Convergence Rates for Inverse Problems with Impulsive Noise

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Outline

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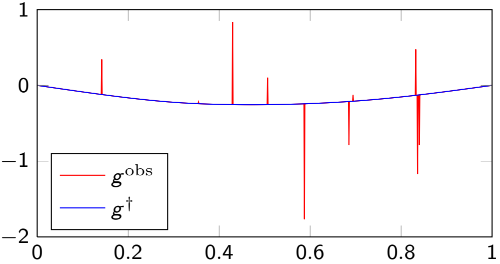
What is Impulsive Noise?

- noise is small in large parts of the domain, but large on small parts of the domain
- occurs e.g. in digital image acquisition
- caused by faulty memory locations, malfunctioning pixels etc.
- popular example: salt-and-pepper noise



Inverse Problems with Impulsive Noise

- we want to reconstruct $f^\dagger$ from

$$g^{\mathrm{obs}} = F\left(f^\dagger\right) + \xi =: g^\dagger + \xi$$

  where ξ is impulsive noise
- natural setup: $F : D(F) \subset \mathcal{X} \to \mathbf{L}^1(\mathbb{M}) \subseteq \mathcal{Y}$, possibly nonlinear
- Favorable method: Tikhonov regularization

$$\widehat{f}_\alpha \in \operatorname*{argmin}_{f \in D(F)} \left[\frac{1}{\alpha r}\left\|F(f) - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^r + \mathcal{R}(f)\right]$$

- Minimizer $\widehat{f}_\alpha$ exists under reasonable assumptions.

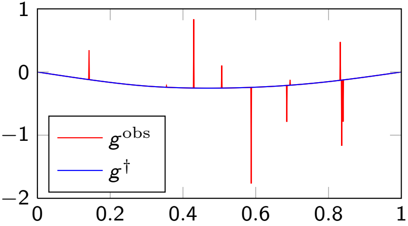
How to choose $\mathcal{Y}$ and r

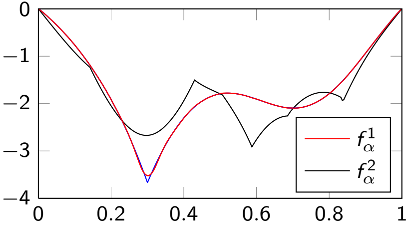
here: F = linear integral operator (two times smoothing) on $\mathbb{M} = [0,1]$

$$f_\alpha^{p} = \underset{f\in\mathbf{L}^2(\mathbb{M})}{\operatorname{argmin}} \left[\left\| F(f) - g^{\mathrm{obs}} \right\|_{\mathbf{L}^{p}(\mathbb{M})}^{p} + \alpha \left\| f \right\|_{\mathbf{L}^2(\mathbb{M})}^2 \right], \qquad p = 1, 2$$

computation of f_α^1 via dual formulation, see e.g.

C. Clason, B. Jin, K. Kunisch.
A semismooth Newton method for $\mathbf{L}^1$ data fitting with automatic choice of regularization parameters and noise calibration.
SIAM J. Imaging Sci., 3:199–231, 2010.





Outline

Theoretical state of the art

- known theory provides rates of convergence as $\|\xi\|_{\mathcal{Y}}$ tends to 0
- this does not fully explain the remarkable quality of the $\mathbf{L}^1$-reconstruction!

Example: 'Most impulsive' noise. $\mathcal{Y} = \mathfrak{M}(\mathbb{M})$ (space of all signed measures) and

$$\xi = \sum_{j=1}^{N} c_j \delta_{x_j}$$

with $N \in \mathbb{N}$, $c_j \in \mathbb{R}$ and $x_j \in \mathbb{M}$ for $1 \leq j \leq N$.

Then $\|\xi\|_{\mathfrak{M}(\mathbb{M})} = \sum_{j=1}^{N} |c_j|$ might be large! However

$$\left\| g - g^{\mathrm{obs}} \right\|_{\mathfrak{M}(\mathbb{M})} = \left\| g - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} + \sum_{j=1}^{N} |c_j| = \left\| g - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} + \|\xi\|_{\mathfrak{M}(\mathbb{M})} .$$

So ξ does not influence the minimizer $\widehat{f}_\alpha$!

Improving the noise level

'Most impulsive' noise ξ influences $g \mapsto \left\|g - g^{\mathrm{obs}}\right\|_{\mathfrak{M}(\mathbb{M})}$ only as an additive constant, no influence on $\widehat{f}_\alpha$!
Idea: For general ξ study the influence of ξ on the data fidelity term $\left\|g - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^r$ for all g.

Variational noise assumption

Suppose there exist $C_{\mathrm{err}} > 0$ and a noise level function $\mathbf{err} : F\left(D\left(F\right)\right) \to [0, \infty]$ such that

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^r - \|\xi\|_{\mathcal{Y}}^r \geq \frac{1}{C_{\mathrm{err}}} \left\|g - g^{\dagger}\right\|_{\mathcal{Y}}^r - \mathbf{err}\left(g\right), \qquad g \in F(D\left(F\right)).$$

Examples for the noise function **err**

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^{r} - \|\xi\|_{\mathcal{Y}}^{r} \geq \frac{1}{C_{\mathrm{err}}} \left\|g - g^{\dagger}\right\|_{\mathcal{Y}}^{r} - \mathbf{err}\,(g)\,, \qquad g \in F(D\,(F)).$$

1. It follows from the triangle inequality that the Assumption is always fulfilled with

$$C_{\mathrm{err}} = 2^{r-1} \qquad \text{and} \qquad \mathbf{err} \equiv 2\,\|\xi\|_{\mathcal{Y}}^{r}\,.$$

2. In the Example of 'most impulsive' noise ($\mathcal{Y} = \mathfrak{M}\,(\mathbb{M})$, $r = 1$) the Assumption holds true with the optimal parameters

$$C_{\mathrm{err}} = 1 \qquad \text{and} \qquad \mathbf{err} \equiv 0.$$

Convergence analysis under the variational noise assumption

- Bregman distance:

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right) := \mathcal{R}\left(f\right) - \mathcal{R}\left(f^\dagger\right) - \left\langle f^*, f - f^\dagger \right\rangle$$

  where $f^* \in \partial\mathcal{R}\left(f^\dagger\right) \subset \mathcal{X}'$.

- use a variational inequality as source condition:

$$\beta\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right) \leq \mathcal{R}\left(f\right) - \mathcal{R}\left(f^\dagger\right) + \varphi\left(\left\|F\left(f\right) - g^\dagger\right\|_{\mathcal{Y}}^{r}\right)$$

  for all $f \in D\left(F\right)$ with $\beta > 0$. φ is assumed to fulfill
  - $\varphi\left(0\right) = 0$,
  - $\varphi \nearrow$,
  - φ concave.

deterministic convergence analysis

suppose

- the noise assumption is fulfilled with a function **err** ≥ 0 and
- the variational inequality holds true.

Theorem (error decomposition)

$$\beta \mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq \frac{\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right)}{r\alpha} + (-\varphi)^*\left(-\frac{1}{rC_{\mathrm{err}}\alpha}\right),$$

$$\left\|F\left(\widehat{f}_\alpha\right) - g^\dagger\right\|_{\mathcal{Y}}^r \leq \frac{C_{\mathrm{err}}}{\lambda}\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) + \frac{rC_{\mathrm{err}}\alpha}{\lambda}(-\varphi)^*\left(-\frac{1-\lambda}{rC_{\mathrm{err}}\alpha}\right)$$

for all $\alpha > 0$ and $\lambda \in (0,1)$.

Fenchel conjugate:

$$(-\varphi)^*(s) = \sup_{\tau \geq 0}\left(s\tau + \varphi(\tau)\right).$$

Outline

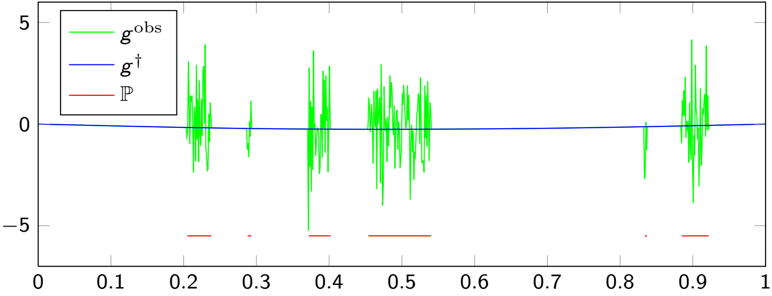
A simple Impulsive Noise model

Suppose $\xi \in \mathbf{L}^1(\mathbb{M})$, $\mathfrak{B}(\mathbb{M}) \hat{=}$ Borel σ-algebra of $\mathbb{M}$.

Noise model

There exist two parameters $\varepsilon, \eta > 0$ such that

$$\exists\, \mathbb{P} \in \mathfrak{B}(\mathbb{M}) : \qquad \|\xi\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} \leq \varepsilon, \qquad |\mathbb{P}| \leq \eta\,.$$



Estimating the error function **err**

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \frac{1}{C_{\mathrm{err}}} \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}\,(g)\,, \qquad g \in F(D\,(F))$$

$$\begin{aligned}
\left\|g - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} &= \int\limits_{\mathbb{M}\setminus\mathbb{P}} \left[\left|g^{\mathrm{obs}} - g\right| - |\xi|\right] \,\mathrm{d}x + \int\limits_{\mathbb{P}} \left[\left|g^{\mathrm{obs}} - g\right| - |\xi|\right] \,\mathrm{d}x \\
&\geq \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} - 2\varepsilon - |\mathbb{P}| \left\|g - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})} \\
&\geq \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\,|\mathbb{P}| \left\|g - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})}
\end{aligned}$$

Here we used

- the first triangle inequality on $\mathbb{M} \setminus \mathbb{P}$ and
- the second triangle inequality on $\mathbb{P}$.

Improving the bound

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \left\|g - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\,|\mathbb{P}|\,\left\|g - g^\dagger\right\|_{\mathbf{L}^\infty(\mathbb{P})}$$

If F is smoothing and $g = F(f)$, then $\left\|g - g^\dagger\right\|_{\mathbf{L}^\infty(\mathbb{P})}$ also decays with η!

Smoothing assumption

$\mathbb{M} \subset \mathbb{R}^d$ is a bounded Lipschitz domain and there exist $k \in \mathbb{N}_0, p \in [1, \infty], k > d/p$ such that

$$F(D(F)) \subset W^{k,p}(\mathbb{M}) \quad \text{and} \quad \left\|F(f) - g^\dagger\right\|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p} \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right)^{\frac{1}{2}}$$

for all $f \in D(F)$ with some $C_{F,k,p} > 0$.

Sampling inequalities

Theorem (Wendland, Rieger '05)

Let $k > d/p$, $p \in [1, \infty)$, $k \in \mathbb{N}$, $X \subset \mathbb{M}$ finite. For all $g \in W^{k,p}(\mathbb{M})$ it holds

$$\|g\|_{\mathbf{L}^\infty(\mathbb{M})} \leq C \left(h_{X,\mathbb{M}}^{k-d/p} |g|_{W^{k,p}(\mathbb{M})} + \max_{x \in X} |g(x)| \right).$$

H. Wendland and C. Rieger
Approximate interpolation with applications to selecting smoothing parameters
Numer. Math., 101 (2005), pp. 729–748.

Theorem (Hohage, W.)

There exist constants $c_1, \eta_0 > 0$ and $c_2 \in [0, 1/2)$ such that

$$\|g\|_{\mathbf{L}^\infty(\mathbb{P})} \leq c_1 \eta^{\frac{k}{d} - \frac{1}{p}} |g|_{W^{k,p}(\mathbb{M})} + \frac{c_2}{\eta} \|g\|_{\mathbf{L}^1(\mathbb{M})}$$

for all $g \in W^{k,p}(\mathbb{M})$, $\eta \in (0, \eta_0)$ and $\mathbb{P} \in \mathfrak{B}(\mathbb{M})$ with $|\mathbb{P}| \leq \eta$.

A better estimate for $\mathbf{err}(F(f))$ I

$$
\begin{aligned}
&\left\|F(f) - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
\geq &\left\|F(f) - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \left\|F(f) - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})}
\end{aligned}
$$

A better estimate for **err** $(F(f))$ II

$$\begin{aligned} & \left\|F(f) - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\ \geq & \left\|F(f) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta\left\|F(f) - g^\dagger\right\|_{\mathbf{L}^\infty(\mathbb{P})} \\ \geq & (1 - 2c_2)\left\|F(f) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1\eta^{\frac{k}{d} - \frac{1}{p} + 1}\left\|F(f) - g^\dagger\right\|_{W^{k,p}(\mathbb{M})} \end{aligned}$$

$$\|g\|_{\mathbf{L}^\infty(\mathbb{P})} \leq c_1\eta^{\frac{k}{d} - \frac{1}{p}}\,|g|_{W^{k,p}(\mathbb{M})} + \frac{c_2}{\eta}\|g\|_{\mathbf{L}^1(\mathbb{M})} \qquad \text{with } g = F(f) - g^\dagger$$

A better estimate for **err** $(F(f))$ III

$$\begin{aligned}
&\left\|F(f) - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
\geq &\left\|F(f) - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \left\|F(f) - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})} \\
\geq &(1 - 2c_2)\left\|F(f) - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1\eta^{\frac{k}{d} - \frac{1}{p} + 1}\left\|F(f) - g^{\dagger}\right\|_{W^{k,p}(\mathbb{M})} \\
\geq &(1 - 2c_2)\left\|F(f) - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1 C_{F,k,p}\eta^{\frac{k}{d} - \frac{1}{p} + 1}\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^{\dagger}\right)
\end{aligned}$$

$$\left\|F(f) - g^{\dagger}\right\|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p}\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^{\dagger}\right)^{\frac{1}{2}}$$

A better estimate for $\mathbf{err}(F(f))$ IV

$$
\begin{aligned}
&\left\|F(f) - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
\geq &\left\|F(f) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \left\|F(f) - g^\dagger\right\|_{\mathbf{L}^\infty(\mathbb{P})} \\
\geq &(1 - 2c_2)\left\|F(f) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1\eta^{\frac{k}{d}-\frac{1}{p}+1}\left\|F(f) - g^\dagger\right\|_{W^{k,p}(\mathbb{M})} \\
\geq &(1 - 2c_2)\left\|F(f) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1 C_{F,k,p}\eta^{\frac{k}{d}-\frac{1}{p}+1}\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right) \\
\stackrel{!}{\geq} &\frac{1}{C_{\mathrm{err}}}\left\|F(f) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}(F(f))
\end{aligned}
$$

Thus we can choose

$$C_{\mathrm{err}} = (1 - 2c_2)^{-1} \qquad \text{and} \qquad \mathbf{err}(F(f)) = 2\varepsilon + 2c_1 C_{F,k,p}\eta^{\frac{k}{d}-\frac{1}{p}+1}\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right)^{\frac{1}{2}}$$

Recursive estimate for $\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right)$

Calculation above:

$$\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) = 2\varepsilon + 2c_1 C_{F,k,p}\eta^{\frac{k}{d}-\frac{1}{p}+1}\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right)^{\frac{1}{2}}$$

General convergence analysis:

$$\beta\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq \frac{\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right)}{\alpha} + (-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

This implies

$$\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) \leq C_1'\varepsilon + C_2'\frac{\eta^{\frac{2k}{d}+2\frac{p-1}{p}}}{\alpha} + C_3'\eta^{\frac{k}{p}+\frac{p-1}{p}}\sqrt{(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)}$$

Improved convergence analysis

$$\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) \leq C_1'\varepsilon + C_2'\frac{\eta^{\frac{2k}{d}+2\frac{p-1}{p}}}{\alpha} + C_3'\eta^{\frac{k}{p}+\frac{p-1}{p}}\sqrt{(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)}$$

Insert this estimate into the general error decomposition to obtain

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq C_1\frac{\varepsilon}{\alpha} + C_2\frac{\eta^{\frac{2k}{d}+\frac{2(p-1)}{p}}}{\alpha^2} + \frac{3}{2\beta}(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

with constants $C_1, C_2 > 0$.

Optimal parameter choice I

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq C_1\frac{\varepsilon}{\alpha} + C_2\frac{\eta^{\frac{2k}{d}+\frac{2(p-1)}{p}}}{\alpha^2} + \frac{3}{2\beta}\left(-\varphi\right)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

Let

- $\theta\left(\alpha\right) := \alpha \cdot \left(-\varphi\right)^*\left(-\frac{1}{\alpha}\right)$
- $\tilde{\theta}\left(\alpha\right) := \alpha^2\left(-\varphi\right)^*\left(-\frac{1}{\alpha}\right)$

If α is chosen such that

$$\alpha \sim \max\left\{\theta^{-1}\left(\varepsilon\right), \tilde{\theta}^{-1}\left(\eta^{\frac{2k}{d}+\frac{2(p-1)}{p}}\right)\right\},$$

then we obtain the convergence rate

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\left(-\varphi\right)^*\left(-\frac{1}{\max\left\{\theta^{-1}\left(\varepsilon\right), \tilde{\theta}^{-1}\left(\eta^{\frac{2k}{d}+\frac{2(p-1)}{p}}\right)\right\}}\right)\right)$$

as $\max\left\{\varepsilon,\eta\right\} \searrow 0$.

Optimal parameter choice II

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq C_1 \frac{\varepsilon}{\alpha} + C_2 \frac{\eta^{\frac{2k}{d}+\frac{2(p-1)}{p}}}{\alpha^2} + \frac{3}{2\beta}(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

If $\varphi(t) = c \cdot t^\kappa$ with $c > 0$ and $\kappa \in (0,1]$, then

- $(-\varphi)^*\left(-\frac{1}{\alpha}\right) = C \cdot t^{\frac{\kappa}{1-\kappa}}$,
- $\theta(\alpha) = \alpha^{\frac{1}{1-\kappa}}$ and
- $\tilde{\theta}(\alpha) = \alpha^{\frac{2-\kappa}{1-\kappa}}$.

So for $\alpha \sim \max\left\{\varepsilon^{1-\kappa}, \eta^{\left(\frac{1-\kappa}{2-\kappa}\right)\left(\frac{2k}{d}+\frac{2(p-1)}{p}\right)}\right\}$ we obtain

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\max\left\{\varepsilon^\kappa, \eta^{\frac{\kappa}{2-\kappa}\left(\frac{2k}{d}+\frac{2(p-1)}{p}\right)}\right\}\right)$$

as $\max\{\varepsilon, \eta\} \searrow 0$.

Improvement of our new analysis

- Consider $|\mathbb{P}| = \eta_0^2, \qquad \xi_{|_\mathbb{P}} = \frac{1}{\eta_0}, \qquad \xi_{|_{\mathbb{M}\setminus\mathbb{P}}} = 0.$
- Old noise level: $\|\xi\|_{\mathbf{L}^1(\mathbb{M})} = \|\xi\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} + \|\xi\|_{\mathbf{L}^1(\mathbb{P})} = |\mathbb{P}|\,\eta_0^{-1} = \eta_0.$
- Formerly known rate:

$$\mathcal{D}_\mathcal{R}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\eta_0^\kappa\right).$$

- Optimal choice of ε and η upto constants: $\eta = \eta_0^2$ and $\varepsilon = 0$. Convergence rate:

$$\mathcal{D}_\mathcal{R}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\eta_0^{\frac{2\kappa}{2-\kappa}\left(\frac{2k}{d} + \frac{2(p-1)}{p}\right)}\right)$$

Outline

Considered operator

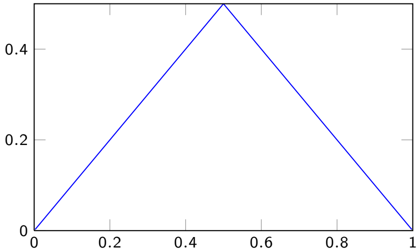
- $\mathbb{M} = [0,1]$ and $T : \mathbf{L}^2(\mathbb{M}) \to \mathbf{L}^2(\mathbb{M})$ defined by

$$(Tf)(x) = \int_0^1 k(x,y) f(y) \,\mathrm{d}y, \qquad x \in \mathbb{M}$$

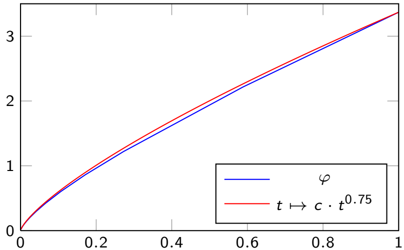
  with kernel $k(x,y) = \min\{x \cdot (1-y), y \cdot (1-x)\}, x, y \in \mathbb{M}$.
- Then $(Tf)'' = f$ for any $f \in \mathbf{L}^2(\mathbb{M})$ and T is 2 times smoothing ($k = 2$ and arbitrary $p \geq 1$).
- The smoothing Assumption is valid with any exponent $\gamma := 2k/d + 2(p-1)/p < 6$, we use $\gamma = 6$.
- Discretization: equidistant points $x_1 = \frac{1}{2n}, x_2 = \frac{3}{2n}, \ldots, x_n = \frac{2n-1}{2n}$ and composite midpoint rule

$$(Tf)(x) = \int_0^1 k(x,y) f(y) \,\mathrm{d}y \approx \frac{1}{n} \sum_{i=1}^{n} k(x, x_i) f(x_i).$$

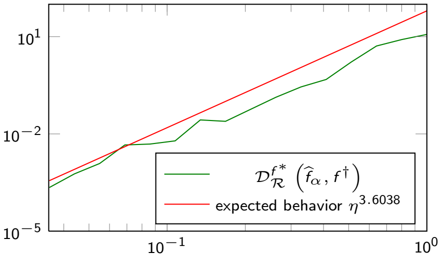
Example 1



(a) Exact solution $f^\dagger$

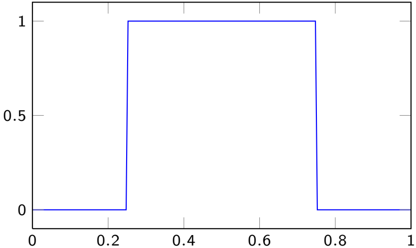


(b) Estimated index function φ

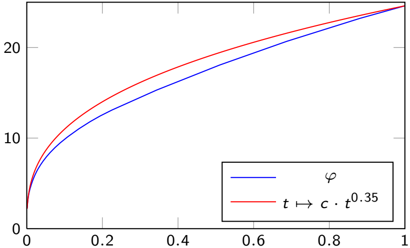


(c) mean convergence in $\mathcal{X}$

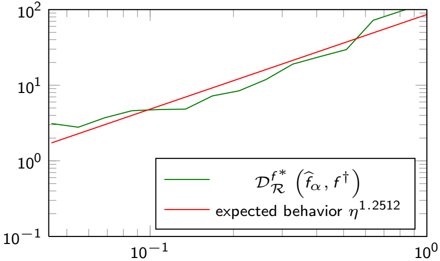
Example 2



(d) Exact solution $f^\dagger$



(e) Estimated index function φ



(f) mean convergence in $\mathcal{X}$

Outline

Presented results and future work

- Inverse Problems with Impulsive noise
  - continuous model for Impulsive noise
  - improved convergence rates
- numerical examples suggest order optimality
- future work: infinitely smoothing operators!

Thank you for your attention!