r^3} - \partial_r U\Big)\partial_w f = 0, \tag{3.15}$$

where in the flat setup, the force $\partial_r U = \mathcal{U}'$ is given by (3.13). For any $r, L \geq 0, w \in \mathbb{R}$, we denote by $[0,T[\ni s \mapsto (R, W, \mathcal{L})(s, r, w, L) = (R, W, \mathcal{L})(s)$ the **characteristics of the flat, axially symmetric Vlasov-Poisson system in** (r, w, L) **coordinates** as the solutions of:

$$\dot{R} = W, \quad \dot{W} = \Big(\frac{L}{R^3} - \mathcal{U}'(s, R)\Big), \quad \dot{\mathcal{L}} = 0, \tag{3.16}$$

with $(R, W, \mathcal{L})(0, r, w, L) = (r, w, L)$. The idea of our numerical algorithm is to find a solution of (3.16) and assume that $(t, r, w, L) \mapsto \mathring{f}(t, R(t), W(t), L)$ solves (3.15). Note, that the flat characteristic system in (x, v) coordinates involves four equations. The property of axial symmetry, however, simplifies the characteristic system, since (3.16) only contains two equations for r, w and the trivial equation for L.

Now we are ready to investigate steady states. For any particle located at (x, v) or (r, w, L), we define its **particle energy** by

$$E = E(x, v) = E(r, w, L) = \frac{1}{2}|v|^2 + \underset{\sim}{U}(x) = \frac{1}{2}\Big(w^2 + \frac{L_z^2}{r^2}\Big) + \mathcal{U}(r). \tag{3.17}$$

Next, note that E and L are constant along solutions $s \mapsto (R, W, \mathcal{L})(s)$ of (3.16), which implies that every reasonable function Φ depending only on E, L is a stationary solution of the Vlasov-Poisson system. In this thesis we will focus on the ansatz function

$$f_0(x, v) := \Phi(E, L_z) := A(E_0 - E)_+^k (1 - Q|L_z|)_+^l\,, \tag{3.18}$$

where the subscript + denotes the positive part. $E_0 < 0$ is the so called **cut-off energy**, which allows the steady state to have finite mass and compact support.

For $Q > 0$, the quantity $1/Q$ can be seen as the **cut-off angular momentum**, where $Q = 0$ lets the ansatz Φ be independent of L_z. In our numerical simulations later on, we will fix some $k, Q \geq 0$, $l \in \mathbb{R}$ and prescribe some $R > 0$ and $M > 0$, such that R is the radius and M the mass of our galaxy, cf. Section 3.4. Fixing these quantities will then determine E_0 and A.

With this ansatz at hand, (3.4) and the change of variables $(v_1, v_2) \mapsto (E, L_z)$ yields the formula for the mass density

$$\Sigma(r) = 2 \int_{\mathcal{U}(r)}^{E_0} \int_{-\sqrt{2r^2(E-\mathcal{U}(r))}}^{\sqrt{2r^2(E-\mathcal{U}(r))}} \frac{\Phi(E, L_z)\,\mathrm{d}L_z\,\mathrm{d}E}{\sqrt{2r^2(E-\mathcal{U}(r)) - L_z^2}} \tag{3.19}$$

for all $r \geq 0$ with $\mathcal{U}(r) < E_0$ and $\Sigma(r) = 0$ else. Hence, we can define $R > 0$ as the smallest number such that $\operatorname{supp} \Sigma \subset [0, R]$. Since Σ is additionally bounded, the steady state has finite mass

$$M_\Sigma := M := 2\pi \int_0^\infty r\,\Sigma(r)\,\mathrm{d}r < \infty. \tag{3.20}$$

Lastly, the formula for the potential and the force field are given by Lemma 3.2 and we are ready to dive into a detailed description of the numerical algorithm.

3.4 The numerical algorithm

Our numerical algorithm has to perform several tasks: First of all, we insert reasonable parameters k, l, Q of the ansatz function, as well as the desired mass M and radius R to obtain a steady state.

Constructing the steady state

The first task is to construct a steady state $(\Sigma, \mathcal{U})$ on $[0, R]$ in the sense that both (3.11) and (3.19) hold. This is done by the following iterative scheme from [12]:

1) As a start, one can choose any ansatz parameter $A_0 > 0$ and any non-negative mass density Σ_0 with mass M and support $[0, R]$. For faster convergence, it is useful to define Σ_0 somewhat close to the resulting mass density Σ. In the algorithm we choose $A_0 := 1$ and the simple linear function

$$\Sigma_0(r) := \frac{3M}{\pi R^2}\big(1 - r/R\big), \quad r \in [0, R], \quad \Sigma_0 := 0, \quad \text{else}.$$

2) If the n-th iterate Σ_n and $A_n > 0$ are given, we can calculate $\mathcal{U}_n$ using (3.11) or (3.12). It turns out that the latter is numerically harder to deal with. However, we use both formulas to keep track of numerical errors. We compute the complete elliptic integrals efficiently using code from the *GSL* package, which uses an arithmetic-geometric mean algorithm, cf. [13]. Next, we define the cut-off energy by $(E_0)_n := \mathcal{U}_n(R)$.

3) For any given $\mathcal{U}_n$, we calculate the corresponding mass density $\tilde{\Sigma}_n$ via (3.19).

4) We define the next iterate by $\Sigma_{n+1} := M/M_{\tilde{\Sigma}_n}\tilde{\Sigma}_n$, hence, Σ_{n+1} has the desired mass M. Lastly, we update our ansatz function by $A_{n+1} := A_n M/M_{\tilde{\Sigma}_n}$.

5) If $\|\Sigma_{n+1} - \Sigma_n\|_1$ is small enough, we calculate $\mathcal{U}_{n+1}$ once again and obtain an approximation of the desired steady state, else we return to Step 2).

It should be noted that in general the error $\|\Sigma_{n+1} - \Sigma_n\|_1 \approx \varepsilon$ does not diminish after a few loops. However, ε is highly dependent on the precision with which we calculate the mass density in Step 3). Thus, in order to obtain better results, one should always begin by increasing the numerical resolution of the appearing integrals in (3.19). On the next page, Figure 3.1 shows some illustrative examples of different steady states with the ansatz (3.18), where we fixed the mass and the radius $M = 0.3, R = 3.0$. For more details on flat steady states, we refer to [12].

The particle-in-cell scheme and initializing numerical particles

The next step is evolving this system in time. As mentioned in Chapter 2, the Vlasov-Poisson system can be reduced to its characteristic system, which has to be solved for all points (x, v) in the phase space. In fact, this is exactly the idea of our numerical approach, the so-called **particle-in-cell** scheme. In this procedure, we cover the phase space according to our ansatz function with numerical particles and evolve them in time according to the characteristic system. Let us go into detail.

Our plan is to initialize the particles in (r, u, α) coordinates. Therefore, we prescribe numbers of steps N_r, N_u and N_α and define the step lengths

$$\Delta r := \frac{R}{N_r}, \quad \Delta u := \frac{u_{max}}{N_u}, \quad \Delta\alpha := \frac{\pi}{N_\alpha},$$

where $u_{max} := \sqrt{2(E_0 - \mathcal{U}(0))}$ based on (3.17). This yields $N_r \cdot N_u \cdot N_\alpha$ particles, which set up the grid of points

$$r_i = \Big(i - \frac{1}{2}\Big)\Delta r, \quad u_j = \Big(j - \frac{1}{2}\Big)\Delta u, \quad \alpha_k = \Big(k - \frac{1}{2}\Big)\Delta\alpha.$$

Every particle (r_i, u_j, α_k) carries the weight f_{ijk}, which is according to the ansatz and the volume element

$$f_{ijk} := \Phi(E_{ijk}, L_{ijk})\, 4\pi r_i \Delta r\, u_j \Delta u\, \Delta\alpha. \tag{3.21}$$

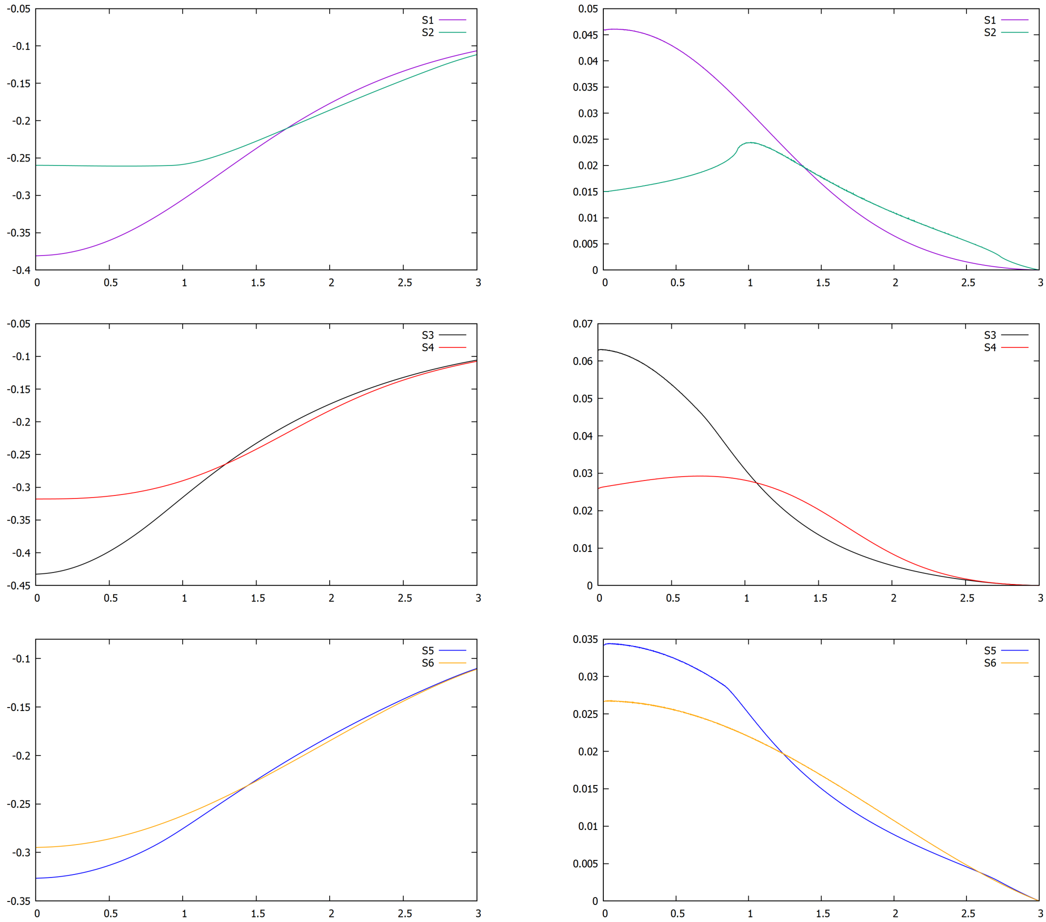


Figure 3.1: Steady states with U on the left, Σ on the right and the following parameters:

	$S1$	$S2$	$S3$	$S4$	$S5$	$S6$
l	-0.5	-0.5	0	-1.0	0	0.5
k	0.5	0	0.5	0.5	0	0
Q	0.5	2.0	2.0	1.0	2.0	0

The particle energy and the modulus of angular momentum is denoted by

$$E_{ijk} := \frac{1}{2}u_j^2 + \mathcal{U}(r_i), \quad L_{ijk} := |r_i u_j \sin \alpha_k|.$$

Particles with weight $f_{ijk} = 0$ will be deleted immediately, and we save the remaining particles in (r, w, L) coordinates. We want to point out that this procedure is quite similar to the more popular techniques in simulations of molecular dynamics. There, one usually works with probability densities, where (up to normalizing) f_{ijk} gives the probability of finding a particle in the area

$$\big[i\Delta r - 1/2, i\Delta r + 1/2\big] \times \big[j\Delta u - 1/2, j\Delta u + 1/2\big] \times \big[k\Delta\alpha - 1/2, k\Delta\alpha + 1/2\big]$$

with energy E_{ijk}; the angular momentum is often not of interest.

Once the particles are initialized, we need to propagate them in time according to the characteristic system.

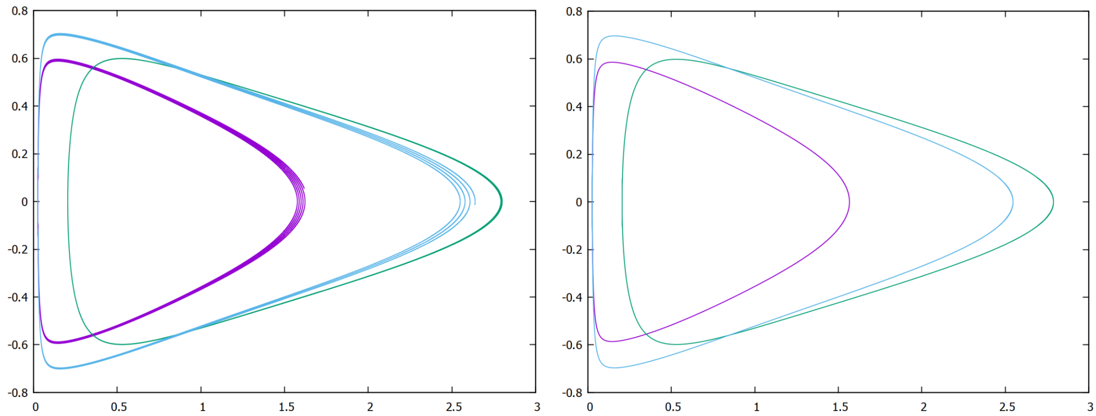
Propagating the particles - solving the characteristic system

The propagation of particles is one of the advantages of using (r, w, L) coordinates, since we only have to solve two differential equations

$$\dot{R} = W, \quad \dot{W} = \Big(\frac{L}{R^3} - \mathcal{U}'(s, R)\Big), \tag{3.22}$$

instead of three in the (r, u, α) coordinates or four in the cartesian coordinates. On the other hand, (3.22) may cause numerical problems if we do not take care of particles close to the origin, which is where $\ddot{r} \propto r^{-3}$ becomes large. This is compensated by choosing a time step Δt accordingly to Δr and by propagating the particles carefully. To illustrate this, we compare three propagation methods: the simple **Euler** algorithm, the so-called **velocity Verlet** algorithm and a fourth order symplectic algorithm, called **4th Yoshida**. Euler can be used for all kinds of differential equation, whereas velocity Verlet and 4th Yoshida are specifically made for Hamiltonian systems of the form $\ddot{x} = F(x)$ and $\ddot{x} = F(x, \dot{x})$ respectively. (3.22) is of this form since our force field $\mathcal{U}'$ is not being updated during one time step. Denoting $x_n = x(n\Delta t)$ and $F_n = F(x_n), F_n^i = F(x_n^i)$, these algorithms have the following schemes:

Euler	velocity Verlet	4th Yoshida
$x_{n+1} = x_n + v_n\Delta t$	$x_{n+1} = x_n + v_n\Delta t + \frac{1}{2}F_n\Delta t^2$	$x_n^{i+1} = x_n^i + c_{i+1}v_n^i\Delta t$
$v_{n+1} = v_n + F_n\Delta t$	$v_{n+1} = v_n + \frac{1}{2}(F_n + F_{n+1})\Delta t$	$v_n^{i+1} = v_n^i + d_{i+1}F_n^{i+1}\Delta t,$
return x_{n+1}, v_{n+1}	return $x_{n+1}, v_{n+1}, F_{n+1}$	for $i = 0, 1, 2, 3,\ x_n^0 := x_n, v_n^0 := v_n$
		return $x_{n+1} := x_n^4, v_{n+1} := v_n^4$



(a) Euler, $\Delta t = 10^{-4}$, total time: $T = 60$ (b) 4th Yoshida, $\Delta t = 10^{-3}$, total time: $T = 130$

Figure 3.2: Comparison between Euler and velocity Verlet/4th Yoshida

The constants appearing in the Yoshida algorithm are defined as

$$c_1 := c_4 := \frac{1}{2(2-\beta)}, \quad c_2 := c_3 := \frac{1-\beta}{2(2-\beta)},$$
$$d_1 := d_3 := \frac{1}{2-\beta}, \quad d_2 := \frac{-\beta}{2-\beta}, \quad d_4 := 0, \quad \text{with } \beta := \sqrt[3]{2}.$$

Note that $\sum_{i=1}^{4} c_i = 1 = \sum_{i=1}^{4} d_i$; for more details we refer to [14, 15]. Figure 3.2 shows the trajectories of three particles in each case, where the radius r is used for the x-axis and the radial velocity $\dot{r} = w$ for the y-axis. It is evident that in Figure 3.2 a) the two trajectories closer to $r = 0$ do not form a closed curve. This should be impossible due to the conservation of the particle energy and thus indicates serious numerical problems. On the contrary, both the velocity Verlet and 4th Yoshida algorithm yield closed trajectories as desired.

Since we know exactly which particles are troublesome, we can use a simple but effective adaptive step size $\Delta t(r, w)$. For particles close to zero or particles approaching zero at high speeds we choose small step sizes, whereas relatively large step sizes can be used for all other particles.

Updating the induced quantities

After the particles have been propagated, we have to update the mass density and the induced force field $\mathcal{U}_n'$ at time $t = n\Delta t$ on all grid points in space. In contrast to the spherically symmetric case, the latter calculation is numerically expensive. Let us go into detail on how the force field is obtained.

First, the mass density can be easily updated with updated values of (r_i, w_j, L_k) and the corresponding weights of the particles f_{ijk} given in (3.21). Note that the weights do not change due to the fact that the characteristic flow conserves phase space volume.

The force field can be calculated after rewriting (3.13) as follows

$$\mathcal{U}'(r) = 4\left(\int_0^r s\Sigma(s)\frac{\mathbb{E}\big(s/r\big)}{r^2 - s^2}\,\mathrm{d}s - \int_r^R \frac{s^2}{r}\Sigma(s)\frac{\mathbb{E}\big(r/s\big)}{s^2 - r^2}\,\mathrm{d}s\right) + \frac{4}{r}\int_r^R \Sigma(s)\mathbb{K}\Big(\frac{r}{s}\Big)\,\mathrm{d}s$$

For the moment, let us assume that r is neither close to 0 nor close to R. The third integral is easy to calculate precisely; the difficulty lies in handling the first two integrals, where we have to deal with the nonintegrable singularities $\pm(r^2 - s^2)^{-1}$. We begin by cutting out a small interval with length Δx around r, assuming that the first and the second integrand cancel each other for $s \in [r - \Delta x, r]$ and $s \in [r, r + \Delta x]$. We can calculate the remaining integrals

$$\int_0^{r-\Delta x} s\Sigma(s)\frac{\mathbb{E}\big(s/r\big)}{r^2 - s^2}\,\mathrm{d}s - \int_{r+\Delta x}^R \frac{s^2}{r}\Sigma(s)\frac{\mathbb{E}\big(r/s\big)}{s^2 - r^2}\,\mathrm{d}s$$

precisely with the fourth order Milne rule. Due to the nonintegrable singularities, it is important to use the same resolution for both integrals. If r is on the other hand close to zero or close to R, we can interpolate the force using $\mathcal{U}'(0) = 0$ and

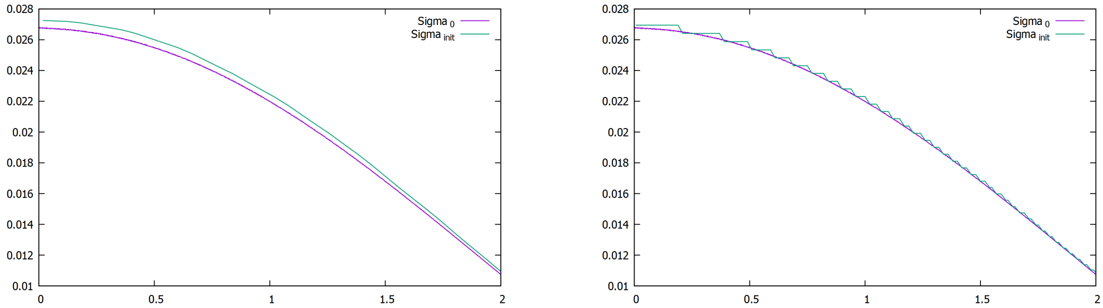
$$\mathcal{U}'(r) = 4\int_0^R s\Sigma(s)\frac{\mathbb{E}\big(s/r\big)}{r^2 - s^2}\,\mathrm{d}s, \quad r > R.$$

3.5 Numerical errors, observations and outlook

Finally, we need to address the following important question: How do we make sure our algorithm works correctly? To this end, let us go through the previous steps:
The construction of the steady state can be verified after the initialization of the particles. For this purpose, remember that $(\Sigma_0, \mathcal{U}_0)$ is a steady state if (3.11) and (3.19) hold. The initialization of the particles depends on the potential $\mathcal{U}_0$ but not on Σ_0. Hence, we can compute the mass density Σ_{init} induced by the numerical particles and compare the difference to Σ_0. Note that even if $(\Sigma_0, \mathcal{U}_0)$ is an exact steady state, we get a small **initialization error** $\mathcal{E}_{\text{init}} := \|\Sigma_0 - \Sigma_{\text{init}}\|_1 / \|\Sigma_0\|_1$ due to the fact that we only use a finite number of particles. Nevertheless, this type of error is usually of higher order. Let us briefly illustrate what both of these errors look like. In Figure 3.3, the parameters $k = 0$ and $Q = 0$ were used which leads to a discontinuity of the ansatz

$$\Phi(E, L) = \begin{cases} A, & E \le E_0, \\ 0, & E > E_0 \end{cases}$$

at the boundary of its support. In Figure 3.3 a), about $8 \cdot 10^7$ numerical particles were used, which seems to be enough for the difficult $k = 0$ case. In b) we constructed Σ_{init} with only 10^7 particles in order to illustrate the discontinuity of Φ. Unless stated otherwise, the following computations use about 15 loops for the fixpoint iteration and about 10^8 numerical particles with an initialization error of the order $\mathcal{E}_{\text{init}} \approx 10^{-2}$.



(a) 4 loops, $\mathcal{E}_{init} \approx 0.6$

(b) 10 loops, $\mathcal{E}_{init} \approx 0.04$

Figure 3.3: Two types of initialization errors

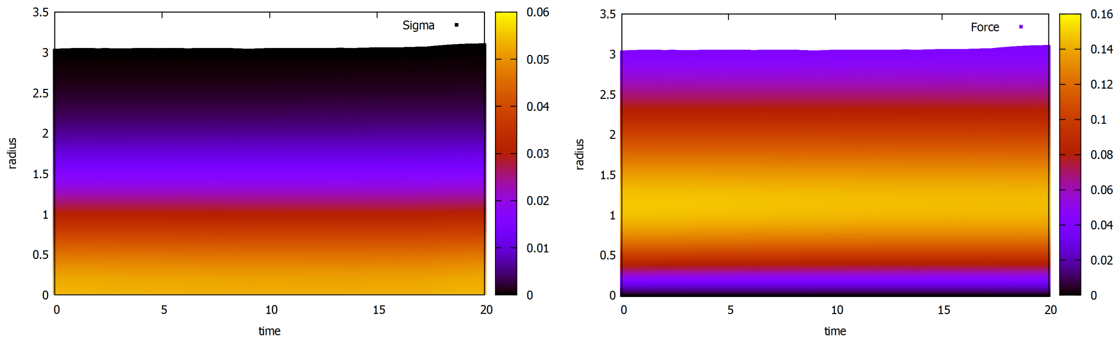


Figure 3.4: $\Sigma(t,r)$ and $\mathcal{U}'(t,r)$ for $k = 0.5$, $Q = 0$.

Next, let us investigate the errors that occur by evolving the system in time. An illustrative method is taking a steady state $(\Sigma_0, \mathcal{U}_0)$ as initial datum and checking if our algorithm leaves these macroscopic quantities unchanged. To be more precise, we have to investigate the difference of the particle distribution functions, the **distribution error** $\mathcal{E}_f(t) := \|f_0 - f(t)\|_{L^1}/\|f_0\|_{L^1}$, which is calculated using (3.21) and the updated values of (r_i, w_j, L_k), to check if the solution is truly time independent. This was indeed the case during the tests of several parameters in the range of $l \in [-0.75, 0.5]$, $k \in [0, 0.5]$ and $Q \in [0, 3]$. In fact, the distribution error $\mathcal{E}_f(t)$ was in general of the same order as the initialization error, which seems to be reasonable.

However, a popular approach for non-stationary solutions uses the conservation of the total energy and considers the **energy error** $\mathcal{E}_{\text{tot}}(t) := |E_{\text{tot}}(0) - E_{\text{tot}}(t)|/|E_{\text{tot}}(0)|$, where we define the kinetic and potential energy analogously to Chapter 2. Figure 3.4 shows the mass density and the force field for a steady state. All choices of time steps $\Delta t \in \{10^{-5}, 10^{-4}, 10^{-3}\}$ and every number of numerical particles $N \in \{8 \cdot 10^7, 10 \cdot 10^7, 12 \cdot 10^7\}$ lead to these pictures of Σ and $\mathcal{U}'$ with error $\mathcal{E}_{\text{tot}}(t = 20) \approx 5 \cdot 10^{-4}$. This value seems to be small, however, in the algorithm for the easier three dimensional Vlasov-Poisson system the energy error is smaller and it becomes even smaller as one decreases Δt from

10^{-3} to 10^{-5}, whilst increasing the numbers of numerical particles accordingly, cf. [16]. This is due to the fact, that the energy error in the three dimensional algorithm arises mainly from small inaccuracies when propagating particles close to zero. This does not seem to be the case here. It is still an open task to improve the conservation of the energy for our algorithm. The author assumes that errors in the calculation of the force field, especially for r close to the boundary R, play an important role.

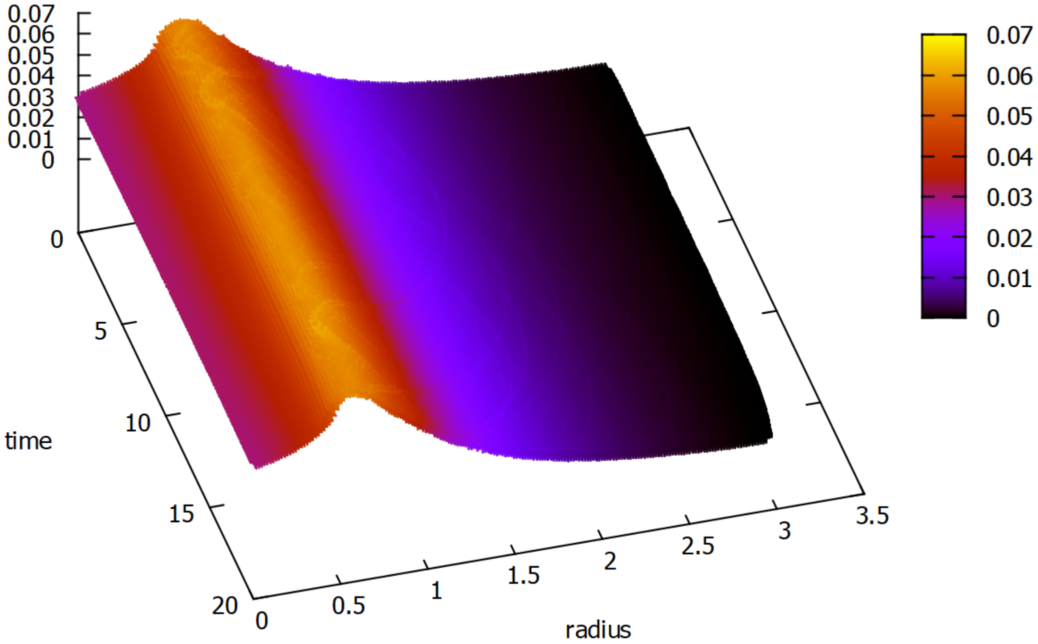


Figure 3.5: Mass density Σ for $(t, r) \in [0, 20] \times [0, 3.5]$ with $k = 0.5, l = -0.6, Q = 3.0$

Figures 3.5 and 3.6 show the evolution of a certain steady state, in which the gravitational force pushes particles in a small region away from the origin. Note that this scenario cannot occur in the three dimensional spherically symmetric Vlasov-Poisson system. In our computations, we find that $\mathcal{U}'(r) < 0$ on $]0, 0.50[$ with Σ taking its maximum at $r \approx 0.58$. About $8 \cdot 10^7$ particles were used with time step $\Delta t = 5 \cdot 10^{-4}$, which yielded an error $\mathcal{E}_{\text{tot}}(t = 20) \approx 10^{-3}$.

For the sophisticated reader who may not be impressed by these unchanging plots, we lastly want to investigate the evolution of a steady state on the particle level (r, w, L) and therefore recall the characteristic system

$$\dot{r} = w, \quad \dot{w} = \left(\frac{L}{r^3} - \mathcal{U}'(s, r)\right), \quad \dot{L} = 0.$$

In Figure 3.7, we plotted the motion of about 10^4 randomly chosen numerical particles at six different times, where the radius is used for the x-axis, the radial velocity w for the

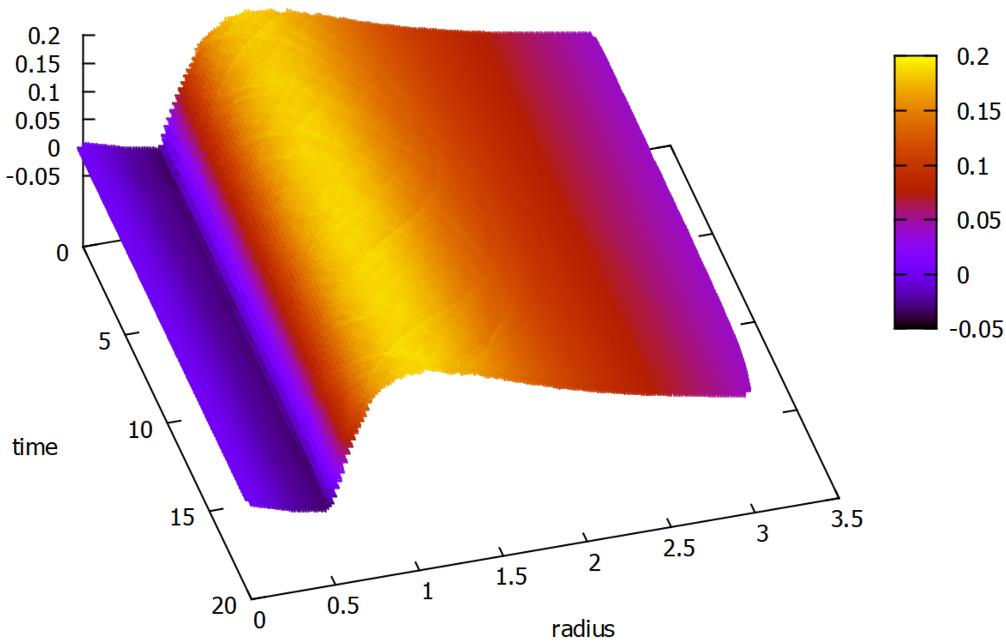


Figure 3.6: Force $\mathcal{U}'$ for $(t, r) \in [0, 20] \times [0, 3.5]$ with $k = 0.5, l = -0.6, Q = 3.0$

y-axis and the colour indicates the modulus of the angular momentum L_z. Note that the corresponding distribution function f and the macroscopic quantities like Σ and $\mathcal{U}'$ from Figure 3.4 are (aside from small numerical errors) constant in time. Furthermore, notice that the moving black spiral-arms of particles appearing in Figure 3.7 do not necessarily contradict the time independence of the mass density. This is due to the fact that the (numerical) particles may have different (numerical) weights which they contribute to the phase-space density f. Thus, the density of numerical particles from Figure 3.7 does not yield in general information on the mass density. For further prospective investigations on steady states or on the oscillating behaviour of perturbed steady states, the author recommends to combine observations of the particle distribution function f, the motion of particles and derived macroscopic quantities like Σ or the kinetic and potential energy.

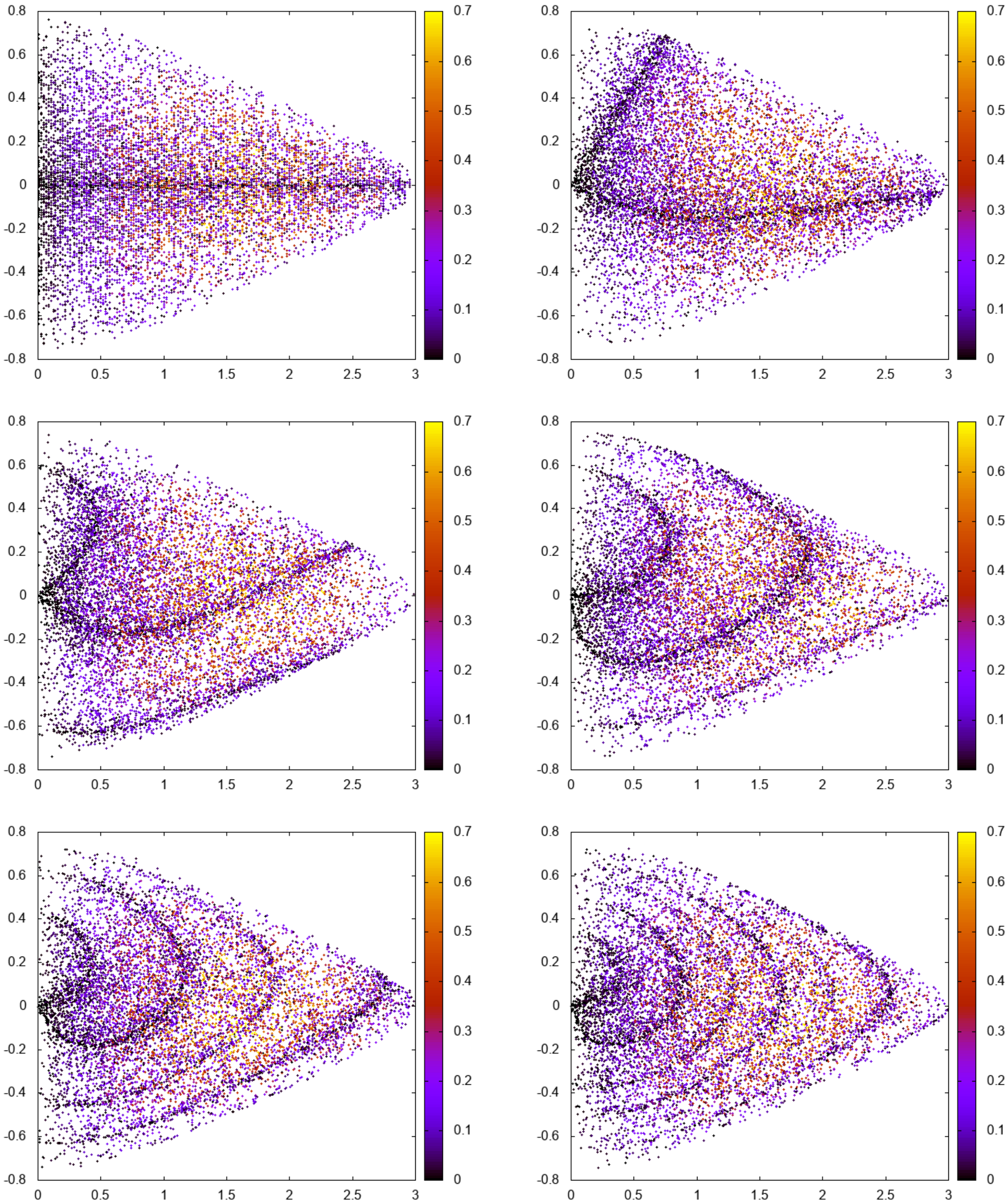


Figure 3.7: Motion of particles for $k = 0.5$, $Q = 0$ at $t = 0, 1, 5, 10, 16, 30$.

Outlook

In [16], perturbed spherically symmetric steady states of the three dimensional gravitational Vlasov-Poisson system have been investigated numerically. For different ansatz functions, numerical evidence is given that these solutions oscillate in time. The author strongly supposes that oscillating behaviour can be found in the axially symmetric Vlasov-Poisson system as well. With the just presented algorithm, one can properly address this issue. Moreover, an intriguing question is how flat steady states with positive gravitational force, i.e. on some region particles are pulled away from the center of mass, behave under certain perturbations, since this phenomenon does not exist in three dimensional spherically symmetric Vlasov-Poisson system.

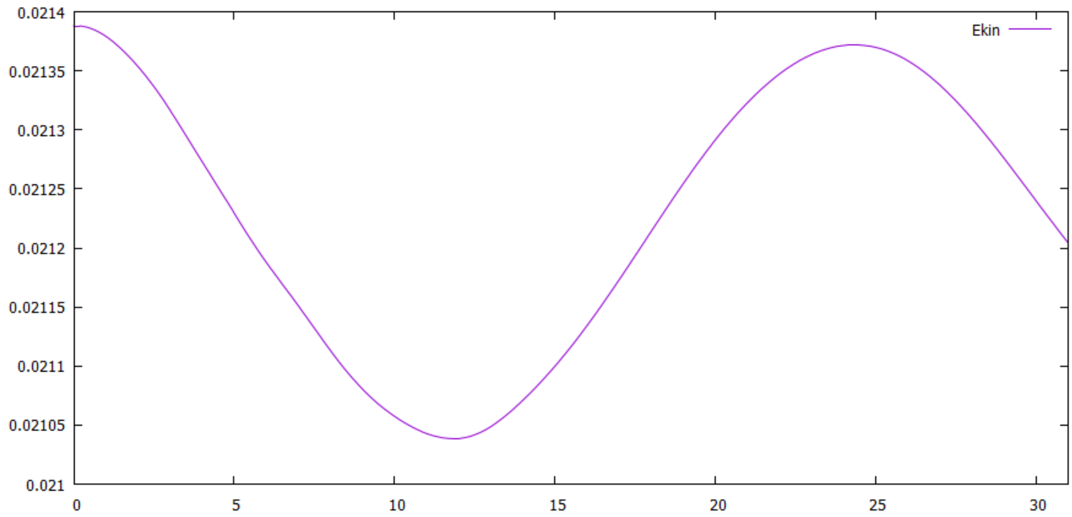


Figure 3.8: Kinetic energy for unperturbed steady state with $k = 0.5$, $Q = 0$ and with a rather large initialization error $\mathcal{E}_{\text{init}} \approx 0.01$.

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