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Institut für Numerische und Angewandte Mathematik

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regularization of Inverse Problems with Poisson data**

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Nr. 8

Preprint-Serie des
Instituts für Numerische und Angewandte Mathematik
Lotzestr. 16-18
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Convergence rates in expectation for Tikhonov-type regularization of Inverse Problems with Poisson data

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April 10, 2012

In this paper we study a Tikhonov-type method for ill-posed nonlinear operator equations $g^\dagger = F(u^\dagger)$ where g^\dagger is an integrable, non-negative function. We assume that data are drawn from a Poisson process with density tg^\dagger where $t > 0$ may be interpreted as an exposure time. Such problems occur in many photonic imaging applications including positron emission tomography, confocal fluorescence microscopy, astronomic observations, and phase retrieval problems in optics. Our approach uses a Kullback-Leibler-type data fidelity functional and allows for general convex penalty terms. We prove convergence rates of the expectation of the reconstruction error under a variational source condition as $t \rightarrow \infty$ both for an a priori and for a Lepskiĭ-type parameter choice rule.

1. Introduction

We consider inverse problems where the ideal data can be interpreted as a photon density $g^\dagger \in \mathbf{L}^1(\mathbb{M})$ on some manifold \mathbb{M} . The unknown will be described by an element u^\dagger of a subset \mathfrak{B} of a Banach space \mathcal{X} , and u^\dagger and g^\dagger are related by a forward operator F mapping from \mathfrak{B} to $\mathbf{L}^1(\mathbb{M})$:

$$F(u^\dagger) = g^\dagger. \quad (1)$$

The data will be drawn from a Poisson process with density tg^\dagger where $t > 0$ can often be interpreted as an exposure time. Such data can be seen as a random collection of points on the manifold \mathbb{M} on which measurements are taken (see section 2 for a precise definition of Poisson processes). Hence unlike the common deterministic setup the data do not belong to the same space as the ideal data g^\dagger .

Such inverse problems occur naturally in photonic imaging since photon count data are Poisson distributed for fundamental physical reasons. Examples include inverse problems in astronomy [4], fluorescence microscopy, in particular 4Pi microscopy [27], coherent X-ray imaging [17], and positron emission tomography [8].

In this paper we study a penalized likelihood or Tikhonov-type estimator

$$u_\alpha \in \underset{u \in \mathfrak{B}}{\operatorname{argmin}} [\mathcal{S}(G_t; F(u)) + \alpha \mathcal{R}(u)] . \quad (2)$$

Here G_t describes the observed data, \mathcal{S} is a Kullback-Leibler type data misfit functional derived in section 2, $\alpha > 0$ is a regularization parameter, and $\mathcal{R} : \mathcal{X} \rightarrow (-\infty, \infty]$ is a convex penalty term, which may incorporate *a priori* knowledge about the unknown solution u^\dagger . If $\mathcal{S}(g_1; g_2) = \|g_1 - g_2\|_{\mathcal{Y}}^2$ and $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$ with Hilbert space norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, then (2) is standard Tikhonov regularization. In many cases estimators of the form (2) can be interpreted as maximum a posteriori (MAP) estimators in a Bayesian framework, but our convergence analysis will follow a frequent paradigm, and in particular u^\dagger will be considered as a deterministic quantity. Our data misfit functional \mathcal{S} will be convex in its second argument, so the minimization problem (2) will be convex if F is linear.

Recently, considerable progress has been achieved in the deterministic analysis of variational regularization methods in Banach spaces [6, 7, 10, 13–15, 26]. In particular, a number of papers have been devoted to the Kullback-Leibler divergence as data fidelity term in (2), motivated by the case of Poisson data (see [2, 3, 10, 11, 23, 24]), but all of them under deterministic error assumptions. On the statistical side, inverse problem for Poisson data have been studied by Antoniadis & Bigot [1] by wavelet Galerkin methods. Their study is restricted to linear operators with favorable mapping properties in certain function spaces. Therefore, there is a need for a statistical convergence analysis for inverse problems with Poisson data involving more general and in particular nonlinear forward operators. This is the aim of the present paper.

Our convergence analysis of the estimator (2) is based on two basic ingredients: The first is a uniform concentration inequality for Poisson data (Theorem 2.1), which will be formulated together with some basic properties of Poisson processes in Section 2. The proof of Theorem 2.1, which is based on results by Reynaud-Bouret [25] is given in an appendix. The second ingredient, presented in Section 3, is a deterministic error analysis of (2) for general \mathcal{S} under a variational source condition (Theorem 3.3). Our main results are two estimates of the expected reconstruction error as the exposure time t tends to ∞ : For an a-priori choice of α , which requires knowledge of the smoothness of u^\dagger , such a result is shown in Theorem 4.3. Finally, a convergence rate result for a completely adaptive method, where α is chosen by a Lepskiĭ-type balancing principle, is presented in Theorem 5.1.

2. Results on Poisson processes

Let $\mathbb{M} \subset \mathbb{R}^d$ be a submanifold where measurements are taken, and let $\{x_1, \dots, x_N\} \subset \mathbb{M}$ denote the positions of the detected photons. Both the total number N of observed photons and the positions $x_i \in \mathbb{M}$ of the photons are random, and it is physically evident that the following two properties hold true:

1. For all measurable subsets $\mathbb{M}' \subset \mathbb{M}$ the integer valued random variable $G(\mathbb{M}') := \#\{i \mid x_i \in \mathbb{M}'\}$ has expectation $\mathbf{E}[G(\mathbb{M}')] = \int_{\mathbb{M}'} g^\dagger dx$ where $g^\dagger \in L^1(\mathbb{M})$ denotes the underlying photon density.
2. For any choice of m disjoint measurable subsets $\mathbb{M}'_1, \dots, \mathbb{M}'_m \subset \mathbb{M}$ the random variables $G(\mathbb{M}'_1), \dots, G(\mathbb{M}'_m)$ are stochastically independent.

By definition, this means that $G := \sum_{i=1}^N \delta_{x_i}$ is a Poisson process with intensity g^\dagger . It follows from these properties that $G(\mathbb{M}')$ for any measurable $\mathbb{M}' \subset \mathbb{M}$ is Poisson distributed with mean $\lambda := \mathbf{E}[G(\mathbb{M}')] = \int_{\mathbb{M}'} g^\dagger dx$, i.e. $\mathbf{P}[G(\mathbb{M}') = n] = \exp(-\lambda) \frac{\lambda^n}{n!}$ for all $n \in \{0, 1, \dots\}$ (see e.g. [18, Thm 1.11.8]). Moreover, for any measurable $\psi : \Omega \rightarrow \mathbb{R}$ we have

$$\mathbf{E} \left[\int_{\mathbb{M}} \psi dG \right] = \int_{\mathbb{M}} \psi g^\dagger dx, \quad \mathbf{Var} \left[\int_{\mathbb{M}} \psi dG \right] = \int_{\mathbb{M}} \psi^2 g^\dagger dx \quad (3)$$

whenever the right hand sides are well defined (see [19]).

Let us introduce for each exposure time $t > 0$ a Poisson process \tilde{G}_t with intensity tg^\dagger and define $G_t := \frac{1}{t}\tilde{G}_t$. We will study error estimates for approximate solutions to the inverse problem (1) with data G_t in the limit $t \rightarrow \infty$. For this end it will be necessary to derive estimates on the distribution of the log-likelihood functional

$$\mathcal{S}(G_t; g) := \int_{\mathbb{M}} g \, dx - \int_{\mathbb{M}} \ln g \, dG_t = \int_{\mathbb{M}} g \, dx - \frac{1}{t} \sum_{i=1}^N \ln g(x_i), \quad (4)$$

which is defined for functions g fulfilling $g \geq 0$ a.e. We set $\ln 0 := -\infty$, so $\mathcal{S}(G_t; g) = \infty$ if $g(x_i) = 0$ for some $i = 1, \dots, N$. Using (3) we obtain

$$\mathbf{E}[\mathcal{S}(G_t; g)] = \int_{\mathbb{M}} [g - g^\dagger \ln(g)] \, dx \quad \text{and} \quad \mathbf{Var}[\mathcal{S}(G_t; g)] = \frac{1}{t} \int_{\mathbb{M}} \ln(g)^2 g^\dagger \, dx \quad (5)$$

if the integrals exist. Moreover, we have

$$\mathbf{E}[\mathcal{S}(G_t; g)] - \mathbf{E}[\mathcal{S}(G_t; g^\dagger)] = \int_{\mathbb{M}} \left[g - g^\dagger - g^\dagger \ln\left(\frac{g}{g^\dagger}\right) \right] \, dx,$$

and the right hand side (if well-defined) is known as *Kullback-Leibler divergence*

$$\mathbb{KL}(g^\dagger; g) := \int_{\{g^\dagger > 0\}} \left[g - g^\dagger - g^\dagger \ln\frac{g}{g^\dagger} \right] \, dx. \quad (6)$$

$\mathbb{KL}(g^\dagger; g)$ can be seen as the ideal data misfit functional if the exact data g^\dagger were known. Since only G_t is given, we approximate $\mathbb{KL}(g^\dagger; g)$ by $\mathcal{S}(G_t; g)$ up to the additive constant $\mathbf{E}[\mathcal{S}(G_t; g^\dagger)]$, which is independent of g . The error between the estimated and the ideal data misfit functional is given by

$$|\mathcal{S}(G_t; g) - \mathbf{E}[\mathcal{S}(G_t; g^\dagger)] - \mathbb{KL}(g^\dagger; g)| = \left| \int_{\mathbb{M}} \ln(g) (dG_t - g^\dagger \, dx) \right|. \quad (7)$$

Based on results by Reynaud-Bouret [25], which can be seen as an analogue to Talagrand's concentration inequalities for empirical processes, we will derive the following concentration inequality for such error terms in the appendix:

Theorem 2.1. *Let $\mathbb{M} \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary ∂D , $R \geq 1$ and $s > \frac{d}{2}$. Consider the ball*

$$B_s(R) := \left\{ \mathbf{g} \in H^s(\mathbb{M}) \mid \|\mathbf{g}\|_{H^s(\mathbb{M})} \leq R \right\}.$$

Then there exists a constant $C_{\text{conc}} \geq 1$ depending only on \mathbb{M} , s and $\|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})}$ such that

$$\mathbf{P} \left[\sup_{\mathbf{g} \in B_s(R)} \left| \int_{\mathbb{M}} \mathbf{g} (dG_t - g^\dagger \, dx) \right| \leq \frac{\rho}{\sqrt{t}} \right] \geq 1 - \exp\left(-\frac{\rho}{RC_{\text{conc}}}\right)$$

for all $t \geq 1$ and $\rho \geq RC_{\text{conc}}$.

To apply this concentration inequality to the right hand side of (7), we would need that $\ln(F(u)) \in B_s(R)$ for all $u \in \mathfrak{B}$. However, since $\sup_{\mathbf{g} \in B_s(R)} \|\mathbf{g}\|_\infty < \infty$ by Sobolev's embedding theorem, zeros of $F(u)$ for some $u \in \mathfrak{B}$ would not be admissible, which is a quite restrictive assumption. Therefore, we use a shifted version of the data fidelity term with an offset parameter $\sigma > 0$:

$$\mathcal{S}(G_t; g) := \int_{\mathbb{M}} g \, dx - \int_{\mathbb{M}} \ln(g + \sigma) (dG_t + \sigma \, dx) \quad (8)$$

$$\mathcal{T}(g^\dagger; g) := \mathbb{KL}(g^\dagger + \sigma; g + \sigma) \quad (9)$$

Then the error is given by

$$Z(g) := |\mathcal{S}(G_t; g) - \mathbf{E}[\mathcal{S}(G_t; g^\dagger)] - \mathcal{T}(g^\dagger; g)| = \left| \int_{\mathbb{M}} \ln(g + \sigma) (dG_t - g^\dagger dx) \right|. \quad (10)$$

We will show in Section 4 that Theorem 2.1 can be used to estimate the concentration of $\sup_{u \in \mathfrak{B}} Z(F(u))$ under certain assumptions.

3. A deterministic convergence rate result

In this section we will perform a convergence analysis for the method (2) with general \mathcal{S} under a deterministic noise assumption. Similar results have been obtained by Flemming [10,11], Grasmair [13], and Bot & Hofmann [6] under different assumptions on \mathcal{S} .

As in Section 2 we will consider the "distance" between the estimated and the ideal data misfit functional as noise level:

Assumption 1. *Let $u^\dagger \in \mathfrak{B} \subset \mathcal{X}$ be the exact solution and denote by $g^\dagger := F(u^\dagger) \in \mathcal{Y}$ the exact data. Let \mathcal{Y}^{obs} be a set containing all possible observations and $g^{\text{obs}} \in \mathcal{Y}^{\text{obs}}$ the observed data. Assume that:*

1. *The exact data fidelity functional $\mathcal{T} : F(\mathfrak{B}) \times \mathcal{Y} \rightarrow [0, \infty]$ is non-negative, and $\mathcal{T}(g^\dagger, g^\dagger) = 0$.*
2. *For the approximate data fidelity term $\mathcal{S} : F(\mathfrak{B}) \times \mathcal{Y} \rightarrow [0, \infty]$ there exist constants $\mathbf{err} \geq 0$ and $C_{\text{err}} \geq 1$ such that*

$$\mathcal{S}(g^{\text{obs}}; g) - \mathcal{S}(g^{\text{obs}}; g^\dagger) \geq \frac{1}{C_{\text{err}}} \mathcal{T}(g^\dagger; g) - \mathbf{err} \quad (11)$$

for all $g \in F(\mathfrak{B})$.

Example 3.1. • *Classical deterministic noise model: If $\mathcal{S}(g; \hat{g}) = \mathcal{T}(g; \hat{g}) = \|g - \hat{g}\|_{\mathcal{Y}}^r$, then we obtain from $|a - b|^r \geq 2^{1-r}a^r - b^r$ that (11) holds true with $C_{\text{err}} = 2^{r-1}$ and $\mathbf{err} = 2\|g^\dagger - g^{\text{obs}}\|_{\mathcal{Y}}^r$. Thus Assumption 1 covers the classical deterministic noise model.*

- *Poisson data: For the case of \mathcal{S} and \mathcal{T} as in (8) and (9) it can be seen from elementary calculations that (11) requires $C_{\text{err}} = 1$ and*

$$\mathbf{err} \geq - \int_{\mathbb{M}} \ln(g^\dagger + \sigma) (dG_t - g^\dagger dx) + \int_{\mathbb{M}} \ln(F(u) + \sigma) (dG_t - g^\dagger dx) \quad (12)$$

for all $u \in \mathfrak{B}$. Consequently (11) holds true with $C_{\text{err}} = 1$ if $\mathbf{err}/2$ is an upper bound for the integrals in (10) with $g = F(u)$, $u \in \mathfrak{B}$. We will show that Theorem 2.1 ensures that this holds true for $\mathbf{err}/2 = \frac{\rho}{\sqrt{t}}$ with probability $\geq 1 - \exp(-c\rho)$ for some constant $c > 0$ (cf. Corollary 4.2).

In a previous study of Newton-type methods for inverse problems with Poisson data [17] the authors had to use a slightly stronger assumption on the noise level involving a second inequality. [17, Assumption 2] implies (11) with $\mathbf{err} = (1 + C_{\text{err}}) \sup_{u \in \mathfrak{B}} \mathbf{err}(g)$ provided this value is finite. On the other hand, (11) allows that $\mathcal{S}(g^{\text{obs}}; g) = \infty$ even if $\mathcal{T}(g^\dagger; g) < \infty$, which is impossible in [17, Assumption 2] if $\mathbf{err}(g) < \infty$.

To measure the smoothness of the unknown solution, we will use a source condition in the form of a variational inequality, which was introduced by Hofmann et al [15] for the case of a Hölder-type source condition with index $\nu = \frac{1}{2}$ and generalized in many recent publications [6, 10, 12, 13, 16]. For their formulation we need the *Bregman distance*. For a subgradient $u^* \in \partial \mathcal{R}(u^\dagger) \subset \mathcal{X}^*$ (e.g. $u^* = u^\dagger - u_0$ if $\mathcal{R}(u) = 1/2 \|u - u_0\|_{\mathcal{X}}^2$ with a Hilbert norm $\|\cdot\|_{\mathcal{X}}$) the Bregman distance of \mathcal{R} between u and u^\dagger w.r.t. u^* is given by

$$\mathcal{D}_{\mathcal{R}}^{u^*}(u, u^\dagger) := \mathcal{R}(u) - \mathcal{R}(u^\dagger) - \langle u^*, u - u^\dagger \rangle.$$

In the aforementioned example of $\mathcal{R}(u) = 1/2 \|u - u_0\|_{\mathcal{X}}^2$ for a Hilbert space norm $\|\cdot\|_{\mathcal{X}}$ we have $\mathcal{D}_{\mathcal{R}}^{u^*}(u, u^\dagger) = 1/2 \|u - u^\dagger\|_{\mathcal{X}}^2$. In this sense, the Bregman distance is a natural generalization of the norm. We will use the Bregman distance also to measure the error of our approximate solutions. Now we are able to formulate our assumption on the smoothness of u^\dagger :

Assumption 2 (variational source condition). $\mathcal{R} : \mathcal{X} \rightarrow (-\infty, \infty]$ is a proper convex functional and there exist $u^* \in \partial\mathcal{R}(u^\dagger)$, a parameter $\beta > 0$ and an index function φ (i.e. φ monotonically increasing, $\varphi(0) = 0$) such that φ^2 is concave and

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}(u, u^\dagger) \leq \mathcal{R}(u) - \mathcal{R}(u^\dagger) + \varphi(\mathcal{T}(g^\dagger; F(u))) \quad \text{for all } u \in \mathfrak{B}. \quad (13)$$

Example 3.2. Let ψ be an index function, ψ^2 concave and $F : \mathfrak{B} \subset \mathcal{X} \rightarrow \mathcal{Y}$ Fréchet differentiable between Hilbert spaces \mathcal{X} and \mathcal{Y} with Fréchet derivative $F'[\cdot]$. Flemming [11, 12] has shown that

$$u^\dagger - u_0 = \psi \left(F' [u^\dagger]^* F' [u^\dagger] \right) \omega \quad (14)$$

together with the tangential cone condition $\|F' [u^\dagger] (u - u^\dagger)\|_{\mathcal{Y}} \leq \eta \|F(u) - F(u^\dagger)\|_{\mathcal{Y}}$ implies the variational inequality

$$\beta \|u - u^\dagger\|_{\mathcal{X}}^2 \leq \|u\|_{\mathcal{X}}^2 - \|u^\dagger\|_{\mathcal{X}}^2 + \varphi_\psi \left(\|F(u) - g^\dagger\|_{\mathcal{Y}}^2 \right). \quad (15)$$

for all $u \in \mathfrak{B}$. Here φ_ψ is another index function depending on ψ , and for the most important cases of Hölder-type and logarithmic source conditions the implications

$$\psi(\tau) = \tau^\nu \quad \Rightarrow \quad \varphi_\psi(\tau) = \tilde{\beta} \tau^{\frac{2\nu}{2\nu+1}}, \quad (16a)$$

$$\psi(\tau) = -(\ln(\tau))^{-p} \quad \Rightarrow \quad \varphi_\psi(\tau) = \bar{\beta} (-\ln(\tau))^{-2p} \quad (16b)$$

hold true with some constants $\tilde{\beta}, \bar{\beta}$ where $p > 0$ and $\nu \in (0, \frac{1}{2}]$ (see Hofmann & Yamamoto [16, Prop. 6.6] and Flemming [11, Sec. 13.5.2] respectively).

With the notation (10) of the error, we are able to perform a deterministic convergence analysis including an error decomposition. Following Grasmair [13] we use the Fenchel conjugate of ϕ to bound the approximation error. Recall that the Fenchel conjugate ϕ^* of a function $\phi : \mathbb{R} \rightarrow (-\infty, \infty]$ is given by

$$\phi^*(s) = \sup_{\tau \in \mathbb{R}} (s\tau - \phi(\tau)).$$

ϕ^* is always convex as supremum over the affine-linear (and hence convex) functions $s \mapsto s\tau - \phi(\tau)$. Setting $\varphi(\tau) := -\infty$ for $\tau < 0$ we obtain

$$(-\varphi)^*(s) = \sup_{\tau \geq 0} (s\tau + \varphi(\tau)). \quad (17)$$

This allows us to apply tools from convex analysis: For convex and continuous ϕ Young's inequality holds true (see e.g. [9, eq. (4.1) and Prop. 5.1]), which states

$$\begin{aligned} s\tau &\leq \phi(\tau) + \phi^*(s) & \text{for all } s, \tau \in \mathbb{R}, \\ s\tau &= \phi(\tau) + \phi^*(s) & \Leftrightarrow \tau \in \partial\phi(s). \end{aligned} \quad (18)$$

Moreover for convex and continuous ϕ we have $\phi^{**} = \phi$ (see e.g. [9, Prop. 4.1]).

Now we are in a position to prove our deterministic convergence rates result:

Theorem 3.3. Suppose Assumptions 1 and 2 hold true and the Tikhonov functional has a global minimizer. Then we have the following assertions:

1. For all $\alpha > 0$ and $\mathbf{err} \geq 0$ we have

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger) \leq \frac{\mathbf{err}}{\alpha} + (-\varphi)^* \left(-\frac{1}{C_{\mathbf{err}} \alpha} \right). \quad (19)$$

2. Let $\mathbf{err} > 0$. Then the infimum of the right hand side of (19) is attained at $\alpha = \bar{\alpha}$ if and only if $\frac{-1}{C_{\mathbf{err}} \bar{\alpha}} \in \partial(-\varphi)(C_{\mathbf{err}} \mathbf{err})$, and we have

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}(u_{\bar{\alpha}}, u^\dagger) \leq \sqrt{C_{\mathbf{err}}} \varphi(\mathbf{err}). \quad (20)$$

Proof. (1): By the definition of u_α we have

$$\mathcal{S}(g^{\text{obs}}; F(u_\alpha)) + \alpha \mathcal{R}(u_\alpha) \leq \mathcal{S}(g^{\text{obs}}; g^\dagger) + \alpha \mathcal{R}(u^\dagger). \quad (21)$$

It follows that

$$\begin{aligned} \beta \mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger) &\stackrel{\text{Ass. 2}}{\leq} \mathcal{R}(u_\alpha) - \mathcal{R}(u^\dagger) + \varphi(\mathcal{T}(g^\dagger; F(u_\alpha))) \\ &\stackrel{(21)}{\leq} \frac{1}{\alpha} (\mathcal{S}(g^{\text{obs}}; g^\dagger) - \mathcal{S}(g^{\text{obs}}; F(u_\alpha))) + \varphi(\mathcal{T}(g^\dagger; F(u_\alpha))) \\ &\stackrel{\text{Ass. 1}}{\leq} \frac{\mathbf{err}}{\alpha} - \frac{1}{C_{\mathbf{err}} \alpha} \mathcal{T}(g^\dagger; F(u_\alpha)) + \varphi(\mathcal{T}(g^\dagger; F(u_\alpha))) \\ &\leq \frac{\mathbf{err}}{\alpha} + \sup_{\tau \geq 0} \left[\frac{\tau}{-C_{\mathbf{err}} \alpha} - (-\varphi)(\tau) \right] \\ &\stackrel{(17)}{=} \frac{\mathbf{err}}{\alpha} + (-\varphi)^* \left(-\frac{1}{C_{\mathbf{err}} \alpha} \right). \end{aligned}$$

(2): Using the fact that $(-\varphi)^{**} = -\varphi$ we obtain

$$\begin{aligned} \inf_{\alpha > 0} \left[\frac{\mathbf{err}}{\alpha} + (-\varphi)^* \left(-\frac{1}{C_{\mathbf{err}} \alpha} \right) \right] &= - \sup_{s < 0} [s C_{\mathbf{err}} \mathbf{err} - (-\varphi)^*(s)] \\ &= -(-\varphi)^{**}(C_{\mathbf{err}} \mathbf{err}) = \varphi(C_{\mathbf{err}} \mathbf{err}) \leq \sqrt{C_{\mathbf{err}}} \varphi(\mathbf{err}) \end{aligned}$$

where we used the concavity of φ^2 . By the conditions for equality in Young's inequality (18), the supremum is attained at $\alpha = \bar{\alpha}$ if and only if $\frac{-1}{C_{\mathbf{err}} \bar{\alpha}} \in \partial(-\varphi)(C_{\mathbf{err}} \mathbf{err})$. \blacksquare

Remark 3.4. Since φ is assumed to be finite, we have $\partial(-\varphi)(s) \neq \emptyset$ for all $s > 0$ (see e.g. [9, Cor. 2.3 and Prop. 5.2]), i.e. the parameter choice (25) is feasible. If φ is differentiable, then $\partial(-\varphi)(s) = \{-\varphi'(s)\}$ and (25) is equivalent to $\alpha = 1/(C_{\mathbf{err}} \varphi'(C_{\mathbf{err}} \mathbf{err}))$.

Example 3.5 (Classical case). Let $F = T : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear operator between Hilbert spaces \mathcal{X} and \mathcal{Y} . For $\mathcal{S}(g_2; g_1) = \mathcal{T}(g_2; g_1) = \|g_1 - g_2\|_{\mathcal{Y}}^2$ and $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$ we have $\mathcal{D}_{\mathcal{R}}^{u^*}(u, u^\dagger) = \|u - u^\dagger\|_{\mathcal{X}}^2$, $C_{\mathbf{err}} = 2$ and $\mathbf{err} = 2 \|g^\dagger - g^{\text{obs}}\|_{\mathcal{Y}}^2$. Moreover (14) implies (13) with $\varphi = \varphi_\psi$.

If $\|g^\dagger - g^{\text{obs}}\|_{\mathcal{Y}} \leq \delta$ as mentioned in the introduction, then we obtain for an appropriate parameter choice $\|u_\alpha - u^\dagger\|_{\mathcal{X}} = \mathcal{O}(\sqrt{\varphi_\psi(\delta^2)})$. For the special examples of ψ given in (16) we obtain

$$\|u_\alpha - u^\dagger\|_{\mathcal{X}} = \mathcal{O}\left(\delta^{\frac{2\nu}{2\nu+1}}\right), \quad \|u_\alpha - u^\dagger\|_{\mathcal{X}} = \mathcal{O}\left((-\ln(\delta))^{-p}\right)$$

respectively, and these convergence rates are known to be of optimal order.

4. Convergence rates for Poisson data with a-priori parameter choice rule

In this section we will combine Theorems 2.1 and 3.3 to obtain convergence rates for the method (2) with Poisson data. We need the following properties of the operator F :

Assumption 3 (Assumptions on the forward operator). *Let \mathcal{X} be a Banach space and $\mathfrak{B} \subset \mathcal{X}$ a bounded, closed and convex subset containing the exact solution u^\dagger to (1). Let $\mathbb{M} \subset \mathbb{R}^d$ a bounded domain with Lipschitz boundary ∂D . Assume moreover that the operator $F : \mathfrak{B} \rightarrow \mathcal{Y} := \mathbf{L}^1(\mathbb{M})$ has the following properties:*

1. $F(u) \geq 0$ a.e. for all $u \in \mathfrak{B}$.
2. There exists a Sobolev index $s > \frac{d}{2}$ such that $F(\mathfrak{B})$ is a bounded subset of $H^s(\mathbb{M})$.

Property (1) is natural since photon densities (or more generally intensities of Poisson processes) have to be non-negative. Property (2) is not restrictive for inverse problems since it corresponds to a smoothing property of F which is usually responsible for the ill-posedness of the underlying problem.

Remark 4.1 (Discussion of Assumption 2). *Let Assumption 3 and (14) hold true. Since we have the lower bound*

$$\|g - \hat{g}\|_{\mathbf{L}^2(\mathbb{M})}^2 \leq \left(\frac{4}{3} \|g + \sigma\|_{\mathbf{L}^\infty(\mathbb{M})} + \frac{2}{3} \|\hat{g} + \sigma\|_{\mathbf{L}^\infty(\mathbb{M})} \right) \mathcal{T}(\hat{g}; g) \quad (22)$$

with \mathcal{T} as in (9) at hand (see [5]), (15) obviously implies (13) with \mathcal{T} as in (9) and an index function differing from φ_ψ only by a multiplicative constant.

Thus Assumption 2 is weaker than a spectral source condition. In particular, if $F(u^\dagger) = 0$ on some parts of \mathbb{M} it may happen that (13) holds true with an index function better than φ_ψ .

Assumption 3 moreover allows us to prove the following corollary, which shows that Theorem 2.1 applies for the integrals in (10):

Corollary 4.2. *Let Assumption 3 hold true, set*

$$R := \sup_{u \in \mathfrak{B}} \|F(u)\|_{H^s(\mathbb{M})},$$

and consider Z defined in (10) with $\sigma > 0$. Then there exists $C_{\text{conc}} \geq 1$ depending only on \mathbb{M} and s such that

$$\mathbf{P} \left[\sup_{u \in \mathfrak{B}} Z(F(u)) \leq \frac{\rho}{\sqrt{t}} \right] \geq 1 - \exp \left(- \frac{\rho}{R \max \{ \sigma^{-\lfloor s \rfloor - 1}, |\ln(R)| \} C_{\text{conc}}} \right) \quad (23)$$

for all $t \geq 1, \rho \geq R \max \{ \sigma^{-\lfloor s \rfloor - 1}, |\ln(R)| \} C_{\text{conc}}$.

Proof. W.l.o.g we may assume that $R \geq 1$. Due to Sobolev's embedding theorem and $s > d/2$ we have $\|F(u)\|_{\mathbf{L}^\infty(\mathbb{M})} \leq R \|E_\infty\|$ for all $u \in \mathfrak{B}$.

By an extension argument it can be seen from [22] that for $\mathbb{M} \subset \mathbb{R}^d$ with Lipschitz boundary, $g \in H^s(\mathbb{M}) \cap \mathbf{L}^\infty(\mathbb{M})$ and $\Phi \in C^{\lfloor s \rfloor + 1}(\mathbb{R})$ one has $\Phi \circ g \in H^s(\mathbb{M})$ and

$$\|\Phi \circ g\|_{H^s(\mathbb{M})} \leq C \|\Phi\|_{C^{\lfloor s \rfloor + 1}(\mathbb{R})} \|g\|_{H^s(\mathbb{M})} \quad (24)$$

with $C > 0$ independent of Φ and g . To apply this result, we first extend the function $x \mapsto \ln(x + \sigma)$ from $[0, R \|E_\infty\|]$ (since we have $0 \leq F(u) \leq R \|E_\infty\|$ a.e.) to a function Φ on the whole real line such that $\Phi \in C^{\lfloor s \rfloor + 1}(\mathbb{R})$. Then for any fixed $u \in \mathfrak{B}$ we obtain $\Phi \circ F(u + \sigma) \in H^s(\mathbb{M})$ and since $\Phi|_{[0, R \|E_\infty\|]}(\cdot) = \ln(\cdot + \sigma)$ and $0 \leq F(u) \leq R \|E_\infty\|$ a.e., we have $\Phi \circ (F(u) + \sigma) =$

$\ln(F(u) + \sigma)$ a.e. Since all derivatives up to order $\lfloor s \rfloor + 1$ of $x \mapsto \ln(x + \sigma)$ and hence of Φ on $[0, R \|E_\infty\|]$ can be bounded by some constant of order $\max\{\sigma^{-\lfloor s \rfloor - 1}, \ln(R \|E_\infty\|)\}$, the extension and composition procedure described above is bounded, i.e. there exists by (24) a constant $\tilde{C} > 0$ independent of u, R and σ such that

$$\|\ln(F(u) + \sigma)\|_{H^s(\mathbb{M})} \leq \tilde{C} \max\left\{\sigma^{-\lfloor s \rfloor - 1}, \ln(R)\right\} R$$

for all $u \in \mathfrak{B}$. Now the assertion follows from Theorem 2.1. \blacksquare

Now we are able to present our first main result for Poisson data:

Theorem 4.3. *Let the Assumptions 2 with \mathcal{T} defined in (9) and Assumption 3 be satisfied. Moreover, suppose that (2) with \mathcal{S} in (8) has a global minimizer. If we choose the regularization parameter $\alpha = \alpha(t)$ such that*

$$\frac{1}{\alpha} \in -\partial(-\varphi)\left(\frac{1}{\sqrt{t}}\right) \quad (25)$$

then we obtain the convergence rate

$$\mathbf{E}\left[\mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger)\right] = \mathcal{O}\left(\varphi\left(\frac{1}{\sqrt{t}}\right)\right), \quad t \rightarrow \infty.$$

Proof. First note that Assumption 1 holds true with $C_{\text{err}} = 1$ whenever the bound **err** fulfills (12). By Corollary 4.2 the right-hand side of (12) is bounded by $2\frac{\rho}{\sqrt{t}}$ with probability greater or equal $1 - \exp(-c\rho)$ with $\rho \geq 1/c$,

$$c = \left(R \max\left\{\sigma^{-\lfloor s \rfloor - 1}, |\ln(R)| C_{\text{conc}}\right\}\right)^{-1},$$

and C_{conc} as in Corollary 4.2. Now let $\rho_k := c^{-1}k, k \in \mathbb{N}$ and consider the events

$$E_0 := \emptyset, \quad E_k := \left\{\sup_{u \in \mathfrak{B}} Z(F(u)) \leq \frac{\rho_k}{\sqrt{t}}\right\}, \quad k \in \mathbb{N}$$

with Z as defined in (10). Corollary 4.2 implies

$$\mathbf{P}[E_k^c] \leq \exp(-k)$$

and on E_k Assumption 1 holds true with $C_{\text{err}} = 1$ and **err** = $2 \sup_{u \in \mathfrak{B}} Z(F(u)) \leq 2\rho_k/\sqrt{t}$. Thus Theorem 3.3(1) implies

$$\max_{E_k} \mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger) \leq \frac{1}{\beta} \left((-\varphi)^* \left(-\frac{1}{\alpha} \right) + \frac{2\rho_k}{\alpha\sqrt{t}} \right) \leq \frac{2\rho_k}{\beta} \left((-\varphi)^* \left(-\frac{1}{\alpha} \right) + \frac{1}{\alpha\sqrt{t}} \right)$$

for all $k \in \mathbb{N}$ and $\alpha > 0$. According to Theorem 3.3(2) the infimum of the right hand side is attained at α defined in (25), and

$$\max_{E_k} \mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger) \leq \frac{2\rho_k}{\beta} \varphi\left(\frac{1}{\sqrt{t}}\right)$$

for all $k \in \mathbb{N}$ with $C(k) = \frac{2}{\beta}c^{-1}k$. Now we obtain

$$\begin{aligned} \mathbf{E}\left[\mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger)\right] &= \sum_{k=1}^{\infty} \mathbf{P}[E_k \setminus E_{k-1}] \mathbf{E}\left[\mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger) \mid E_k \setminus E_{k-1}\right] \\ &\leq \sum_{k=1}^{\infty} \mathbf{P}[E_k \setminus E_{k-1}] \max_{E_k} \mathcal{D}_{\mathcal{R}}^{u^*}(u_\alpha, u^\dagger) \\ &\leq \mathbf{P}[E_1] \frac{2\rho_1}{\beta} \varphi\left(\frac{1}{\sqrt{t}}\right) + \sum_{k=2}^{\infty} \mathbf{P}[E_{k-1}^c] \frac{2\rho_k}{\beta} \varphi\left(\frac{1}{\sqrt{t}}\right) \\ &\leq \frac{2}{\beta} c^{-1} \left(\sum_{k=1}^{\infty} \exp(-(k-1)k) \right) \varphi\left(\frac{1}{\sqrt{t}}\right). \end{aligned}$$

The sum converges and the proof is complete. \blacksquare

5. A Lepskiĭ-type parameter choice rule

Usually the parameter choice rule (25) is not implementable since it requires a priori knowledge of the function φ characterizing the smoothness of the unknown solution u^\dagger . To adapt to unknown smoothness of the solution, a posteriori parameter choice rules have to be used. In a deterministic context the most widely used such rule is *discrepancy principle*. However, in our context is not applicable in an obvious way since \mathcal{S} approximates \mathcal{T} only up to the unknown constant $\mathbf{E}[\mathcal{S}(G_t; g^\dagger)]$.

In the following we will describe and analyse the Lepskiĭ principle as described and analyzed in the context of inverse problems by Mathé and Pereverzev [20, 21]. Lepskiĭ's balancing principle requires a metric on \mathcal{X} , and hence we assume in the following that there exists a constant $C_{\text{bd}} > 0$ and a number $q \geq 1$ such that

$$\|u - u^\dagger\|_{\mathcal{X}}^q \leq C_{\text{bd}} \mathcal{D}_{\mathcal{R}}^{u^*}(u, u^\dagger) \quad \text{for all } u \in \mathfrak{B}. \quad (26)$$

This is fulfilled trivially with $q = 2$ and $C_{\text{bd}} = 1$ if \mathcal{X} is a Hilbert space and $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$ (then we have equality in (26)). Moreover for a q -convex Banach space \mathcal{X} and $\mathcal{R}(u) = \|u\|_{\mathcal{X}}^q$ the estimate (26) is valid (see [31]). Besides this special cases of norm powers, (26) can be fulfilled for other choices of \mathcal{R} . E.g. for maximum entropy regularization, i.e. $\mathcal{R}(u) = \int_a^b u \ln(u) \, dx$, the Bregman distance coincides with the Kullback-Leibler divergence, and we have seen in Remark 4.1 that (26) holds true in this situation.

The deterministic convergence analysis from Section 3 already provides an error decomposition. Assuming $\beta \geq 1/2$, Theorem 3.3(1) together with (26) states that

$$\|u_\alpha - u^\dagger\|_{\mathcal{X}} \leq \frac{1}{2} (f_{\text{app}}(\alpha) + f_{\text{noi}}(\alpha)) \quad \text{for all } \alpha > 0 \quad (27)$$

with the *approximation error* $f_{\text{app}}^\beta(\alpha)$ and the *propagated data noise error* $f_{\text{noi}}^\beta(\alpha)$ defined by

$$f_{\text{app}}(\alpha) := 2 \left(2C_{\text{bd}} (-\varphi)^* \left(-\frac{1}{\alpha} \right) \right)^{\frac{1}{q}} \quad \text{and} \quad f_{\text{noi}}(\alpha) := 2 \left(2C_{\text{bd}} \frac{\mathbf{err}}{\alpha} \right)^{\frac{1}{q}}. \quad (28)$$

Here the constant 2 in front of C_{bd} is an estimate of $1/\beta$. For the error decomposition (27) it is important to note that f_{app} is typically unknown, whereas f_{noi} is known if the upper bound \mathbf{err} is available. But due to Corollary 4.2 the error is bounded by ρ/\sqrt{t} with probability $1 - \exp(-c\rho)$. This observation is fundamental in the proof of the following theorem:

Theorem 5.1. *Let Assumptions 2 and 3 with $\beta \in [\frac{1}{2}, \infty)$ and \mathcal{S} and \mathcal{T} as in (8) and (9) be fulfilled and suppose (26) holds true. Suppose that (2) has a global minimizer and let $\sigma > 0$, $r > 1$, $R := \sup_{u \in \mathfrak{B}} \|F(u)\|_{H^s(\mathbb{M})} < \infty$ and $\tau \geq \frac{1}{4} R \max\{\sigma^{-\lfloor s \rfloor - 1}, |\ln(R)|\} C_{\text{conc}}$. Define the sequence*

$$\alpha_j := \frac{\tau \ln(t)}{\sqrt{t}} r^{2j-2}, \quad j \in \mathbb{N}.$$

Then with $m := \min\{j \in \mathbb{N} \mid \alpha_j \geq 1\}$ the choice

$$j_{\text{bal}} := \min\left\{j \in \{1, \dots, m\} \mid \|u_{\alpha_i} - u_{\alpha_j}\| \leq 4(4C_{\text{bd}})^{\frac{1}{q}} r^{\frac{2-2i}{q}} \text{ for all } i < j\right\} \quad (29)$$

yields

$$\mathbf{E} \left[\|u_{\alpha_{j_{\text{bal}}}} - u^\dagger\|_{\mathcal{X}}^q \right] = \mathcal{O} \left(\varphi \left(\frac{\ln(t)}{\sqrt{t}} \right) \right) \quad \text{as } t \rightarrow \infty.$$

Proof. If $t \geq \exp(4)$, the assumptions of Corollary 4.2 are fulfilled with $\rho(t) := \tau \ln(t)$. Then with Z as in (10) the event

$$A_\rho := \left\{ \sup_{u \in \mathfrak{B}} Z(F(u)) \leq \frac{\rho(t)}{\sqrt{t}} \right\}$$

has probability $\mathbf{P}[A_\rho^c] \leq \exp(-c\rho(t))$ with $c = (R \max\{\sigma^{-\lfloor s \rfloor - 1}, |\ln(R)|\}) C_{\text{conc}}^{-1}$ by Corollary 4.2. Moreover, as we have seen in (27), on A_ρ the error decomposition

$$\|u_j - u^\dagger\|_{\mathcal{X}} \leq \frac{1}{2} (\phi(j) + \psi(j))$$

holds true with

$$\psi(j) = 2(4C_{\text{bd}})^{\frac{1}{q}} \left(\frac{\rho(t)}{\sqrt{t}\alpha_j} \right)^{\frac{1}{q}} = 2(4C_{\text{bd}})^{\frac{1}{q}} r^{\frac{2-2j}{q}}$$

and $\phi = f_{\text{app}}$ as in (28). Note that $2\psi(j)$ corresponds to the required bound for $\|u_{\alpha_j} - u_{\alpha_j}\|_{\mathcal{X}}$ in (29). The function ψ is obviously non-increasing and fulfills $\psi(j) \leq r^{2/q}(j+1)$ and it can be seen by elementary computations that ϕ is monotonically increasing. Now [20, Cor. 1] implies the so-called oracle inequality

$$\max_{A_\rho} \|u_{\alpha_{j_{\text{bal}}}} - u^\dagger\|_{\mathcal{X}} \leq 3r^{\frac{2}{q}} \min\{\phi(j) + \psi(j) \mid j \in \{1, \dots, m\}\}.$$

By inserting the definitions of ϕ and ψ we find

$$\max_{A_\rho} \|u_{\alpha_{j_{\text{bal}}}} - u^\dagger\|_{\mathcal{X}}^q \leq 4r^2 12^q C_{\text{bd}} \min_{j=1, \dots, m} \left((-\varphi)^* \left(-\frac{1}{\alpha_j} \right) + \frac{\rho(t)}{\sqrt{t}\alpha_j} \right) \quad (30)$$

and obviously the minimum over $\alpha_1, \dots, \alpha_m$ can be replaced up to some constant depending only on r by the infimum over $\alpha \geq \alpha_1$ if t is sufficiently large. By Theorem 3.3(2) the sum $(-\varphi)^*(-1/\alpha) + \rho(t)/(\sqrt{t}\alpha)$ attains its minimum over $\alpha \in (0, \infty)$ at $\alpha = \alpha_{\text{opt}}$ if and only if $1/\alpha_{\text{opt}} \in -\partial(-\varphi)(\rho(t)/\sqrt{t})$. Note that $\rho(t)/\sqrt{t} = \alpha_1$. By elementary arguments from convex analysis we find using the concavity of φ that

$$\frac{1}{\alpha_{\text{opt}}} \leq -\inf \partial(-\varphi)(\alpha_1) = \lim_{h \searrow 0} \frac{\varphi(\alpha_1) - \varphi(\alpha_1 - h)}{h} \leq \frac{\varphi(\alpha_1) - \varphi(s)}{\alpha_1 - s}$$

for all $0 \leq s \leq \alpha_1$. Thus choosing $s = 0$ shows that $\alpha_1/\alpha_{\text{opt}} \leq \varphi(\alpha_1) = \varphi(\rho(t)/\sqrt{t})$ for all $t > 0$. As the right-hand side decays to 0 as $t \rightarrow \infty$, we have $\alpha_1 \leq \alpha_{\text{opt}}$ for t sufficiently large. Therefore, the minimum in (30) can indeed be replaced (up to some constant) by the infimum over all $\alpha > 0$ (see [29, Lem. 3.42] for details). Defining $\text{diam}(\mathfrak{B}) := \sup_{u, v \in \mathfrak{B}} \|u - v\|_{\mathcal{X}}$ which is finite by Assumption 3 we find from (30) and Theorem 3.3 that

$$\begin{aligned} \mathbf{E} \left[\|u_{n_{\text{bal}}} - u^\dagger\|_{\mathcal{X}}^q \right] &\leq \mathbf{P}[A_\rho] \max_{A_\rho} \|u_{\alpha_{j_{\text{bal}}}} - u^\dagger\|_{\mathcal{X}}^q + \mathbf{P}[A_\rho^c] \max_{A_\rho^c} \|u_{\alpha_{j_{\text{bal}}}} - u^\dagger\|_{\mathcal{X}}^q \\ &\leq C\varphi \left(\frac{\ln(t)}{\sqrt{t}} \right) + \exp(-c\rho(t)) \text{diam}(\mathfrak{B})^q. \end{aligned}$$

with some constant $C > 0$. Due to the definition of ρ , $2\tau c \geq \frac{1}{2}$, $\ln(t) \geq 1$ and $\ln(t)/\sqrt{t} < 1$ we obtain

$$\exp(-c\rho(t)) = \left(\frac{1}{\sqrt{t}} \right)^{2\tau c} \leq \left(\frac{\ln(t)}{\sqrt{t}} \right)^{2\tau c} \leq \sqrt{\frac{\ln(t)}{\sqrt{t}}} \leq \frac{1}{\varphi(1)} \varphi \left(\frac{\ln(t)}{\sqrt{t}} \right)$$

using the concavity of φ^2 . This proves the assertion. \blacksquare

Note that the constants R and C_{conc} - which are necessary to ensure a proper choice of the sequence α_j and hence for the implementation of this Lepskiĭ-type balancing principle - can be calculated in principle (assuming e.g. the scaling condition $\|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} = 1$). Thus Theorem 5.1 yields convergence rates in expectation for a completely adaptive algorithm.

Comparing the rates in Theorems 4.3 and 5.1 note that we have to pay a logarithmic factor for adaptation to unknown smoothness by the Lepskiĭ principle. It is known (see [28]) that in some cases the loss of such a logarithmic factor is inevitable.

A. Proof of Theorem 2.1

In this section we will prove the uniform concentration inequality stated in Theorem 2.1. Our result is based on the work of Reynaud-Bouret [25] who proved the following concentration inequality:

Lemma A.1 ([25, Corollary 2]). *Let N be a Poisson process with finite mean measure ν . Let $\{f_a\}_{a \in A}$ be a countable family of functions with values in $[-b, b]$ and define*

$$Z := \sup_{a \in A} \left| \int_{\mathbb{M}} f_a(x) (dN - d\nu) \right| \quad \text{and} \quad v_0 := \sup_{a \in A} \int_{\mathbb{M}} f_a^2(x) d\nu.$$

Then for all positive numbers ρ and ε it holds

$$\mathbf{P} \left[Z \geq (1 + \varepsilon) \mathbf{E}[Z] + \sqrt{12v_0\rho} + \kappa(\varepsilon)b\rho \right] \leq \exp(-\rho)$$

where $\kappa(\varepsilon) = 5/4 + 32/\varepsilon$.

We will use a denseness argument to apply Lemma A.1 to $t\tilde{Z}$ with

$$\tilde{Z} := \sup_{\mathbf{g} \in B_s(R)} \left| \int_{\mathbb{M}} \mathbf{g}(x) (dG_t - g^\dagger dx) \right|.$$

The properties derived in the following lemma will be sufficient to bound $\mathbf{E}[\tilde{Z}]$:

Lemma A.2. *Let $\mathbb{M} \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $R > 0$ and suppose $s > \frac{d}{2}$. Then there exists a countable family of real-valued functions $\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathcal{J}\}$, numbers $\gamma_{\mathbf{j}}$, $\mathbf{j} \in \mathcal{J}$ and constants $c_1, c_2 > 0$ depending only on s and \mathbb{M} such that*

$$\sum_{\mathbf{j} \in \mathcal{J}} \gamma_{\mathbf{j}}^2 \int_{\mathbb{M}} \phi_{\mathbf{j}}^2 g^\dagger dx \leq c_1 \|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})}, \quad (31)$$

and for all $\mathbf{g} \in B_s(R)$ there exists real numbers $\beta_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}$ such that

$$\mathbf{g} = \sum_{\mathbf{j} \in \mathcal{J}} \beta_{\mathbf{j}} \phi_{\mathbf{j}} \quad \text{and} \quad \sum_{\mathbf{j} \in \mathcal{J}} \left(\frac{\beta_{\mathbf{j}}}{\gamma_{\mathbf{j}}} \right)^2 \leq c_2^2 R^2. \quad (32)$$

Proof. Choose some $\kappa > 0$ such that $\overline{\mathbb{M}} \subset (-\kappa, \kappa)^d$. Then there exists a continuous extension operator $E : H^s(\mathbb{M}) \rightarrow H_0^s([-\kappa, \kappa]^d)$ (see e.g. [30, Cor. 5.1]). Consider the following orthonormal bases $\{\varphi_j : j \in \mathbb{Z}\}$ of $\mathbf{L}^2([-\kappa, \kappa])$ and $\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$ of $\mathbf{L}^2([-\kappa, \kappa]^d)$:

$$\varphi_j(x) := \frac{1}{\sqrt{\kappa}} \begin{cases} \sin(\pi j x / \kappa), & j > 0, \\ 1/\sqrt{2}, & j = 0, \\ \cos(\pi j x / \kappa), & j < 0, \end{cases} \quad \phi_{\mathbf{j}}(x_1, \dots, x_d) := \prod_{l=1}^d \varphi_{j_l}(x_l).$$

We introduce the norm $\|\mathbf{g}\|_{H_{\text{per}}^s} = \left(\sum_{\mathbf{j} \in \mathbb{Z}^d} (1 + |\mathbf{j}|^2)^s |\langle \mathbf{g}, \phi_{\mathbf{j}} \rangle|^2\right)^{1/2}$ and the periodic Sobolev space $H_{\text{per}}^s([- \kappa, \kappa]^d) := \{\mathbf{g} \in \mathbf{L}^2([- \kappa, \kappa]^d) \mid \|\mathbf{g}\|_{H_{\text{per}}^s} < \infty\}$. The embedding $J : H_0^s([- \kappa, \kappa]^d) \hookrightarrow H_{\text{per}}^s([- \kappa, \kappa]^d)$ is well defined and continuous as the norms of both spaces are equivalent (see e.g. [30, Exercise 1.13]), so the extension operator

$$E_{\text{ext}} := J \circ E : H^s(\mathbb{M}) \longrightarrow H_{\text{per}}^s([- \kappa, \kappa]^d).$$

is continuous. In particular,

$$E_{\text{ext}}(B_s(R)) \subset \left\{ \mathbf{g} \in H_{\text{per}}^s([- \kappa, \kappa]^d) \mid \|\mathbf{g}\|_{H_{\text{per}}^s([- \kappa, \kappa]^d)} \leq c_2 R \right\} \quad \text{with } c_2 := \|E_{\text{ext}}\|$$

and (32) holds true with $\beta_{\mathbf{j}} := \langle \mathbf{g}, \phi_{\mathbf{j}} \rangle$ and $\gamma_{\mathbf{j}} := (1 + |\mathbf{j}|^2)^{-s/2}$. Moreover, as $\|\phi_{\mathbf{j}}^2\|_{\infty} \leq \kappa^{-d}$ for all $\mathbf{j} \in \mathbb{Z}^d$ we obtain

$$\sum_{\mathbf{j} \in \mathbb{Z}^d} \gamma_{\mathbf{j}}^2 \int_{\mathbb{M}} \phi_{\mathbf{j}}^2 g^\dagger dx \leq c_1 \int_{\mathbb{M}} g^\dagger dx \quad \text{with } c_1 := \kappa^{-d} \sum_{\mathbf{j} \in \mathbb{Z}^d} (1 + |\mathbf{j}|^2)^{-s},$$

and majorization of the sum by an integral shows that $c_1 < \infty$ as $s > d/2$. Therefore, (31) holds true, and the proof is complete. \blacksquare

Lemma A.3. *Under the assumptions of Lemma A.2 we have*

$$\mathbf{E} \left[\tilde{Z} \right] \leq \frac{c_1 c_2 R}{\sqrt{t}} \|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})}.$$

Proof. With the help of Lemma A.2 we can now insert (32) and apply Hölder's inequality for sums to find

$$\begin{aligned} \tilde{Z} &\leq \sup_{\sum_{\mathbf{j} \in J} \left(\frac{\beta_{\mathbf{j}}}{\gamma_{\mathbf{j}}}\right)^2 \leq (c_2 R)^2} \left| \sum_{\mathbf{j} \in J} \frac{\beta_{\mathbf{j}}}{\gamma_{\mathbf{j}}} \int_{\mathbb{M}} \gamma_{\mathbf{j}} \phi_{\mathbf{j}} (dG_t - g^\dagger dx) \right| \\ &\leq c_2 R \sqrt{\sum_{\mathbf{j} \in J} \gamma_{\mathbf{j}}^2 \left(\int_{\mathbb{M}} \phi_{\mathbf{j}} (dG_t - g^\dagger dx) \right)^2} \end{aligned}$$

where we used that the functions $\phi_{\mathbf{j}}$ are real-valued. Hence by Jensen's inequality

$$\mathbf{E} \left[\tilde{Z} \right] \leq \sqrt{\mathbf{E} \left[\tilde{Z}^2 \right]} \leq c_2 R \sqrt{\sum_{\mathbf{j} \in \mathcal{J}} \gamma_{\mathbf{j}}^2 \mathbf{E} \left[\left(\int_{\mathbb{M}} \phi_{\mathbf{j}} (dG_t - g^\dagger dx) \right)^2 \right]}. \quad (33)$$

Using (3) we obtain

$$\mathbf{E} \left[\left(\int_{\mathbb{M}} \phi_{\mathbf{j}} (dG_t - g^\dagger dx) \right)^2 \right] = \frac{1}{t^2} \mathbf{E} \left[\left(\int_{\mathbb{M}} \phi_{\mathbf{j}} (t dG_t - t g^\dagger dx) \right)^2 \right] = \frac{1}{t} \int_{\mathbb{M}} \phi_{\mathbf{j}}^2 g^\dagger dx$$

and plugging this into (33) and using (31) we obtain

$$\mathbf{E} \left[\tilde{Z} \right] \leq \frac{c_2 R}{\sqrt{t}} \sqrt{\sum_{\mathbf{j} \in \mathcal{J}} \gamma_{\mathbf{j}}^2 \int_{\mathbb{M}} \phi_{\mathbf{j}}^2 g^\dagger dx} \leq \frac{\sqrt{c_1} c_2 R}{\sqrt{t}} \sqrt{\|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})}}.$$

\blacksquare

Proof of Theorem 2.1. By Sobolev's embedding theorem the embedding operator $E_\infty : H^s(\mathbb{M}) \hookrightarrow \mathbf{L}^\infty(\mathbb{M})$ is well defined and continuous, so

$$\|\mathbf{g}\|_{\mathbf{L}^\infty(\mathbb{M})} \leq R \|E_\infty\| \quad \text{for all } \mathbf{g} \in B_s(R). \quad (34)$$

Now we choose a countable subset $\{\mathbf{g}_a\}_{a \in A} \subset B_s(R)$ which is dense in $B_s(R)$ w.r.t. the H^s -norm, and hence also the \mathbf{L}^∞ -norm and set $N = tG_t$ and $d\nu = tg^\dagger dx$ in Lemma A.1 to obtain

$$\mathbf{P} \left[\tilde{Z} \geq (1 + \varepsilon) \mathbf{E} \left[\tilde{Z} \right] + \frac{\sqrt{12v_0\bar{\rho}}}{t} + \frac{\kappa(\varepsilon) \|E_\infty\| R\bar{\rho}}{t} \right] \leq \exp(-\bar{\rho}) \quad (35)$$

for all $\bar{\rho} > 0$. Choosing $\varepsilon = 1$ and using Lemma A.3 and the simple estimate

$$\tilde{v}_0 \leq tR^2 \|E_\infty\|^2 \|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})},$$

yields

$$\mathbf{P} \left[\tilde{Z} \leq \frac{C_1 R}{\sqrt{t}} + \frac{C_2 R \sqrt{\bar{\rho}}}{\sqrt{t}} + \frac{C_3 R \bar{\rho}}{t} \right] \geq 1 - \exp(-\bar{\rho}) \quad (36)$$

for all $\bar{\rho}, t > 0$ with $C_1 := 2\sqrt{c_1}c_2\sqrt{\|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})}}$, $C_2 := \sqrt{12} \|E_\infty\| \sqrt{\|g^\dagger\|_{\mathbf{L}^1(\mathbb{M})}}$ and $C_3 := (32 + \frac{5}{4}) \|E_\infty\|$. If $t, \bar{\rho} \geq 1$, we have $\frac{1}{t} \leq \frac{1}{\sqrt{t}}$ and $\sqrt{\bar{\rho}} \leq \bar{\rho}$, so

$$\mathbf{P} \left[\tilde{Z} \leq (C_1 + C_2 + C_3) \frac{\bar{\rho} R}{\sqrt{t}} \right] \geq 1 - \exp(-\bar{\rho}) \quad \text{for } \bar{\rho}, t \geq 1.$$

Setting $C_{\text{conc}} := \max\{C_1 + C_2 + C_3, 1\}$ and $\rho := \bar{\rho} R C_{\text{conc}}$ this shows the assertion. \blacksquare

Acknowledgments

We would like to thank Patricia Reynaud-Bouret for fruitful discussions on the concentration inequality. Financial support by the German Research Foundation DFG through the Research Training Group 1023 and CRC 755 is gratefully acknowledged.

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