The Sequential Quadratic Hamiltonian Method

by

Alfio Borzì

Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

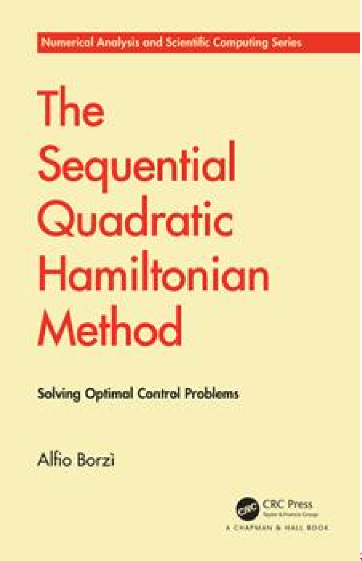
Framework

- Investigate the Pontryagin maximum principle (PMP) to analyse optimal control problems governed by differential models (ODEs, PDEs) with different cost functionals (non-smooth, non-convex, discontinuous, mixed-integer, multi-objective) and control mechanisms (distributed linear, bilinear, on boundary, in the coefficients, relaxed, etc.).
- Develop an efficient and robust PMP-based numerical strategy, iterative, allowing point-wise (or local-wise) updates, for solving optimal control problems of large-size.
- The Sequential Quadratic Hamiltonian Method

Joint works with Stefana Anita, Tim Breitenbach, Sebastian Hofmann, Souvik Roy, Anwesa Dey, Mario Annunziato, Francesca Cala'-Campana.
And strong support by M.I. Sumin and D. Wachsmuth.

The book and the software

Software:
`https://github.com/alfioborzi/SQHmethod`



An optimal control problem

Consider the following ODE optimal control problem

$$
\begin{aligned}
&\min J(y,u) := \int_{t_0}^{T} \ell(t, y(t), u(t))\, dt + \gamma(y(T)) \\
&\text{s.t.} \quad y'(t) = f(t, y(t), u(t)), \qquad y(t_0) = y_0, \\
&u \in U_{ad}.
\end{aligned}
\tag{1}
$$

The optimal control is sought in a subset of L^2 given by

$$
U_{ad} = \left\{ u \in L^2(t_0, T; \mathbb{R}^m) \ : \ u(t) \in K_{ad} \text{ a.e.} \right\},
$$

where K_{ad} is a compact subset of $\mathbb{R}^m$.

Subject to appropriate conditions (Carathéodory's, Clarke's, Cesari's) this problems admits a solution.

Extensions: Relaxed controls (Young's measures), quasi- or suboptimal controls (Ekeland).

Characterization: Optimality conditions (Euler-Lagrange's, Pontryagin's)

Euler-Lagrange's optimality

Subject to appropriate convexity and differentiability conditions, the EL optimality system for a solution to (1) is given by

$$
\begin{aligned}
y' &= f(t,y,u), \qquad y(t_0) = y_0 \\
-p' &= \left(\partial_y f(t,y,u)\right)^\top p - \partial_y \ell(t,y,u), \qquad p(T) = -\partial_y \gamma(y(T)) \\
&\Big(-\left(\partial_u f(\cdot,y,u)\right)^\top p + \partial_u \ell(\cdot,y,u),\, v-u \Big) \geq 0, \qquad v \in U_{ad}.
\end{aligned}
$$

For this system there is an equivalent hamiltonian formulation based on the Hamilton-Pontryagin (HP) function

$$\mathcal{H}(t,y,u,p) = p \cdot f(t,y,u) - \ell(t,y,u). \tag{2}$$

Resulting from the transformation:

$$\mathcal{L}(y,u,p) = \gamma(y(T)) + \int_{t_0}^{T} \Big(p(t) \cdot y'(t) - \mathcal{H}(t,y(t),u(t),p(t)) \Big)\, dt.$$

Then the optimality condition above reads $(\partial_u \mathcal{H}, v-u) \leq 0, v \in U_{ad}$.

The hamiltonian formulation

In the ODE case, the forward and adjoint equations are written in the following form

$$\begin{cases} y'(t) = \partial_p \mathcal{H}(t, y(t), p(t), u(t)) \\ p'(t) = -\partial_y \mathcal{H}(t, y(t), p(t), u(t)) \end{cases}$$

with appropriate transversality conditions. Moreover, without control constraints, along the optimal trajectory , it must hold:

$$\partial_u \mathcal{H}(t, y(t), p(t), u(t)) = 0,$$

and

$$\partial^2_{uu} \mathcal{H}(t, y(t), p(t), u(t)) < 0,$$

which corresponds to the Legendre's conditions.

Pontryagin maximum principle

The Pontryagin maximum principle (PMP) includes all previous results by L. Euler, J.-L. Lagrange, K. Weierstrass, A.-M. Legendre, etc., and removes the requirement of differentiability w.r.t. u.

It is replaced by the maximality condition

$$\mathcal{H}(t, y(t), u(t), p(t)) = \max_{v \in K_{ad}} \mathcal{H}(t, y(t), v, p(t)),$$

pointwise, for almost all $t \in [t_0, T]$.

Taking the HP function $\tilde{\mathcal{H}}(t, y, u, \tilde{p}) = \tilde{p} \cdot f(t, y, u) + \ell(t, y, u)$, where $\tilde{p} = -p$, we have $\tilde{\mathcal{H}} = -\mathcal{H}$, and a minimality condition.

The set K_{ad} may not be convex: it could represent a finite set of discrete values.

V. G. Boltyanskii, R. V. Gamkrelidze, and L. S. Pontryagin. On the theory of optimal processes. Dokl. Akad. Nauk SSSR, 110:7–10, 1956.

Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko. The Mathematical Theory of Optimal Processes, Wiley & Sons, 1962

State-of-the-art in PMP

The maximum principle must be tailored and proved for the given problem at hand.

In the case of ODE models, the PMP has been investigated for virtually all kind of optimal control problems with: control constraints, state constraints, path constraints, mixed-integer controls, relaxed controls, etc.: Pontryagin, Boltyanskii, Gamkrelidze, Rozonoèr, Dubovitskii, Milyutin, Dmitruk, Osmolovskii, Cesari, Sussmann, Ekeland, Vinter, Clarke, Hartl, Halkin, Markus, etc..

In the case of PDE models, much less results are available: Sumin, Plotnikov, Casas, Raymond, Zidani, Li & Yong, Suryanarayana, Mordukhovich, etc..

H.-J. Pesch and M. Plail, The maximum principle of optimal control: A history of ingenious ideas and missed opportunities, Control and Cybernetics, 38 (2009), 973-995.
H.J. Sussmann and J.C. Willems. 300 years of optimal control: from the brachystochrone to the maximum principle. IEEE Control Systems, 17(3):32-44, 1997.
A. V. Dmitruk and N. P. Osmolovskii. On the proof of Pontryagin's maximum principle by means of needle variations. Journal of Mathematical Sciences, 218(5):581-598, 2016.

Rozonoèr's estimate

Soon after the formulation of the PMP, Lev Rozonoèr (1959) proved the following result:

Let $u, v \in U_{ad}$ with K_{ad} compact and convex, then there exists a constant $C > 0$ such that the following holds

$$J(y_v, v) - J(y_u, u) = -\int_{t_0}^{T} \Big(\mathcal{H}\left(t, y_u, v, p_u\right) - \mathcal{H}\left(t, y_u, u, p_u\right) \Big)\, dt + R,$$

where $|R| \leq C \int_{t_0}^{T} |u(t) - v(t)|^2 \, dt$. The constant C depends on the size of the interval and on the Lipschitz constants of f and ℓ with respect to y.

This result and the following ones can be extended to PDE optimal control problems in different settings.

L. I. Rozonoèr. Pontryagin maximum principle in the theory of optimum systems. Avtomat. i Telemeh., 20:1320–1334, 1959.

English transl. in Automat. Remote Control, 20 (1959), 1288–1302.

The intermediate adjoint

Consider two admissible pairs (y_u, u) and (y_v, v), where $u, v \in U_{ad}$.

The intermediate (or average) adjoint variable is the solution to the following problem

$$-\tilde{p}'(t) = \left(\tilde{f}_y(t, y_u(t), y_v(t), v(t))\right)^\top \tilde{p}(t) - \tilde{\ell}_y(t, y_u(t), y_v(t), v(t)),$$

with terminal condition $\tilde{p}(T) = -\tilde{\gamma}_y(y_u(T), y_v(T))$, and the functions $\tilde{f}_y$, $\tilde{\ell}_y$ and $\tilde{\gamma}_y$ are constructed as follows:

$$\tilde{\phi}_y\,(t, y_u, y_v, u) := \int_0^1 \partial_y \phi(t, y_u + s\,(y_v - y_u), u)\, ds.$$

Then the following equality holds

$$J(y_v, v) - J(y_u, u) = -\int_{t_0}^{T} \Big(\mathcal{H}\,(t, y_u, v, \tilde{p}) - \mathcal{H}\,(t, y_u, u, \tilde{p})\Big)\, dt.$$

Successive approximations schemes

Based on the above results, it is natural to design different algorithms based on the pointwise maximization of the HP function. These iterative methods are called successive approximations (SA) schemes. Earlier works: L.I. Rozonoer, I.A. Krylov and F.L. Chernous'ko, H.J. Kelley, R.E. Kopp and H.G. Moyer.

Robustness of SA schemes was achieved by Y. Sakawa and Y. Shindo by introducing a quadratic penalty of the control updates

$$H_\epsilon(t, y, u, w, p) := \mathcal{H}(t, y, u, p) - \epsilon\, |u - w|^2.$$

In Sakawa & Shindo's version of SA, the parameter ϵ is fixed and w represents the value of the control obtained in the previous iterate. (Connection to the works of B. Järmark, R.T. Rockafellar, etc..)

I. A. Krylov and F. L. Chernous'ko. On a method of successive approx- imations for the solution of problems of optimal control. Zh. Vychisl. Mat. Mat. Fiz., 1962, Vol. 2, Nr. 6, 1132-1139.
H. J. Kelley, R. E. Kopp, and H. G. Moyer. Successive approximation techniques for trajectory optimization. In Proc. IAS Symp. on Vehicle System Optimization, pages 10-25, New York, November 1961.
Y. Sakawa and Y. Shindo. On global convergence of an algorithm for optimal control. IEEE Transactions on Automatic Control, 25(6):1149-1153, 1980.
J. F. Bonnans. On an algorithm for optimal control using Pontrya- gin's maximum principle. SIAM Journal on Control and Optimization, 24(3):579–588, 1986.

The SQH algorithm

Input: initial approx. u^0, max. number of iterations k_{max}, tolerance $\kappa > 0$, $\epsilon > 0, \sigma > 1, \eta > 0$, and $\zeta \in (0,1)$; set $\tau > \kappa$, $k := 0$.
Compute the solution y^0 to the governing model with given data and control u^0.

while ($k < k_{max}$ && $\tau > \kappa$) do

1. Compute the solution p^k to the adjoint problem with given data.
2. Determine u^{k+1} that solves the pointwise optimization problem

$$H_\epsilon\left(z, y^k(z), u^{k+1}(z), u^k(t), p^k(z)\right) = \max_{w \in K_{ad}} H_\epsilon\left(z, y^k(z), w, u^k(z), p^k(z)\right)$$

   (where $z = t$, or $z = x$, or $z = (x,t)$, etc.)

3. Compute the solution y^{k+1} to the governing model with given data and control u^{k+1}.
4. Compute $\tau := \|u^{k+1} - u^k\|_{L^2}^2$.
5. If $J\left(y^{k+1}, u^{k+1}\right) - J\left(y^k, u^k\right) > -\eta\,\tau$, then increase $\epsilon = \sigma\,\epsilon$ and go to Step 2. Else if $J\left(y^{k+1}, u^{k+1}\right) - J\left(y^k, u^k\right) \leq -\eta\,\tau$, then decrease $\epsilon = \zeta\,\epsilon$ and continue.
6. Set $k := k + 1$.

end while

The SQH method

The SQH method is a SA scheme with augmented Hamiltonian and a mechanism that adaptively chooses ϵ in order to guarantee a sufficient decrease of the value of the cost functional. In Step 2. also partial maximization would suffice.

Theorem

Let (y^k, u^k) and (y^{k+1}, u^{k+1}) be generated by the SQH algorithm, and u^k, u^{k+1} be measurable. Then, subject to appropriate assumptions on f and ℓ, there exists a $\theta > 0$ independent of ϵ and u^k such that for the $\epsilon > 0$ currently chosen by the SQH algorithm, the following holds

$$J\left(y^{k+1}, u^{k+1}\right) - J\left(y^k, u^k\right) \leq -\left(\epsilon - \theta\right) \|u^{k+1} - u^k\|_{L^2}^2 .$$

In particular, it holds

$$J\left(y^{k+1}, u^{k+1}\right) - J\left(y^k, u^k\right) \leq -\eta\,\tau$$

for $\epsilon \geq \theta + \eta$, and $\tau = \|u^{k+1} - u^k\|_{L^2}^2$.

Some convergence results

Theorem

For the sequence (u^k), generated by the SQH algorithm, it holds:

1. *the sequence $(J(y^k, u^k))_{k=0,1,2,\ldots}$ is monotonically decreasing and converges to some $J^* \geq \inf_{u \in U_{ad}} J(y_u, u)$;*
2. $\lim_{k\to\infty} \|u^{k+1} - u^k\|_{L^2} = 0$.

Theorem

Let K_{ad} be compact and convex, and assume that f and ℓ are continuously differentiable with respect to u. Further, suppose that $\epsilon \leq \bar{\epsilon}$, for some $\bar{\epsilon} > 0$, for all SQH iterates. Then the sequence $(u^k)_{k=0,1,2,\ldots}$ generated by the SQH algorithm, asymptotically satisfies the first-order necessary optimality conditions for the maximisation of the HP function, in the sense that

$$\lim_{k\to\infty} \|u^k - \Pi_K\left(u^k + \partial_u \mathcal{H}(\cdot, y^k, u^k, p^k)\right)\|_{L^2} = 0,$$

where $\Pi_K(v)$ denotes the projection on the convex set K satisfying $|v - \Pi_K(v)| = d_K(v) := \inf_{w\in K} |v - w|$, and $K := K_{ad}$.

An quantum control problem

$$\min J(y,u) := \frac{1}{2}\sum_{i=1}^{n}\left(y_i(T) - (y_d)_i\right)^2 + \int_0^T g_{\alpha,\beta,\delta}(u(t))\,dt$$

$$\text{s.t.} \qquad y' = (A + uB)\,y,\ t \in (0,T), \qquad y(0) = y_0$$

$$u \in U_{ad} := \left\{u \in L^2(0,T) \mid u(t) \in K_U \text{ a.e.}\right\}$$

with a discontinuous cost:

$$g_{\alpha,\beta,\delta}(u) := \frac{\alpha}{2}u^2 + \beta|u| + \delta|u|_s,$$

and

$$|u|_s := \begin{cases} |u| & \text{if } |u| > s \\ 0 & \text{else} \end{cases}, \qquad s > 0.$$

$$H_\epsilon(t,y,u,v,p) = p^T(A + uB)\,y - g_{\alpha,\beta,\delta}(u) - \epsilon(u-v)^2.$$

Optimisation of H_ϵ

Let $\beta = 0$ and $K_{ad} = [\underline{u}, \overline{u}]$. If $|u| \leq s$ then H_ϵ achieves it maximum at

$$u^1 = \min\left(\max\left(-s, \frac{2\,\epsilon\,u^k + (p^k)^T\,B\,y^k}{2\,\epsilon + \alpha}\right), s\right).$$

If $u < -s$ or $u > s$, we obtain two additional points where H_ϵ can attain a maximum

$$u^2 = \min\left(\max\left(\underline{u}, \frac{2\,\epsilon\,u^k + (p^k)^T\,B\,y^k + \delta}{2\,\epsilon + \alpha}\right), -s\right)$$

and

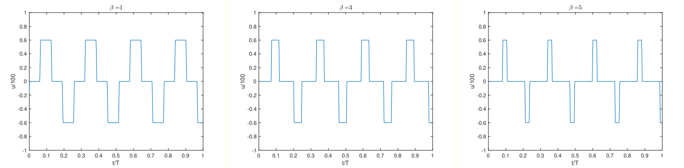
$$u^3 = \min\left(\max\left(s, \frac{2\,\epsilon\,u^k + (p^k)^T\,B\,y^k - \delta}{2\,\epsilon + \alpha}\right), \overline{u}\right).$$

Therefore, in Step 2. of the SQH algorithm, we implement

$$u^{k+1} = \underset{w \in \{u^1, u^2, u^3\}}{\arg\max}\; H_\epsilon\left(t, y^k, w, u^k, p^k\right).$$

Numerical results I - SQH vs. SSN

The case $\alpha = 10^{-3}$, $\beta > 0$, $\delta = 0$, and $K_{ad} = [-60, 60]$.



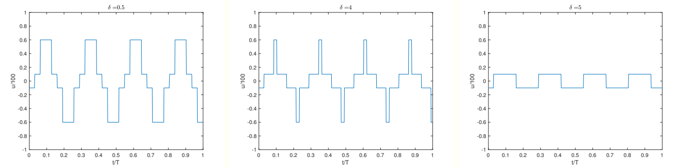
PMP test : $0 \leq \Delta H \leq 10^{-\ell}$ where $\Delta H\,(t) := \left|\,\left(H\,(t, y, u, p) - \max_{w \in K_{ad}} H\,(t, y, w, p)\right)\,\right|$

β	$\frac{N^4_{\%}}{\%}$	$\frac{N^5_{\%}}{\%}$	$\frac{N^8_{\%}}{\%}$	$\frac{N^{15}_{\%}}{\%}$	CPU time/s		# it	# up
					SQH	SSN		
1	96.01	95.88	95.88	90.14	0.77	94	51	24
3	100	100	98.00	82.65	0.78	21	50	25
5	100	100	98.13	84.27	0.66	28	42	23

The number of SQH iterations is denoted with # it, and the number of sweeps of updates of the control is denoted with # up.

Numerical results II - disc. L^1 cost

In the case of discontinuous L^1 costs: $\alpha = 10^{-2}$, $\beta = 0$, $\delta > 0$, $s = 10$, and $K_{ad} = [-60, 60]$.

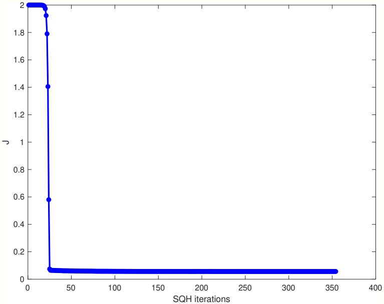


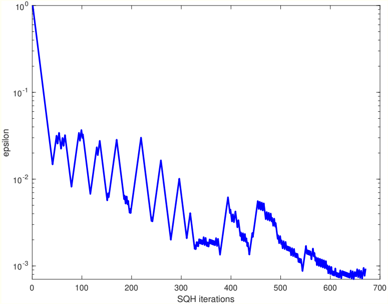
PMP test : $0 \leq \Delta H \leq 10^{-\ell}$ where $\Delta H(t) := |\left(H(t,y,u,p) - \max_{w \in K_{ad}} H(t,y,w,p)\right)|$

δ	$\frac{N^4_\%}{\%}$	$\frac{N^5_\%}{\%}$	$\frac{N^8_\%}{\%}$	$\frac{N^{15}_\%}{\%}$	CPU time/s	# it	# up
0.5	99.63	99.63	99.63	98.88	1.5	77	58
4	98.13	98.13	98.13	98.13	0.33	16	12
5	100	100	100	100	0.17	7	7

Convergence behaviour

Convergence behaviour of the cost functional J and of the value of ϵ along the SQH iterations for Exp II.





SQH initialisation: $u^0 = 0$ and $\epsilon = 1$.
SQH parameters: $\sigma = 1.1, \zeta = 0.9, \eta = 10^{-5}, \kappa = 10^{-8}$.

Elliptic optimal control problems

The weak formulation of a semilinear elliptic optimal control problem is as follows

$$\min J(y,u) := \int_\Omega \Big(h(y(x)) + g(u(x)) \Big)\, dx$$

$$\text{s.t.} \qquad B(y,v) = \int_\Omega f(x, y(x), u(x))\, v(x)\, dx, \qquad v \in \mathcal{V},$$

$$u \in U_{ad}$$

For many choices of J that result in a weakly lower semicontinuous cost functionals in the appropriate space, existence of solutions can be proved by Tonelli's technique of minimising sequences; see, e.g., the books: Lions (1971), Ahmed & Teo (1981), Troeltzsch (2010).

Also in the book: different distributed controls, boundary controls, state constraints, L^1 tracking terms, discrete K_{ad}, time-dependent parabolic and hyperbolic optimal control problems.

PMP minimality condition

Assuming that B is self-adjoint, we have the adjoint problem

$$B(p, v) = \int_\Omega \left(\partial_y h(y(x)) + \partial_y f(x, y(x), u(x))\, p(x)\right) v(x)\, dx,$$

for all $v \in \mathcal{V}$; p denotes the adjoint variable. We introduce the HP function

$$\mathcal{H}(z, y, u, p) := p\, f(x, y, u) + h(y) + g(u).$$

Theorem

Let h and f be continuously differentiable, and let g be continuous and convex. Furthermore, assume that f is Lipschitz in u and $\partial_y f$ be uniformly bounded. Then any solution (y, u) to the above problem fulfils

$$\mathcal{H}(x, y(x), u(x), p(x)) = \min_{w \in K_{ad}} \mathcal{H}(x, y(x), w, p(x)), \qquad x \in \Omega.$$

J.-P. Raymond and H. Zidani. Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations. Applied Mathematics and Optimization, 39(2):143–177, 1999.

Bilinear elliptic optimal control

$$\min J(y,u) := \int_\Omega \left(\frac{1}{2} \left(y(x) - y_d(x)\right)^2 + g(u(x)) \right) dx$$
$$(\nabla y, \nabla v) + (u\,y, v) = (\varphi, v)\,, \qquad v \in H_0^1(\Omega)\,,$$

with target function: $y_d(x) := \sin(2\pi x_1)\cos(2\pi x_2)$ and discontinuous cost

$$g(u) := \frac{\alpha}{2} u^2 + \begin{cases} \beta\,|u| & \text{if } |u| > s \\ 0 & \text{else} \end{cases},$$

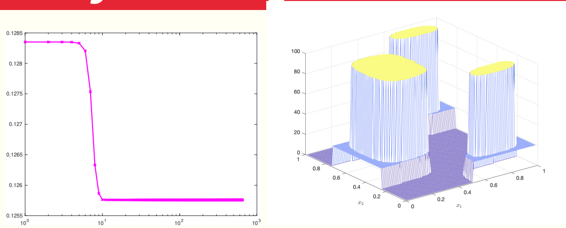
where $\alpha = 10^{-10}$, $\beta = 10^{-5}$ and $s = 20$; $K_{ad} = [0, 100]$. The HP function:

$$\mathcal{H}(x,y,u,p) = \frac{1}{2}(y - y_d)^2 + g(u) + p\,\varphi - u\,y\,p.$$

The adjoint problem is asfollows

$$(\nabla p, \nabla v) + (u\,p, v) = (y - y_d, v)\,, \qquad v \in H_0^1(\Omega)\,.$$

Solution by the SQH method



Minimisation of J (left) and optimal control.

Quantify the accuracy of PMP optimality:

$$\triangle H(x) := \left(H(x, y, u, p) - \min_{w \in K_{ad}} H(x, y, w, p) \right).$$

Report $N^l_{\%}$: the percentage of the grid points at which the inequality $0 \leq \Delta H \leq 10^{-l}$, $l \in \mathbb{N}$, is fulfilled.

Convergence ($\kappa = 10^{-8}$) at $k_{tot} = 1126$, $k_{up} = 622$, we obtain $\frac{N^4_{\%}}{\%} = 99.83$, $\frac{N^8_{\%}}{\%} = 98.16$, $\frac{N^{10}_{\%}}{\%} = 99.14$, and $\max_{x \in \Omega} \Delta H(x) = 1.28 \cdot 10^{-4}$ and

Wave stabilization

Consider the following wave optimal control problem with a Neumann boundary control:

$$\begin{aligned}
\min J(y,u) &:= \frac{1}{2}\|y(\cdot,T)-y_T(\cdot)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Sigma)} + \gamma\,\|u\|_{L^1(\Sigma)} \\
\text{s.t.} \quad & \partial^2_{tt} y(x,t) - v^2\,\partial^2_{xx} y(x,t) = 0, \qquad \text{in } Q = \Omega\times(0,T) \\
& y(x,0) = y_0(x), \qquad \partial_t y(x,0) = y_1(x), \qquad \text{in } \Omega \\
& \partial_n y(x,t) = u(x,t), \qquad \text{on } \Sigma = \partial\Omega\times(0,T) \\
& u \in U_{ad}
\end{aligned}$$

where $\gamma \geq 0$, and $\nu > 0$. Set of admissible controls:

$$U_{ad} = \left\{u \in L^2(\Sigma) \mid u(x,t) \in K_{ad} \text{ a.e. in } \Sigma\right\}.$$

B.S. Mordukhovich, J.-P. Raymond. Neumann boundary control of hyperbolic equations with pointwise state constraints. SIAM J. Contr. Opt., 43 (2004), 1354-1372.

M. Gugat, G. Leugering, G. Sklyar. Lp-optimal boundary control for the wave equation. SIAM J. Contr. Opt., 44 (2005), 49-74.

Setting for stabilization

The HP function is given by

$$\mathcal{H}(x,t,y,u,p) = v^2\,p\,u - \frac{\nu}{2}\,u^2 - \gamma\,|u|,$$

which is defined for the variables p and u on Σ.

A change on the boundary value can influence the solution in the entire domain within a characteristic time $t_0 = L/v$. It is obvious to choose $T > t_0$. We take $L = 10$ and $v = 10$ so that $t_0 = 1$ and $T = 4$.

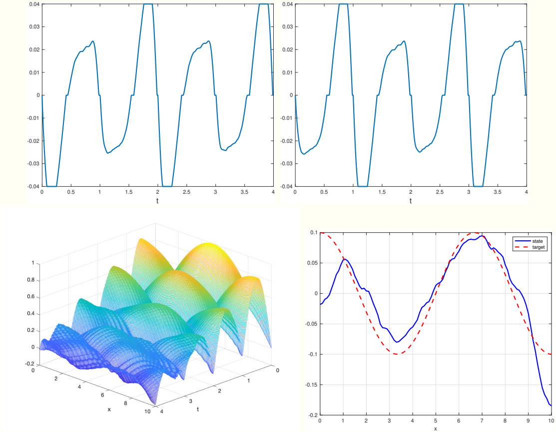
Initial conditions and target function:

$$y_0(x) = \sin(\pi\,x/L), \qquad y_1(x) = 0, \qquad y_T(x) = \cos(3\,\pi\,x/L).$$

The target function is asymmetric with respect to the midpoint of the interval $[0, L]$.
In the cost functional, we set $\nu = 10^{-12}$, and $\gamma = 2\cdot 10^{-1}$. The set of admissible control values is $K_{ad} = [-0.04, 0.04]$.

Solution by the SQH method

The values of the parameters are chosen ad-hoc to show active control constraints. The effect of $L^1(\Sigma)$ penalisation is slightly visible.



The optimal Neumann boundary control function on the left boundary point (left) and on the right boundary point. The wave propagation driven by the optimal Neumann boundary control (left) and comparison at final time of the controlled state with the given target function.

PMP as a sufficient condition

In the case(s) of wave models with linear boundary control mechanisms, the PMP condition is sufficient to characterise an optimal control.

$$
\begin{aligned}
\min J(y,u) &:= \frac{1}{2}\|y(\cdot,T)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t y(\cdot,T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Sigma)} + \gamma\,\|u\|_{L^1(\Sigma)} \\
\text{s.t.} \quad & \partial^2_{tt}\, y(x,t) - v^2\,\partial^2_{xx}\, y(x,t) = 0, \qquad \text{in } Q \\
& y(x,0) = y_0(x), \qquad \partial_t\, y(x,0) = y_1(x), \qquad \text{in } \Omega \\
& y(0,t) = 0, \qquad a\, y(L,t) + b\,\partial_x y(L,t) = u(L,t), \qquad t \in [0,T] \\
& u \in U_{ad}.
\end{aligned}
$$

At $x = 0$, we set homogeneous Dirichlet boundary conditions, and at $x = L$ we consider a general Robin boundary control mechanism.

In this setting, we can prove existence of a subsequence of the SQH iterates $(u^k)_{k=0,1,2,\ldots}$ that converges weakly in $L^2\,(\Sigma)$ to an optimal control. Strong convergence can be proved in specific settings.

Thank you for your attention!

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