Formulation and Monte Carlo Solution of Linear Kinetic Optimal Control Problems

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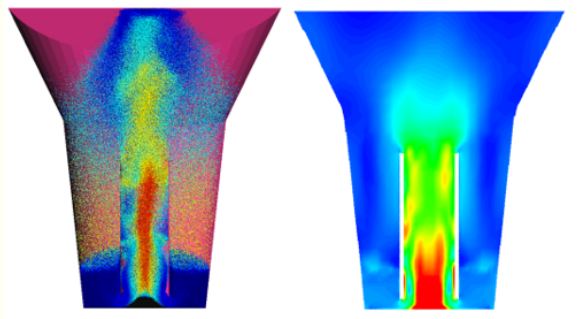
Framework

Many models in sciences and engineering aim at describing the dynamics of multiple agents/particles subject to internal and external forces.
The manipulation of these forces allows to control the systems' dynamics in order to perform desired tasks.

- A convenient description of the configuration of multi-agent (multi-particle) systems is achieved by means of probabilistic or material densities.
- The time evolution of these densities is governed by kinetic models.
- Optimal control theory provides the mathematical tools to formulate and solve control problems.
- Ensemble optimal control problems represent the natural framework for designing control mechanisms and objectives for systems governed by kinetic models.

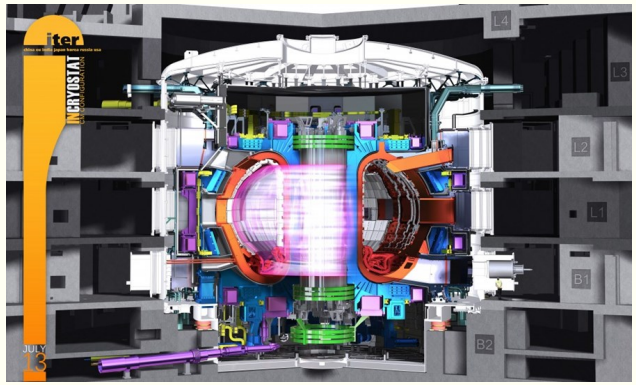
Applications

collective motion, ©STIR



coating and mixing of powder, ©RCPE

space propulsion, ©SPARC



fusion reactor, ©ITER

Density functions

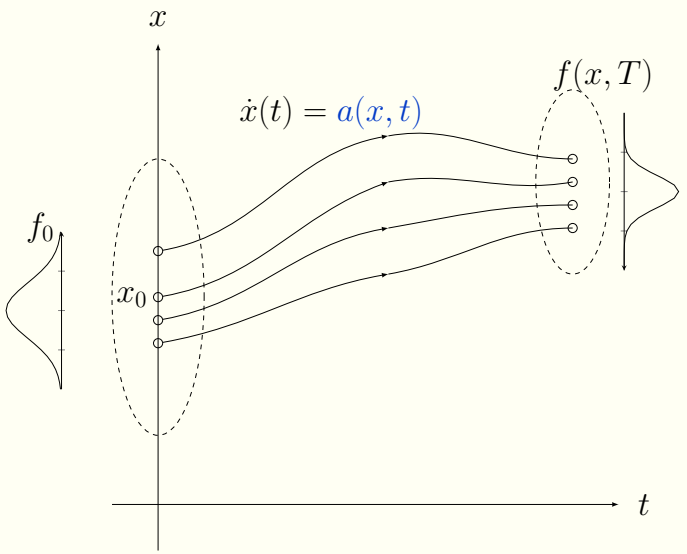
Consideration of all possible trajectories of a multi-particle system is an overwhelming task. For this reason, L. E. Boltzmann introduced the concept of material density $f(x,t)$.

In the non-interacting case, if $f_0(x)$ represents the initial density (configuration) at time $t = 0$, then the evolution of this density is modelled by

the Liouville equation

$$\partial_t f(x,t) + \operatorname{div}\left(a(x,t)\, f(x,t)\right) = 0,$$

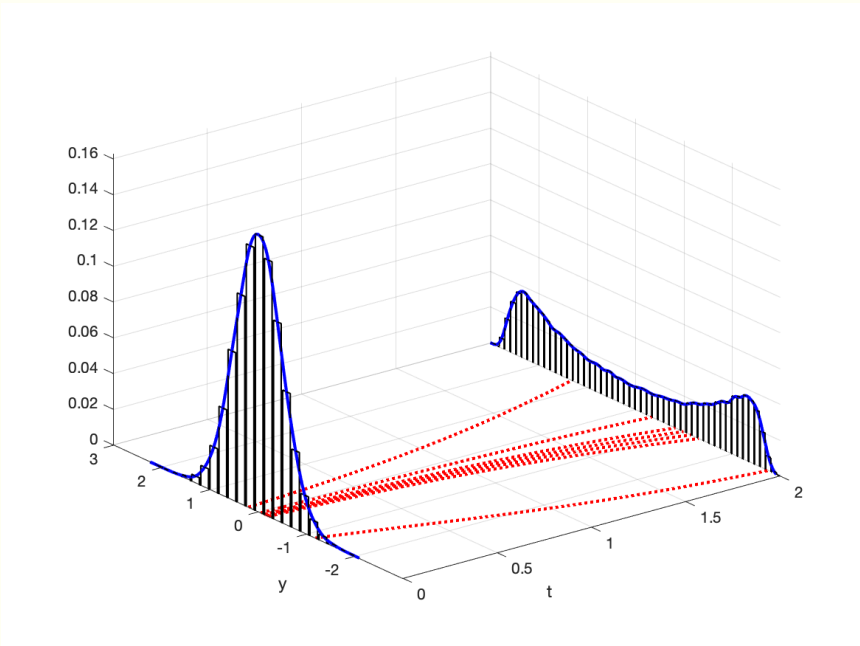
with drift a and initial condition $f(x,0) = f_0(x)$.



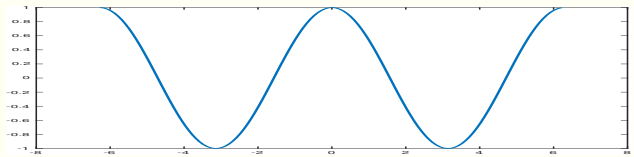
This fundamental result of statistical mechanics leads to the formulation of various kinetic equations.

Role of the drift

The Liouville equation $\partial_t f + \operatorname{div}(a(x,t)\, f) = 0$ is the fundamental continuity equation; the first in the hierarchy of kinetic models.



Take the dynamics
$\dot{X}(t) = \sin(X(t))$
$X(0) = X_0 \sim \mathcal{N}(\mu, \bar{\sigma}^2)$,
$\mu = 0$ and $\bar{\sigma} = 0.5$.



Notice that $a(x,t) = \sin(x) = -\frac{d}{dx}\cos(x) = -\nabla U(x)$ where $U(x) = \cos(x)$. The function U can be interpreted as a potential.

Compare with moments' equations in the case $\dot{X}(t) = [A(t)\, X(t) + b(t)]$:

$$\dot{\mu}(t) = A(t)\, \mu(t) + b(t), \qquad \dot{\Sigma}(t) = \Sigma(t)\, A(t)^\top + A(t)\, \Sigma(t).$$

A control mechanism

Optimal control applications require to identify a control mechanism in the model. A convenient structure with time-dependent controls:

$$a(x,t;u_1,u_2) = a_0(x,t) + a_1\,u_1(t) + a_2\,u_2(t)\,x.$$

$a_0(x,t) \in \mathbb{R}^d$ smooth vector field, $a_1, a_2 \in \mathbb{R}$, $u_1(t), u_2(t) \in \mathbb{R}^d$

Moment equations: Define $m(t)$ as the mean, $v(t)$ as the variance. Choose $a(x,t;u_1,u_2) = u_1(t) + u_2(t)\,x$, and f_0 as normal Gaussian distribution.
From the Liouville equation, we obtain

$$\begin{aligned} \dot{m}(t) &= u_1(t) + m(t)\,u_2(t), & m(0) &= m_0 \\ \dot{v}(t) &= 2\,v(t)\,u_2(t) & v(0) &= v_0. \end{aligned}$$

where
$m(t) = \int x\,f(x,t)\,dx$ and $v(t) = \int (x - m(t))^2\,f(x,t)\,dx$

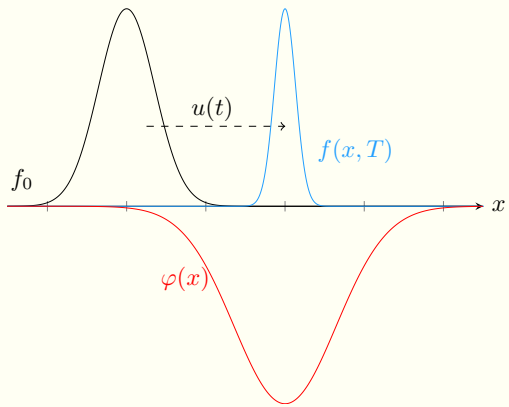
Ensemble cost functional

The particles should follow closely a desired trajectory $x_D(t), t \in [0, T]$, reach a target position x_T at $t = T$.

Ensemble control approach: define "attracting" potentials

$$\theta(x,t) = \Theta(|x - x_D(t)|)$$

$$\varphi(x) = \Phi(|x - x_T|)$$



Ensemble cost functional:

$$J(f,u) = \int_0^T \int_{\mathbb{R}^d} \theta(x,t)\, f(x,t)\, dx\, dt + \int_{\mathbb{R}^d} \varphi(x)\, f(x,T)\, dx + \kappa(u).$$

Cost of control

For space-dependent controls in Ω:

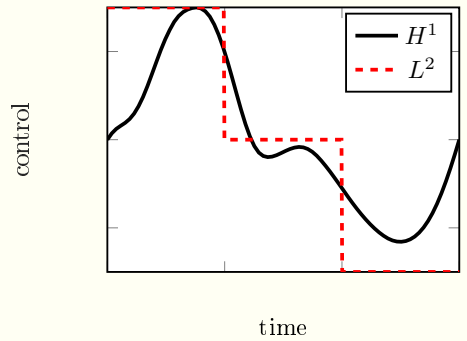
$L^2(\Omega), H^1(\Omega)$

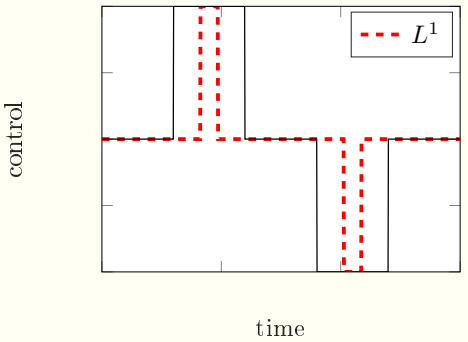
For time-dependent controls in $(0, T)$:

$L^2(0, T)$: standard control cost.

$H^1(0, T)$: includes time-derivative of the control (minimum attention control); turning control on at initial time and off at terminal time.

$L^1(0, T)$: sparse controls (minimum action control)





Cost function:

$$\kappa(u) = \frac{\gamma}{2} \int_0^T \left|u(t)\right|^2 dt + \delta \int_0^T \left|u(t)\right| dt + \frac{\nu}{2} \int_0^T \left|\frac{d}{dt} u(t)\right|^2 dt$$

An ensemble control problem

A Liouville ensemble optimal control problem:

$$\min_{u\in U_{ad}} J(f,u) := \int_0^T \int_{\mathbb{R}^d} \theta(x,t)\, f(x,t)\, dx\, dt + \int_{\mathbb{R}^d} \varphi(x)\, f(x,T)\, dx + \frac{\gamma}{2}\int_0^T \left|u(t)\right|^2 dt + \delta \int_0^T \left|u(t)\right| dt + \frac{\nu}{2}\int_0^T \left|\frac{d}{dt}u(t)\right|^2 dt$$

$$\text{subject to} \quad \begin{cases} \partial_t f(x,t) + \operatorname{div}\big(a(x,t;u)\, f(x,t)\big) = 0 & \text{in } \mathbb{R}^d \times [0,T] \\ f(x,0) = f_0(x) & \text{in } \mathbb{R}^d \end{cases}$$

with the set of admissible controls

$$U_{ad} := \{u = (u_1,u_2) \in U \times U \,|\, u_a \le u(t) \le u_b,\ t \in [0,T]\},$$

$U = H^1([0,T];\mathbb{R}^d)$ or $U = L^2([0,T];\mathbb{R}^d)$.

$\gamma, \delta, \nu \ge 0, \gamma + \delta + \nu > 0, u_a < 0, u_b > 0$

Results with Liouville model

1. For $u \in U_{ad}$ there exists a unique solution $f \in C\big([0,T]; H_k^m(\mathbb{R}^d)\big)$ of the Liouville initial-value problem.
2. The control-to-state map G, $u \mapsto f = G(u)$ is Fréchet differentiable
3. The ensemble optimal control problem admits at least one solution in U_{ad}.
4. Derivation and analysis of the optimality system
5. Approximation by SSP Runge-Kutta, Kurganov-Tadmor and Strang splitting schemes.
6. L^1 stability, second-order accuracy, positivity preserving.
7. Projected semi-smooth Newton method.

A kinetic model with collision

In many physical systems, the density f is defined in the phase space spanned by position $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$, $Q = \Omega \times \mathbb{R}^d$.

In this statistical framework, we assume that the time evolution of f is governed by the following kinetic model

$$\begin{aligned} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) + \nabla_v \cdot [u(x,v,t)\, f] &= C[f](v,t) \\ f|_{t=0} &= f_0, \\ f(x,v,t)|_{\partial\Omega \times \mathbb{R}^d_< \times (0,T]} &= f(x, v - 2n(n \cdot v), t) \end{aligned}$$

where $C[f]$ represents collisions of the particles with an homogeneous background system (as in Brownian motion: colloidals suspended in a bath in thermal equilibrium).
We also have specular reflection space boundaries;
$\mathbb{R}^d_< := \{v \in \mathbb{R}^d \mid v \cdot n(x) < 0\}$, and a control field u.

Keilson-Storer collision model

We consider the Keilson-Storer (KS) collision model:

$$C[f](v,t) := \int f(w,t)\, A(w,v)\, dw - f(v,t) \int A(v,w)\, dw,$$

It has a gain – loss structure. We have $A(v,w) := A_0\, e^{(-\beta|w-\gamma v|^2)}$ and $\gamma \in [-1,1], A_0, \beta > 0$. For post-collision velocity holds $w \sim \mathcal{N}(\gamma v, (2\beta)^{-1})$.

- $\gamma \lessapprox 1$: weak collisions, Brownian motion
- $\gamma \approx 0$: strong collision, Bhatnagar-Gross-Krook (BGK) operator
- collision frequency $\frac{1}{\tau} = A_0 \sqrt{\pi/\beta}$
- detailed balance: $A(w,v)\, f^{eq}(w) = A(v,w)\, f^{eq}(v)$
- equilibrium solution $f^{eq}(v)$ is the Maxwellian distribution
- A_0 and β related to the background density and temperature

A LKS optimal control problem

We consider the following Liouville-Keilson-Storer (LKS) problem

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + u(x) \cdot \nabla_v f &= C[f] \\ f|_{t=0} &= f_0, \\ f(x,v,t)|_{\partial\Omega \times \mathbb{R}^d_< \times (0,T]} &= f(x, v - 2n(n \cdot v), t) \end{aligned}$$

with specular reflection space boundaries; $\mathbb{R}^d_< := \{v \in \mathbb{R}^d \mid v \cdot n(x) < 0\}$. The control field u is sought in $H^1_0(\Omega)$.

Suppose a desired trajectory in the phase space $z_D(t)$, $t \in [0,T]$, and desired final configuration z_T. We choose the potentials θ and φ.

Our problem is to find $u \in H^1_0(\Omega)$ such that the following ensemble cost functional is minimised

$$J(f,u) := \int_0^T \int_Q \theta(x,v,t) f(x,v,t)\, dx\, dv\, dt + \int_Q \varphi(x,v) f(x,v,T)\, dx\, dv + \frac{\nu}{2} \|u\|^2_{H^1}$$

where $Q = \Omega \times \mathbb{R}^d$.

The LKS adjoint equation

The LKS adjoint equation is given by

$$-\partial_t q(x,v,t) - v \cdot \nabla_x q(x,v,t) - u(x) \cdot \nabla_v q(x,v,t) = \tilde{C}[q](x,v,t) - \theta(x,v,t)$$

with

$$\tilde{C}[q](x,v,t) = \int A(v,w)\, q(x,w,t)\, dw - q(x,v,t) \int A(v,w)\, dw.$$

The operator $\tilde{C}[q]$ has not a gain-loss structure, but such a structure can be partially recovered defining

- $C^*[q](x,v,t) = \int A^*(w,v)\, q(x,w,t)\, dw - q(x,v,t) \int A^*(v,w)\, dw,$
- $A^*(w,v) = \frac{1}{\gamma} A(v,w)$
- $\int \big(A(w,v) - A(v,w)\big)\, dw = \frac{1-\gamma}{\tau_q} =: C_0^*.$
- 'adjoint' mean free time $\tau_q = \gamma\, \tau$

Reformulation of the LKS adjoint

We choose θ and φ as follows

$$\theta(z,t) := -\frac{C_\theta}{2\pi\sigma_\theta^2}\exp\left(-\frac{|z-z_D(t)|^2}{2\sigma_\theta^2}\right), \qquad \sigma_\theta > 0.$$

and

$$\varphi(z) := -\frac{C_\varphi}{2\pi\sigma_\varphi^2}\exp\left(-\frac{|z-z_T)|^2}{2\sigma_\varphi^2}\right), \qquad \sigma_\varphi > 0.$$

With this choice, θ and φ play the role of sources (or sinks) of particles.

The resulting adjoint LKS model is given by

$$-\partial_t q - v\cdot\nabla_x q - u(x)\cdot\nabla_v q = C^*[q] + C_0^* q - \theta, \qquad q_{|t=T} = -\varphi.$$

The forward and adjoint problems can be written introducing the free-streaming operators

$$L_u = v\cdot\nabla_x + u\cdot\nabla_v, \qquad \text{and} \qquad L_u^* = -L_u.$$

LKS optimality system

$$\partial_t f(x,v,t) + L_u f(x,v,t) = C[f](x,v,t),$$
$$f(x,v,0) = f_0(x,v)$$
$$f(x,v,t)|_{\partial\Omega\times\mathbb{R}^d_<} = f(x, v - 2n(n\cdot v), t)$$

$$-\partial_t q(x,v,t) + L_u^* q(x,v,t) = C^*[q](x,v,t) + C_0^*\, q(x,v,t) - \theta(x,v,t),$$
$$q(x,v,T) = -\varphi(x,v)$$
$$q(x,v,t)|_{\partial\Omega\times\mathbb{R}^d_>} = q(x, v - 2n(n\cdot v), t)$$

$$-\nu\,\Delta u(x) + \nu\, u(x) + \int_0^T \int_{\mathbb{R}^d} q(x,v,t)\,\nabla_v f(x,v,t)\,dv\,dt = 0$$
$$u_{|\partial\Omega} = 0.$$

The reduced gradient in H^1

The L^2 gradient of the reduced cost functional $J_r(u) := J(f(u), u)$ is given by

$$\nabla J_r(u)\big|_{L^2}(x) = -\nu\,\Delta u(x) + \nu\,u(x) + \int_0^T \int_{\mathbb{R}^d} q(x, v, t)\,\nabla_v f(x, v, t)\,dv\,dt.$$

However, the update for the control needs the H^1 reduced gradient. Considering the Riesz representative of $J_r'(u)$ on different Hilbert spaces:

$$\left(\nabla J_r(u)\big|_{L^2}, \delta u\right)_{L^2} = \left(\nabla J_r(u)\big|_{H^1}, \delta u\right)_{H^1}, \qquad \delta u \in H^1.$$

Thus, the H^1 gradient is obtained as the solution to the following boundary-value problem

$$-\Delta\psi + \psi = \nabla J_r(u)\big|_{L^2}, \qquad \psi\big|_{\partial\Omega} = 0.$$

That is, $\nabla J_r(u)\big|_{H^1} = \psi$.

Numerical optimization

Calculate H^1 gradient

Require: control u, f_0, θ, φ.

Solve forward LKS equation with inputs: f_0, u

Solve backward the adjoint LKS equation with inputs: θ, φ, u

Assemble the L^2 gradient $\nabla J_r(u)|_{L^2}$.

Compute the H^1 gradient $\nabla J_r(u)|_{H^1}$.

return $\nabla J(u)|_{H^1}(x)$

Nonlinear conjugate gradient (NCG) method

Require: $u^0, f_0, \theta, \varphi$.

$n = 0$

Compute gradient $g^0 = \nabla J_r(u^0)|_{H^1}$; set $d^0 = -g^0$.

while $\|g^n\|_{H^1} > tol$ **and** $n < n_{\max}$ **do**

Use linesearch to determine α_n

Update control: $u^{n+1} = u^n + \alpha_n d^n$

Compute gradient $g^{n+1} = \nabla J_r(u^{n+1})|_{H^1}$

Calculate the new descent direction $d^{n+1} = -g^{n+1} + \beta_n d^n$

Set $n = n + 1$

end while

return $u^n(x)$

Simulation scales & Numerics

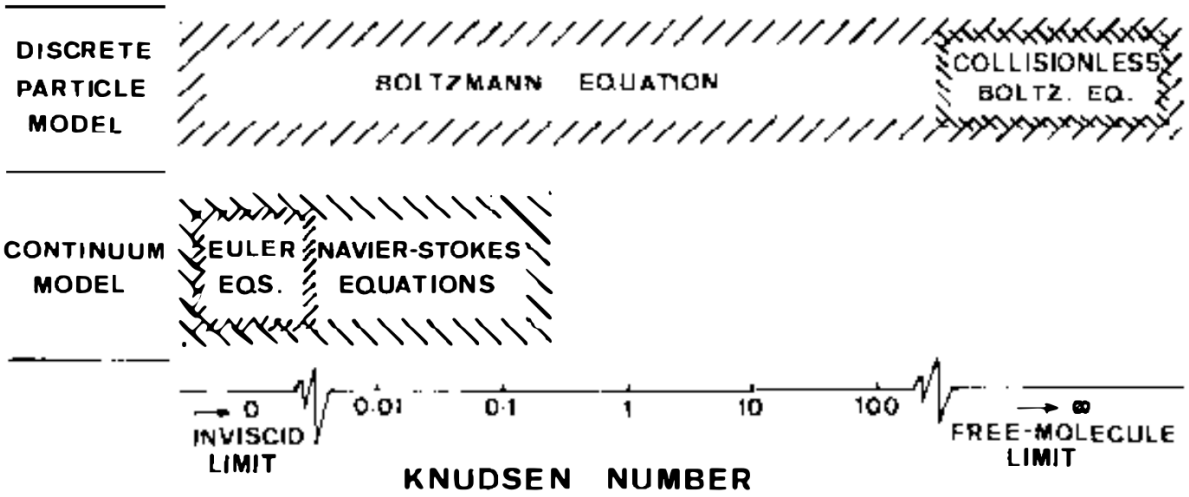


Figure 1 The Knudsen-number limits on the conventional mathematical models of neutral gas flows.

In the long term, we are concerned with methods for calibration, control, and optimization of kinetic models in the mesoscopic regime where probabilistic aspects of the evolution of particles play an essential role.

This is the case in the simulation of rarefied gases with high Knudsen number (the ratio of the mean-free path to the characteristic length of the problem).

The mesoscopic setting accommodates the case where the coefficients of the model are prescribed probabilistically by some distribution functions.

Although kinetic models are partial-integro differential equations, methods developed in a deterministic context cannot always be applied and computation by Monte Carlo methods could be required.

Monte Carlo method

Split the kinetic operator: free flight & collision.

The deterministic free flight of each particle (Newton's law of motion) between two collisions: $\dot{x} = v$ and $\dot{v} = u(x)$. It is usually computed with the Störmer–Verlet method.

The probabilistic collision is estimated according to the collision frequency $1/\tau$. The free streaming time is given by

$$\delta t = -\tau \log(r),$$

where $r \in [0, 1]$ is a uniform random number.

Specular reflection boundary conditions are taken into account in this step.

Particles and their adjoint counterpart are managed in a list of pointers F^k, Q^k storing position and velocity at time step k.

MC collision

At $t + \delta t$, a collision changes the velocity v of the particle while not changing position x. Notice that we refer to collision of the particle with much smaller particles in a bath.

The KS collision kernel can be written as a normal distribution:

$$A(v, w) = \mathcal{N}\left(\gamma v, \frac{1}{2\beta}\right),$$

Thus, in our case the new velocity is given by

$$w = \gamma v + \frac{p}{2\beta}.$$

where $p \sim \mathcal{N}\left(0, (2\beta)^{-1}\right)$ is a normal random number, sampled using the Box-Muller formula.

LKS adjoint and Monte Carlo

$$\partial_s q - v \cdot \nabla_x q - u(x) \cdot \nabla_v q = C^*[q] + C_0^* q - \theta, \qquad q_{|s=0} = -\varphi.$$

The LKS adjoint model consists out of a free-streaming part, a collision part, a reaction term and a source term. We have collision frequency $(\gamma\tau)^{-1}$ and post-collision velocities $w^* \sim \mathcal{N}\left(v/\gamma, (2\beta\gamma^2)^{-1}\right)$.

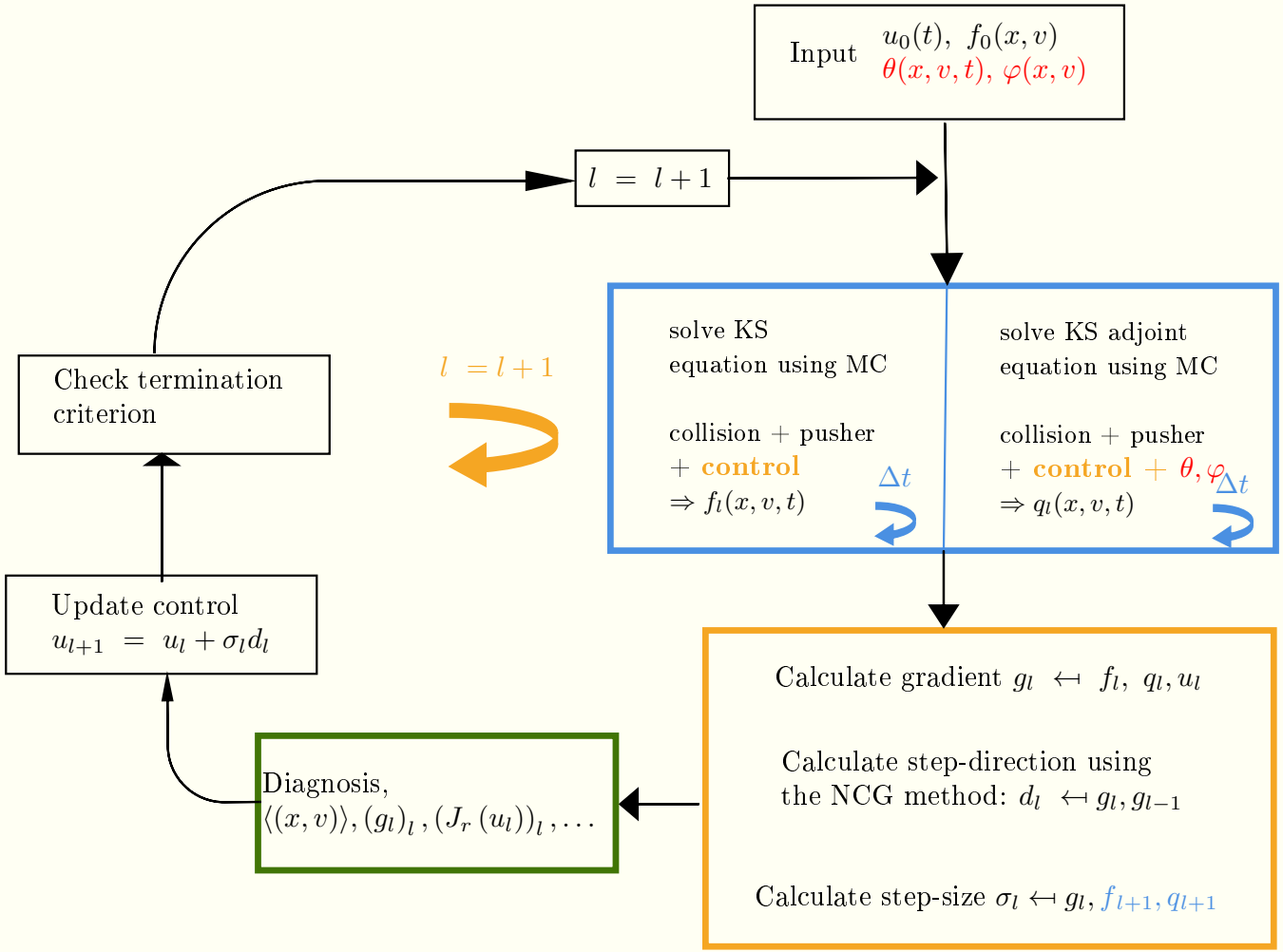
For reaction term $C_0^* q$:

- For all particles p in Q^k: Generate $r_* := \lfloor \Delta t\, C_0^* \rfloor$ particles with the velocity $Q^k[p].v$ and position $Q^k[p].x$.
- Add these particles to the existing ones in Q^k.

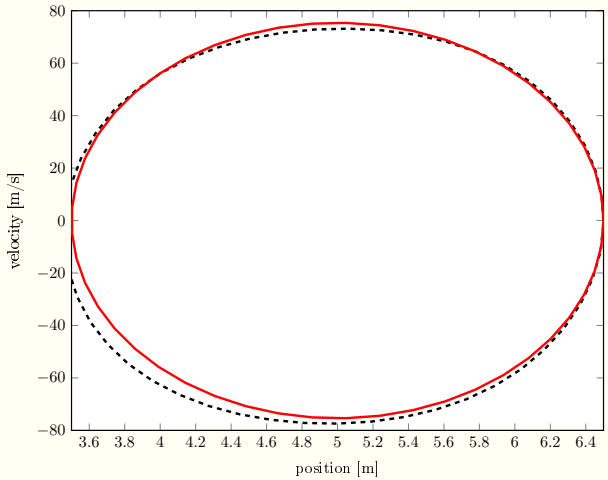
For the source term $-\theta$:

- Generate N_{frac} new particles with phase space components having the normal distribution with mean $z_D(t^k)$ and variance σ_θ^2: $v \sim \mathcal{N}\left(z_D(t^k), \sigma_\theta^2\right)$.
- Add these particles to the existing ones in Q^k

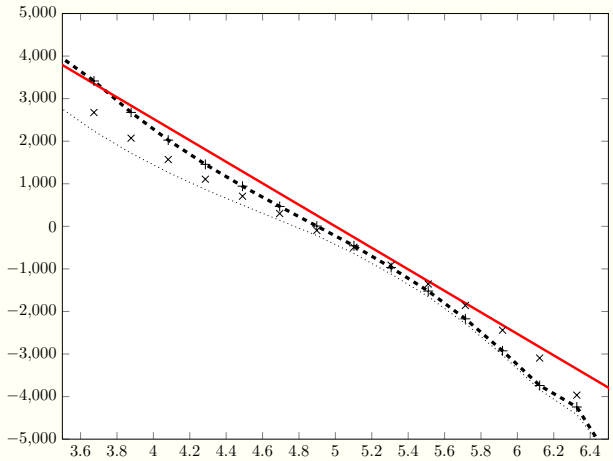
Monte Carlo algorithm



Num. exp. - Harmonic oscillator



Plot of mean values $(\langle x \rangle, \langle v \rangle)$ (- - -) and desired z_D (——).



Optimal control (- - -) and elastic force $F(x)$ of the harmonic oscillator(——). Comparison of results with number of particles N_f and with two times N_f (· · · · ·), four times N_f (× × ×), eight times N_f (+ + +).

$$f_0(x,v) = \frac{1}{2\pi \cdot 0.15 \cdot 5.0} \exp\left(-\frac{1}{2}\left[\left(\frac{x-5.0}{0.15}\right)^2 + \left(\frac{v-0.0}{5.0}\right)^2\right]\right),$$
$$z_D(t) = (1.5\cos(\omega t) + 5.0, -1.5\sin(\omega t))^T, F(x) = -\omega^2(x-5)$$

A stabilization problem

Now, we consider the more general model:

$$\begin{aligned} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) + \nabla_v [F(x,v,t;u(x,v,t))\, f] &= C[f](x,v,t) \\ f(x,v,0) &= f_0(x,v), \\ f(x,v,t)|_{\mathcal{Q}^-} &= \alpha\, f(x, v - 2n(n\cdot v), t), \end{aligned}$$

where $\mathcal{Q}^- := \partial\Omega \times \mathbb{R}^d_< \times (0,T]$, and $\alpha \in [0,1]$.

Our purpose is to design a control field capable of driving an initial density of particles randomly distributed in the phase space to reach a desired cyclic trajectory on the phase space and follow it in a stable way.

The desired trajectory $z_D(t) = (x_D(t), v_D(t))$ is the solution to

$$X'(t) = V(t), \qquad V'(t) = F_0(X(t), V(t)),$$

where the dynamics F_0 and the initial condition $X(0) = X_0$ and $V(0) = V_0$, $(X_0, V_0) \in \Omega \times \mathbb{R}^d$ are chosen such that the resulting periodic trajectory satisfies $(X(t), V(t)) \in \Omega \times \mathbb{R}^d$, $t \in [0,T]$.

A Markov control field

We choose F_0 corresponding to Hooke's law, which does not result in a stable limit cycle dynamics: starting with a distributed f_0 will result in particles following different trajectories.

In order to drive and maintain the particles, subject to collisions, close to the desired trajectory, we augment F_0 with a control field as follows:

$$F(x, v, t; u(x, v, t)) = F_0(x, v) + u(x, v, t).$$

Our ensemble cost functional is given by

$$J(f, u) = \int_0^T \int_{\Omega \times \mathbb{R}^d} \left[\theta(x, v, t) + \frac{\nu}{2}|u(x, v, t)|^2\right] f(x, v, t)\, dx\, dv\, dt$$
$$+ \int_{\Omega \times \mathbb{R}^d} \varphi(x, v)\, f(x, v, T)\, dx\, dv.$$

Adjoint problem

The adjoint kinetic model is given by

$$\begin{aligned} -\partial_t q(x,v,t) + L_u^* \, q(x,v,t) &= C^*[q](x,v,t) + C_0^* \, q(x,v,t) \\ &\quad - \theta(x,v,t) - \frac{\nu}{2}|u(x,v,t)|^2, \\ q(x,v,T) &= -\varphi(x,v), \\ q(x,v,t)|_{\Sigma^+} &= \alpha \, q(x, v - 2\, n(x)\,(n(x)\cdot v), t) \end{aligned}$$

where $\Sigma^+ := \partial\Omega \times \mathbb{R}^d_> \times (0,T]$, and the adjoint free-streaming operator L_u^* is given by

$$L_u^* := -v \cdot \nabla_x - F(x,v,t;u) \cdot \nabla_v.$$

Further, we have C_0^* and $C^*[q](x,v,t)$, θ and φ as previously defined.

Optimality condition

The optimality system is completed with specification of the optimality condition equation:

$$\nabla_u J_r(u) := f(x,v,t)\Big(\nu\, u(x,v,t) - \partial_u F(x,v,t;u)\nabla_v q(x,v,t)\Big) = 0.$$

However, recall our aim to construct a control field on the entire phase space. Such a control would be required if the density f is everywhere positive, in which case a necessary and sufficient condition for optimality is given by

$$u(x,v,t) = \frac{1}{\nu}\nabla_v q(x,v,t),$$

since in our setting $\partial_u F(x,v,t;u)$ is the identity matrix.

A similar result would be obtained in the case of constraints on the control and based on the PMP framework.

Feedback control

The $u = q/\nu$ replaced in the adjoint kinetic model gives a new equation for the adjoint variable that does not depend on the density f nor on f_0. It depends only on θ and φ.

The adjoint kinetic model becomes

$$\begin{aligned} -\partial_t q(x,v,t) + L_u^* \, q(x,v,t) &= C^*[q](x,v,t) + C_0^* \, q(x,v,t) \\ &\quad - \theta(x,v,t) - \frac{1}{2\nu} |\nabla_v q(x,v,t)|^2, \\ q(x,v,T) &= -\varphi(x,v), \\ q(x,v,t)|_{\Sigma^+} &= \alpha \, q(x, v - 2\, n(x)\,(n(x)\cdot v), t) \end{aligned}$$

Once q is computed, we obtain a control $u = q/\nu$ having all characterizing features of a feedback control. We solve this model with our Monte Carlo strategy.

Our modified adjoint model has similarities with the Hamilton-Jacobi-Bellman equation arising in the dynamic programming approach.

A stationary control field

A periodic orbit admits different time parametrizations. For this reason, we construct a time-independent θ corresponding to the entire trajectory as follows:

$$\bar{\theta}(x, v) = \frac{1}{T} \int_0^T \theta(x, v, t)\, dt.$$

Hence, $\bar{\theta}$ represents a closed valley with the bottom line corresponding to the desired orbit.
We construct a stationary feedback law as follows

$$\bar{u}(x, v) = \frac{1}{T} \int_0^T u(x, v, t)\, dt.$$

This approach is motivated by works on optimal control of periodic processes in the field of engineering of chemical plants with cyclic regimes.

Experimental setting

We consider a two dimensional phase-space domain $\Omega = [0, p_{\max}] \times [-v_{\max}, v_{\max}]$ with positive $p_{\max}, v_{\max}$. We set f_0 equal to a uniform distribution.

The desired orbit corresponds to a harmonic oscillator of unit mass and force corresponding to Hooke's law as follows

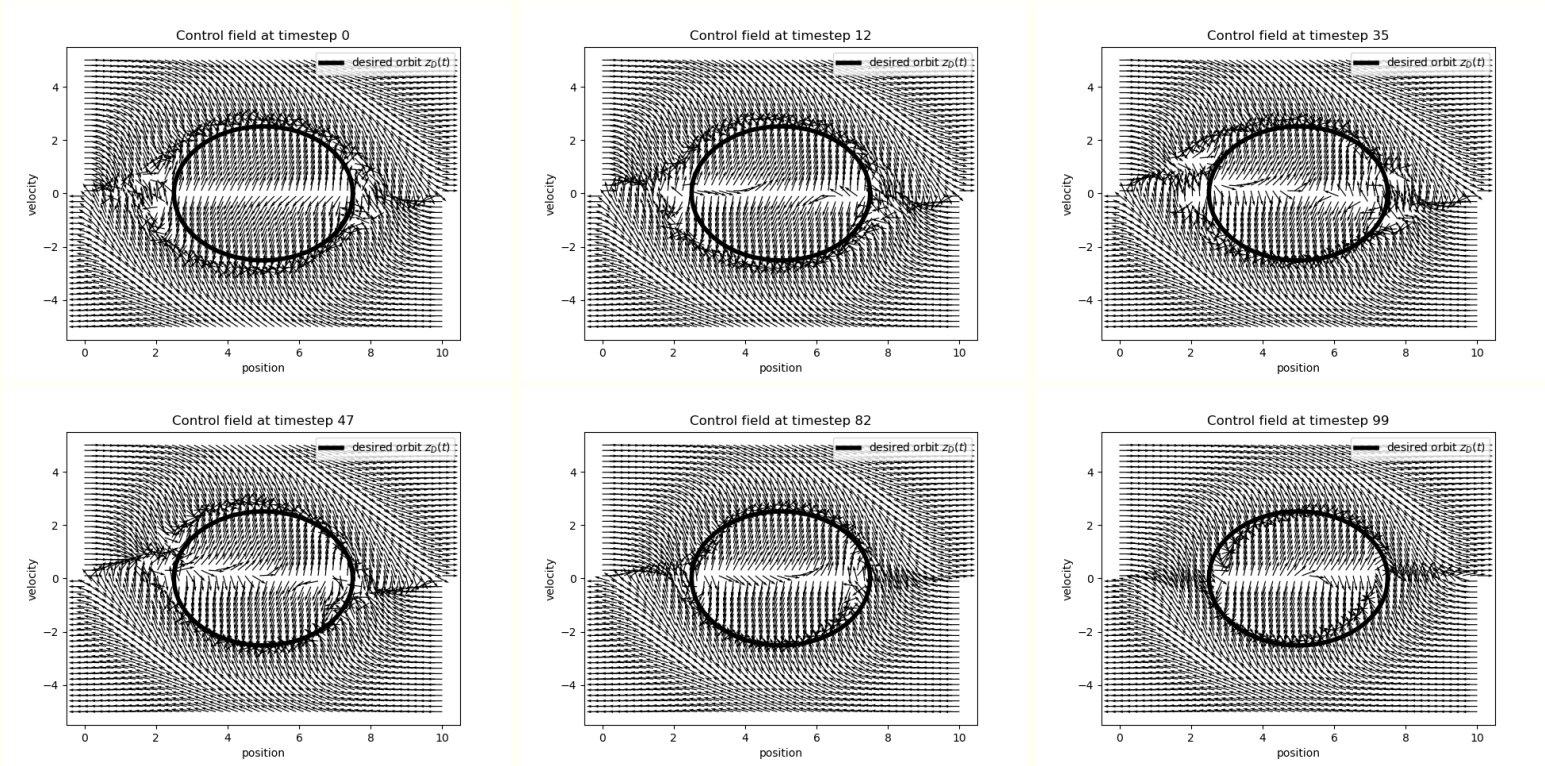
$$F_0(x, v) = -\omega^2 \left(x - \frac{p_{\max}}{2} \right).$$

The resulting trajectory is given by

$$z_D(t) = \begin{pmatrix} x_D(t) \\ v_D(t) \end{pmatrix} = \begin{pmatrix} 2.5\cos(\omega t) + x_0 \\ -2.5\omega \sin(\omega t) - v_0 \end{pmatrix}, \qquad \omega = \frac{2\pi}{T},$$

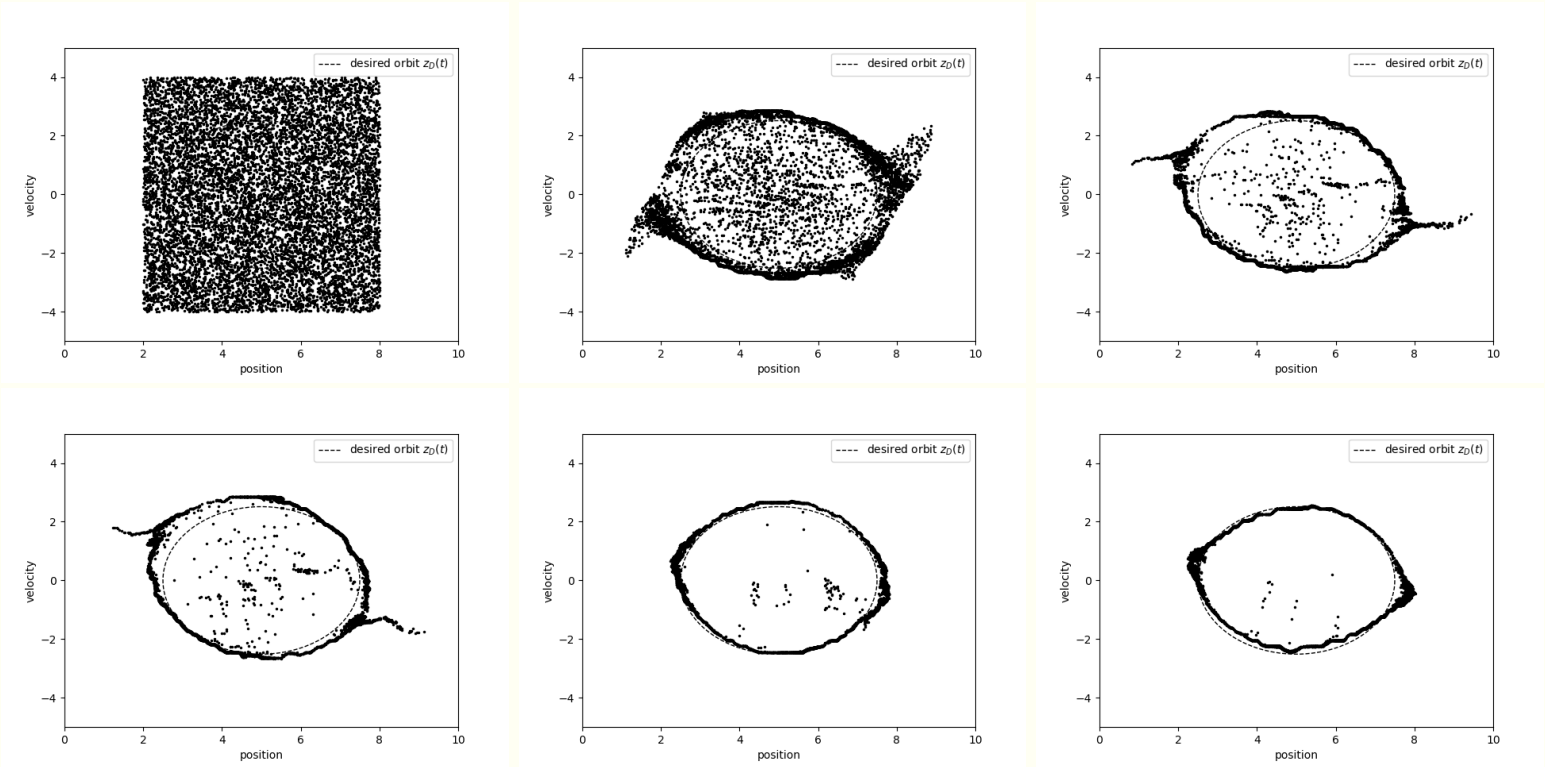
where T is the period of the orbit, and $x_0 = p_{\max}/2$, $v_0 = 0$.
We have $p_{\max} = 10$, $v_{\max} = 5$, $T = 2.5$, $\gamma = 0.9999$ and $\nu = 10$.

A Markov control function



Quiver plot of the calculated control. The solid ellipse is the curve $z_D(t)$, $t \in [0, T]$. The arrows are given by the scaled vector $(v, u(x, v, t))^T$.

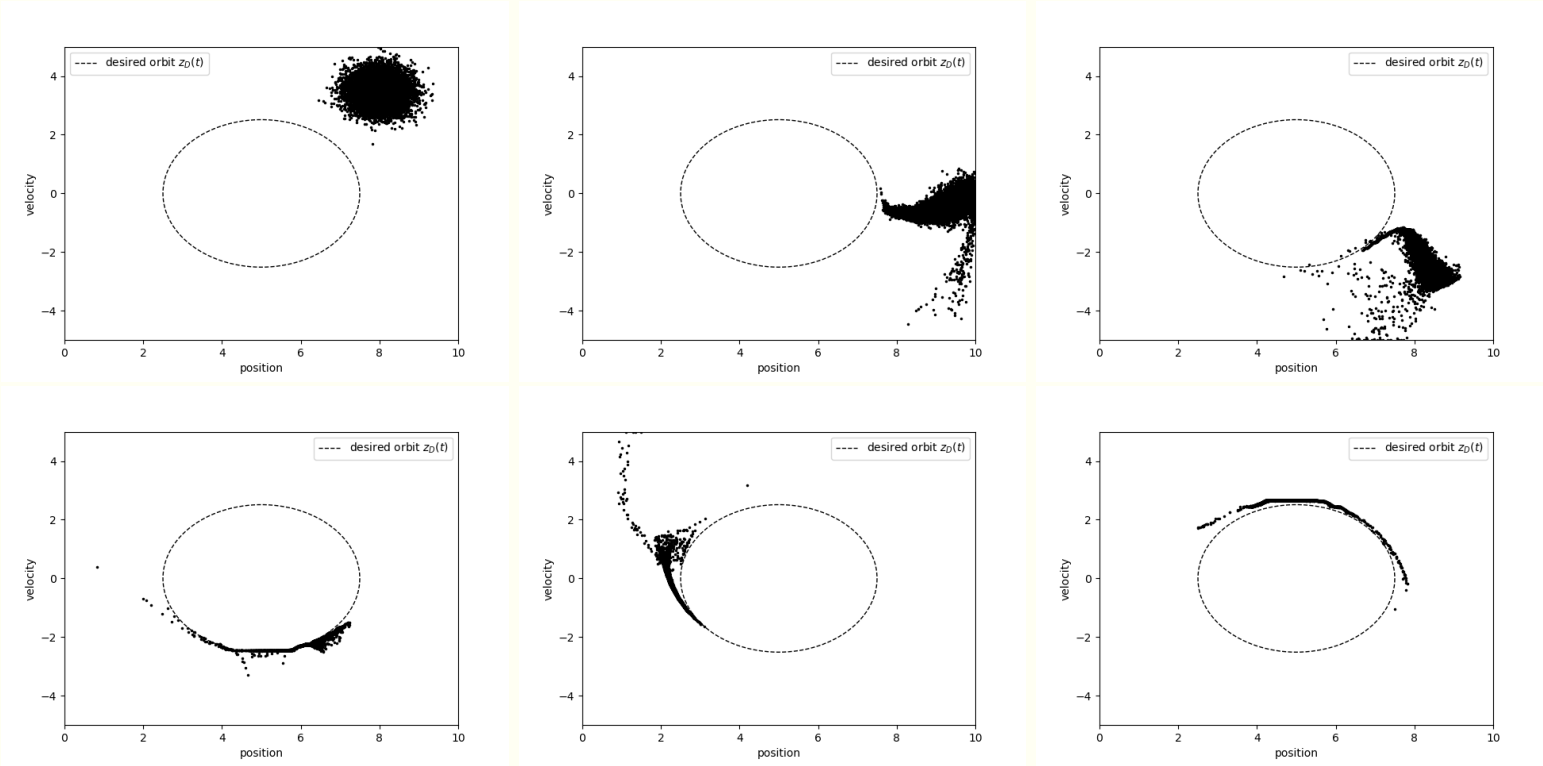
Particles' evolution



Evolution of f starting from an uniform initial distribution and subject to the control field u.

Stationary control and feedback

Starting from a Gaussian initial distribution, we simulate the evolution of these particles subject to the time-averaged control $\bar{u}$.



Evolution of f, starting with an initial Gaussian distribution and subject to the averaged control $\bar{u}$.

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Thank you for inviting me to give this talk.

Please visit my homepage for updates:
`https://www.mathematik.uni-wuerzburg.de/en/scientificcomputing/team/borzi-alfio/`

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