

Convergence Rates for Inverse Problems with Impulsive Noise

Frank Werner (joint work with Thorsten Hohage)

Institute for Numerical and Applied Mathematics
University of Göttingen, Germany

29^o Colóquio Brasileiro de Matemática, Rio de Janeiro, Brazil
July 30th, 2013



Outline

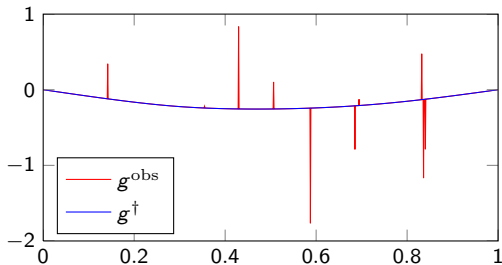
- ① Impulsive Noise
- ② Analysis of Tikhonov regularization
- ③ Application to Impulsive Noise
- ④ Numerical simulations
- ⑤ Conclusion

Outline

- ① Impulsive Noise
- ② Analysis of Tikhonov regularization
- ③ Application to Impulsive Noise
- ④ Numerical simulations
- ⑤ Conclusion

What is Impulsive Noise?

- noise is small in large parts of the domain, but large on small parts of the domain
- occurs e.g. in digital image acquisition
- caused by faulty memory locations, malfunctioning pixels etc.
- popular example: salt-and-pepper noise



Inverse Problems with Impulsive Noise

- we want to reconstruct f^\dagger from

$$g^{\text{obs}} = F(f^\dagger) + \xi =: g^\dagger + \xi$$

where ξ is impulsive noise

- natural setup: $F : D(F) \subset \mathcal{X} \rightarrow \mathbf{L}^1(\mathbb{M}) \subseteq \mathcal{Y}$, possibly nonlinear
- Favorable method: Tikhonov regularization

$$\hat{f}_\alpha \in \operatorname{argmin}_{f \in D(F)} \left[\frac{1}{\alpha r} \left\| F(f) - g^{\text{obs}} \right\|_{\mathcal{Y}}^r + \mathcal{R}(f) \right]$$

- Minimizer \hat{f}_α exists under reasonable assumptions.

How to choose \mathcal{Y} and r

here: $F =$ linear integral operator (two times smoothing) on $\mathbb{M} = [0, 1]$

$$f_{\alpha}^p = \operatorname{argmin}_{f \in \mathbf{L}^2(\mathbb{M})} \left[\left\| F(f) - g^{\text{obs}} \right\|_{\mathbf{L}^p(\mathbb{M})}^p + \alpha \|f\|_{\mathbf{L}^2(\mathbb{M})}^2 \right], \quad p = 1, 2$$

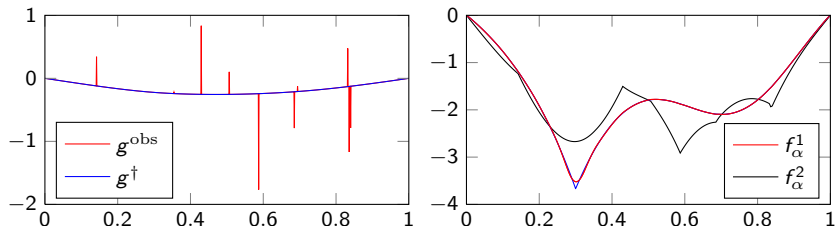
computation of f_{α}^1 via dual formulation, see e.g.



C. Clason, B. Jin, K. Kunisch.

A semismooth Newton method for \mathbf{L}^1 data fitting with automatic choice of regularization parameters and noise calibration.

SIAM J. Imaging Sci., 3:199–231, 2010.



Outline

- ① Impulsive Noise
- ② Analysis of Tikhonov regularization
- ③ Application to Impulsive Noise
- ④ Numerical simulations
- ⑤ Conclusion

Theoretical state of the art

- known theory provides rates of convergence as $\|\xi\|_{\mathcal{Y}}$ tends to 0
- this does not fully explain the remarkable quality of the \mathbf{L}^1 -reconstruction!

Example: 'Most impulsive' noise. $\mathcal{Y} = \mathfrak{M}(\mathbb{M})$ (space of all signed measures) and

$$\xi = \sum_{j=1}^N c_j \delta_{x_j}$$

with $N \in \mathbb{N}$, $c_j \in \mathbb{R}$ and $x_j \in \mathbb{M}$ for $1 \leq j \leq N$.

Then $\|\xi\|_{\mathfrak{M}(\mathbb{M})} = \sum_{j=1}^N |c_j|$ might be large! However

$$\|g - g^{\text{obs}}\|_{\mathfrak{M}(\mathbb{M})} = \|g - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} + \sum_{j=1}^N |c_j| = \|g - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} + \|\xi\|_{\mathfrak{M}(\mathbb{M})}.$$

So ξ does not influence the minimizer \hat{f}_α !

Improving the noise level

'Most impulsive' noise ξ influences $g \mapsto \|g - g^{\text{obs}}\|_{\mathfrak{M}(\mathbb{M})}$ only as an additive constant, no influence on \hat{f}_α !

Idea: For general ξ study the influence of ξ on the data fidelity term $\|g - g^{\text{obs}}\|_{\mathcal{Y}}^r$ for all g .

Variational noise assumption

Suppose there exist $C_{\text{err}} > 0$ and a noise level function $\text{err} : F(D(F)) \rightarrow [0, \infty]$ such that

$$\|g - g^{\text{obs}}\|_{\mathcal{Y}}^r - \|\xi\|_{\mathcal{Y}}^r \geq \frac{1}{C_{\text{err}}} \|g - g^\dagger\|_{\mathcal{Y}}^r - \text{err}(g), \quad g \in F(D(F)).$$

Examples for the noise function **err**

$$\left\| g - g^{\text{obs}} \right\|_{\mathcal{Y}}^r - \|\xi\|_{\mathcal{Y}}^r \geq \frac{1}{C_{\text{err}}} \left\| g - g^\dagger \right\|_{\mathcal{Y}}^r - \mathbf{err}(g), \quad g \in F(D(F)).$$

- ① It follows from the triangle inequality that the Assumption is always fulfilled with

$$C_{\text{err}} = 2^{r-1} \quad \text{and} \quad \mathbf{err} \equiv 2 \|\xi\|_{\mathcal{Y}}^r.$$

- ② In the Example of 'most impulsive' noise ($\mathcal{Y} = \mathfrak{M}(\mathbb{M})$, $r = 1$) the Assumption holds true with the optimal parameters

$$C_{\text{err}} = 1 \quad \text{and} \quad \mathbf{err} \equiv 0.$$

Convergence analysis under the variational noise assumption

- **Bregman distance:**

$$D_{\mathcal{R}}^{f^*} (f, f^\dagger) := \mathcal{R}(f) - \mathcal{R}(f^\dagger) - \langle f^*, f - f^\dagger \rangle$$

where $f^* \in \partial \mathcal{R}(f^\dagger) \subset \mathcal{X}'$.

- use a **variational inequality** as source condition:

$$\beta D_{\mathcal{R}}^{f^*} (f, f^\dagger) \leq \mathcal{R}(f) - \mathcal{R}(f^\dagger) + \varphi \left(\|F(f) - g^\dagger\|_{\mathcal{Y}}^r \right)$$

for all $f \in D(F)$ with $\beta > 0$. φ is assumed to fulfill

- $\varphi(0) = 0$,
- $\varphi \nearrow$,
- φ concave.

deterministic convergence analysis

suppose

- the noise assumption is fulfilled with a function $\mathbf{err} \geq 0$ and
- the variational inequality holds true.

Theorem (error decomposition)

$$\beta \mathcal{D}_{\mathcal{R}}^{f^*}(\hat{f}_\alpha, f^\dagger) \leq \frac{\mathbf{err}(F(\hat{f}_\alpha))}{r\alpha} + (-\varphi)^* \left(-\frac{1}{rC_{\text{err}}\alpha} \right),$$

$$\|F(\hat{f}_\alpha) - g^\dagger\|_Y^r \leq \frac{C_{\text{err}}}{\lambda} \mathbf{err}(F(\hat{f}_\alpha)) + \frac{rC_{\text{err}}\alpha}{\lambda} (-\varphi)^* \left(-\frac{1-\lambda}{rC_{\text{err}}\alpha} \right)$$

for all $\alpha > 0$ and $\lambda \in (0, 1)$.

Fenchel conjugate:

$$(-\varphi)^*(s) = \sup_{\tau \geq 0} (s\tau + \varphi(\tau)).$$

Outline

- 1 Impulsive Noise
- 2 Analysis of Tikhonov regularization
- 3 Application to Impulsive Noise**
- 4 Numerical simulations
- 5 Conclusion

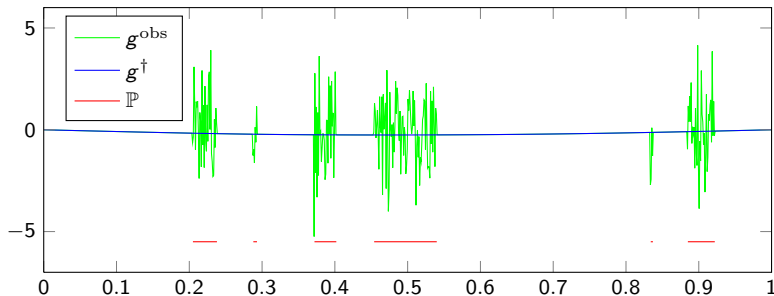
A simple Impulsive Noise model

Suppose $\xi \in \mathbf{L}^1(\mathbb{M})$, $\mathfrak{B}(\mathbb{M}) \hat{=} \text{Borel } \sigma\text{-algebra of } \mathbb{M}$.

Noise model

There exist two parameters $\varepsilon, \eta \geq 0$ such that

$$\exists \mathbb{P} \in \mathfrak{B}(\mathbb{M}) : \quad \|\xi\|_{\mathbf{L}^1(\mathbb{M} \setminus \mathbb{P})} \leq \varepsilon, \quad |\mathbb{P}| \leq \eta.$$



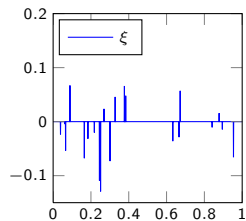
Functional dependence of ε and η

$$\exists \mathbb{P} \in \mathfrak{B}(\mathbb{M}) : \quad \|\xi\|_{\mathbf{L}^1(\mathbb{M} \setminus \mathbb{P})} \leq \varepsilon, \quad |\mathbb{P}| \leq \eta \quad (1)$$

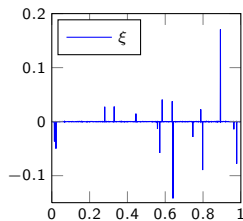
- model allows for different choices of ε and η which depend on each other
- study the dependence function

$$\varepsilon_\xi(\eta) := \inf \left\{ \|\xi\|_{\mathbf{L}^1(\mathbb{M} \setminus \mathbb{P})} \mid \mathbb{P} \in \mathfrak{B}(\mathbb{M}), |\mathbb{P}| \leq \eta \right\}.$$

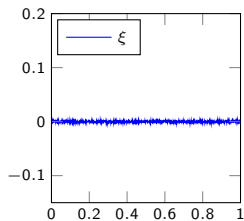
- then for any $\eta \geq 0$ eq. (1) is fulfilled with $\varepsilon = \varepsilon_\xi(\eta)$
- for $\xi \in \mathbf{L}^1(\mathbb{M})$ the following holds true:
 - 1 $\varepsilon_\xi(0) = \|\xi\|_{\mathbf{L}^1(\mathbb{M})}$, $\varepsilon_\xi(|\mathbb{M}|) = 0$
 - 2 ε_ξ is continuous, decreasing, and convex

Examples for ε_ξ 

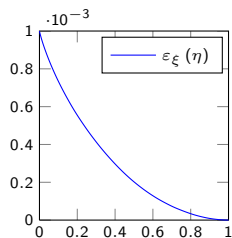
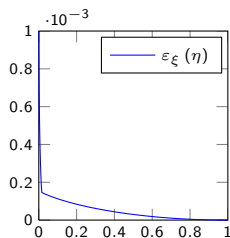
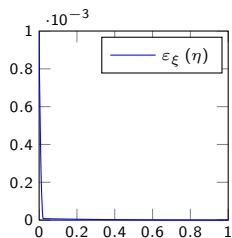
Impulsive noise



Impulsive noise?



Gaussian white noise



Estimating the error function \mathbf{err}

$$\left\| \mathbf{g} - \mathbf{g}^{\text{obs}} \right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \frac{1}{C_{\text{err}}} \left\| \mathbf{g} - \mathbf{g}^\dagger \right\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}(\mathbf{g})$$

$$\begin{aligned} \left\| \mathbf{g} - \mathbf{g}^{\text{obs}} \right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} &= \int_{\mathbb{M} \setminus \mathbb{P}} [|\mathbf{g}^{\text{obs}} - \mathbf{g}| - |\xi|] \, dx + \int_{\mathbb{P}} [|\mathbf{g}^{\text{obs}} - \mathbf{g}| - |\xi|] \, dx \\ &\geq \left\| \mathbf{g} - \mathbf{g}^\dagger \right\|_{\mathbf{L}^1(\mathbb{M} \setminus \mathbb{P})} - 2\varepsilon - |\mathbb{P}| \left\| \mathbf{g} - \mathbf{g}^\dagger \right\|_{\mathbf{L}^\infty(\mathbb{P})} \\ &\geq \left\| \mathbf{g} - \mathbf{g}^\dagger \right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2|\mathbb{P}| \left\| \mathbf{g} - \mathbf{g}^\dagger \right\|_{\mathbf{L}^\infty(\mathbb{P})} \end{aligned}$$

Here we used

- the first triangle inequality on $\mathbb{M} \setminus \mathbb{P}$ and
- the second triangle inequality on \mathbb{P} .

Improving the bound

$$\left\| g - g^{\text{obs}} \right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \left\| g - g^\dagger \right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2|\mathbb{P}| \left\| g - g^\dagger \right\|_{\mathbf{L}^\infty(\mathbb{P})}$$

If F is smoothing and $g = F(f)$, then $\|g - g^\dagger\|_{\mathbf{L}^\infty(\mathbb{P})}$ also decays with η !

Smoothing assumption

$\mathbb{M} \subset \mathbb{R}^d$ is a bounded Lipschitz domain and there exist $k \in \mathbb{N}_0, p \in [1, \infty], k > d/p$ such that

$$F(D(F)) \subset W^{k,p}(\mathbb{M}) \quad \text{and} \quad \left\| F(f) - g^\dagger \right\|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p} \mathcal{D}_{\mathcal{R}}^{f^*}(f, f^\dagger)^{\frac{1}{2}}$$

for all $f \in D(F)$ with some $C_{F,k,p} > 0$.

Sampling inequalities

Theorem (Wendland, Rieger '05)

Let $k > d/p$, $p \in [1, \infty)$, $k \in \mathbb{N}$, $X \subset \mathbb{M}$ finite. For all $g \in W^{k,p}(\mathbb{M})$ it holds

$$\|g\|_{\mathbf{L}^\infty(\mathbb{M})} \leq C \left(h_{X,\mathbb{M}}^{k-d/p} |g|_{W^{k,p}(\mathbb{M})} + \max_{x \in X} |g(x)| \right).$$



H. Wendland and C. Rieger

Approximate interpolation with applications to selecting smoothing parameters
 Numer. Math., 101 (2005), pp. 729–748.

Theorem (Hohage, W.)

There exist constants $c_1, \eta_0 > 0$ and $c_2 \in [0, 1/2)$ such that

$$\|g\|_{\mathbf{L}^\infty(\mathbb{P})} \leq c_1 \eta^{\frac{k}{d} - \frac{1}{p}} |g|_{W^{k,p}(\mathbb{M})} + \frac{c_2}{\eta} \|g\|_{\mathbf{L}^1(\mathbb{M})}$$

for all $g \in W^{k,p}(\mathbb{M})$, $\eta \in (0, \eta_0)$ and $\mathbb{P} \in \mathfrak{B}(\mathbb{M})$ with $|\mathbb{P}| \leq \eta$.

A better estimate for $\mathbf{err}(F(f))$ I

$$\begin{aligned} & \|F(f) - g^{\text{obs}}\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\ & \geq \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \|F(f) - g^\dagger\|_{\mathbf{L}^\infty(\mathbb{P})} \end{aligned}$$

A better estimate for $\mathbf{err}(F(f))$ II

$$\begin{aligned}
& \|F(f) - g^{\text{obs}}\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
& \geq \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \|F(f) - g^\dagger\|_{\mathbf{L}^\infty(\mathbb{P})} \\
& \geq (1 - 2c_2) \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1\eta^{\frac{k}{d} - \frac{1}{p} + 1} \|F(f) - g^\dagger\|_{W^{k,p}(\mathbb{M})}
\end{aligned}$$

$$\|g\|_{\mathbf{L}^\infty(\mathbb{P})} \leq c_1\eta^{\frac{k}{d} - \frac{1}{p}} \|g\|_{W^{k,p}(\mathbb{M})} + \frac{c_2}{\eta} \|g\|_{\mathbf{L}^1(\mathbb{M})} \quad \text{with } g = F(f) - g^\dagger$$

A better estimate for $\text{err}(F(f))$ III

$$\begin{aligned}
& \|F(f) - g^{\text{obs}}\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
& \geq \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \|F(f) - g^\dagger\|_{\mathbf{L}^\infty(\mathbb{P})} \\
& \geq (1 - 2c_2) \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1 \eta^{\frac{k}{d} - \frac{1}{p} + 1} \|F(f) - g^\dagger\|_{W^{k,p}(\mathbb{M})} \\
& \geq (1 - 2c_2) \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1 C_{F,k,p} \eta^{\frac{k}{d} - \frac{1}{p} + 1} \mathcal{D}_{\mathcal{R}}^{f^*}(f, f^\dagger)
\end{aligned}$$

$$\|F(f) - g^\dagger\|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p} \mathcal{D}_{\mathcal{R}}^{f^*}(f, f^\dagger)^{\frac{1}{2}}$$

A better estimate for $\mathbf{err}(F(f))$ IV

$$\begin{aligned}
& \|F(f) - g^{\text{obs}}\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
& \geq \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \|F(f) - g^\dagger\|_{\mathbf{L}^\infty(\mathbb{P})} \\
& \geq (1 - 2c_2) \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1 \eta^{\frac{k}{d} - \frac{1}{p} + 1} \|F(f) - g^\dagger\|_{W^{k,p}(\mathbb{M})} \\
& \geq (1 - 2c_2) \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2c_1 C_{F,k,p} \eta^{\frac{k}{d} - \frac{1}{p} + 1} \mathcal{D}_{\mathcal{R}}^{f^*}(f, f^\dagger) \\
& \stackrel{!}{\geq} \frac{1}{C_{\text{err}}} \|F(f) - g^\dagger\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}(F(f))
\end{aligned}$$

Thus we can choose

$$C_{\text{err}} = (1 - 2c_2)^{-1} \quad \text{and} \quad \mathbf{err}(F(f)) = 2\varepsilon + 2c_1 C_{F,k,p} \eta^{\frac{k}{d} - \frac{1}{p} + 1} \mathcal{D}_{\mathcal{R}}^{f^*}(f, f^\dagger)^{\frac{1}{2}}$$

Recursive estimate for $\mathbf{err} \left(F \left(\widehat{f}_\alpha \right) \right)$

Calculation above:

$$\mathbf{err} \left(F \left(\widehat{f}_\alpha \right) \right) = 2\varepsilon + 2c_1 C_{F,k,p} \eta^{\frac{k}{d} - \frac{1}{p} + 1} \mathcal{D}_{\mathcal{R}}^{f^*} \left(\widehat{f}_\alpha, f^\dagger \right)^{\frac{1}{2}}$$

General convergence analysis:

$$\beta \mathcal{D}_{\mathcal{R}}^{f^*} \left(\widehat{f}_\alpha, f^\dagger \right) \leq \frac{\mathbf{err} \left(F \left(\widehat{f}_\alpha \right) \right)}{\alpha} + (-\varphi)^* \left(-\frac{1}{C_{\text{err}} \alpha} \right)$$

This implies

$$\mathbf{err} \left(F \left(\widehat{f}_\alpha \right) \right) \leq C'_1 \varepsilon + C'_2 \frac{\eta^{\frac{2k}{d} + 2\frac{p-1}{p}}}{\alpha} + C'_3 \eta^{\frac{k}{p} + \frac{p-1}{p}} \sqrt{(-\varphi)^* \left(-\frac{1}{C_{\text{err}} \alpha} \right)}$$

Improved convergence analysis

$$\mathbf{err} \left(F \left(\widehat{f}_\alpha \right) \right) \leq C'_1 \varepsilon + C'_2 \frac{\eta^{\frac{2k}{d} + 2\frac{p-1}{p}}}{\alpha} + C'_3 \eta^{\frac{k}{p} + \frac{p-1}{p}} \sqrt{(-\varphi)^* \left(-\frac{1}{C_{\text{err}} \alpha} \right)}$$

Insert this estimate into the general error decomposition to obtain

Theorem (Hohage, W.)

Under the above assumptions we have for all $\alpha > 0$ and $\eta > 0$ sufficiently small

$$\mathcal{D}_{\mathcal{R}}^{f^*} \left(\widehat{f}_\alpha, f^\dagger \right) \leq C_1 \frac{\varepsilon}{\alpha} + C_2 \frac{\eta^{\frac{2k}{d} + \frac{2(p-1)}{p}}}{\alpha^2} + \frac{3}{2\beta} (-\varphi)^* \left(-\frac{1}{C_{\text{err}} \alpha} \right)$$

with constants $C_1, C_2 > 0$.

Optimal parameter choice I

$$\mathcal{D}_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \leq C_1 \frac{\varepsilon}{\alpha} + C_2 \frac{\eta^{\frac{2k}{d} + \frac{2(p-1)}{p}}}{\alpha^2} + \frac{3}{2\beta} (-\varphi)^* \left(-\frac{1}{C_{\text{err}} \alpha} \right)$$

Let

- $\theta(\alpha) := \alpha \cdot (-\varphi)^* \left(-\frac{1}{\alpha} \right)$
- $\tilde{\theta}(\alpha) := \alpha^2 (-\varphi)^* \left(-\frac{1}{\alpha} \right)$

If α is chosen such that

$$\alpha \sim \max \left\{ \theta^{-1}(\varepsilon), \tilde{\theta}^{-1} \left(\eta^{\frac{2k}{d} + \frac{2(p-1)}{p}} \right) \right\},$$

then we obtain the convergence rate

$$\mathcal{D}_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) = \mathcal{O} \left((-\varphi)^* \left(-\frac{1}{\max \left\{ \theta^{-1}(\varepsilon), \tilde{\theta}^{-1} \left(\eta^{\frac{2k}{d} + \frac{2(p-1)}{p}} \right) \right\}} \right) \right)$$

as $\max \{\varepsilon, \eta\} \searrow 0$.

Optimal parameter choice II

$$\mathcal{D}_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \leq C_1 \frac{\varepsilon}{\alpha} + C_2 \frac{\eta^{\frac{2k}{d} + \frac{2(p-1)}{p}}}{\alpha^2} + \frac{3}{2\beta} (-\varphi)^* \left(-\frac{1}{C_{\text{err}} \alpha} \right)$$

If $\varphi(t) = c \cdot t^\kappa$ with $c > 0$ and $\kappa \in (0, 1]$, then

- $(-\varphi)^* \left(-\frac{1}{\alpha} \right) = C \cdot t^{\frac{\kappa}{1-\kappa}},$
- $\theta(\alpha) = \alpha^{\frac{1}{1-\kappa}}$ and
- $\tilde{\theta}(\alpha) = \alpha^{\frac{2-\kappa}{1-\kappa}}.$

So for $\alpha \sim \max \left\{ \varepsilon^{1-\kappa}, \eta^{\left(\frac{1-\kappa}{2-\kappa}\right)\left(\frac{2k}{d} + \frac{2(p-1)}{p}\right)} \right\}$ we obtain

$$\mathcal{D}_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) = \mathcal{O} \left(\max \left\{ \varepsilon^\kappa, \eta^{\frac{\kappa}{2-\kappa} \left(\frac{2k}{d} + \frac{2(p-1)}{p} \right)} \right\} \right)$$

as $\max \{ \varepsilon, \eta \} \searrow 0$.

Improvement of our new analysis

- Consider $|\mathbb{P}| = \eta_0^2$, $\xi|_{\mathbb{P}} = \frac{1}{\eta_0}$, $\xi|_{\mathbb{M} \setminus \mathbb{P}} = 0$.
- Old noise level: $\|\xi\|_{\mathbf{L}^1(\mathbb{M})} = \|\xi\|_{\mathbf{L}^1(\mathbb{M} \setminus \mathbb{P})} + \|\xi\|_{\mathbf{L}^1(\mathbb{P})} = |\mathbb{P}| \eta_0^{-1} = \eta_0$.
- Formerly known rate:

$$\mathcal{D}_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) = \mathcal{O}(\eta_0^\kappa).$$

- Optimal choice of ε and η upto constants: $\eta = \eta_0^2$ and $\varepsilon = 0$.
Convergence rate:

$$\mathcal{D}_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) = \mathcal{O}\left(\eta_0^{\frac{2\kappa}{2-\kappa}\left(\frac{2k}{d} + \frac{2(p-1)}{p}\right)}\right)$$

Outline

- ① Impulsive Noise
- ② Analysis of Tikhonov regularization
- ③ Application to Impulsive Noise
- ④ Numerical simulations**
- ⑤ Conclusion

Considered operator

- $\mathbb{M} = [0, 1]$ and $T : \mathbf{L}^2(\mathbb{M}) \rightarrow \mathbf{L}^2(\mathbb{M})$ defined by

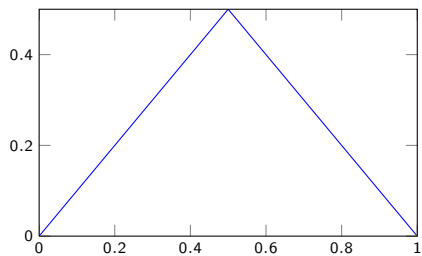
$$(Tf)(x) = \int_0^1 k(x, y) f(y) dy, \quad x \in \mathbb{M}$$

with kernel $k(x, y) = \min\{x \cdot (1 - y), y \cdot (1 - x)\}$, $x, y \in \mathbb{M}$.

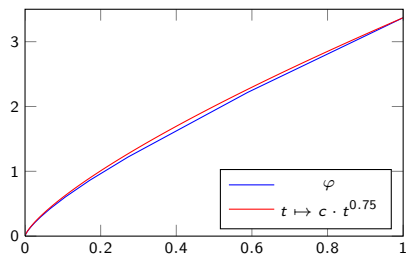
- Then $(Tf)'' = f$ for any $f \in \mathbf{L}^2(\mathbb{M})$ and T is 2 times smoothing ($k = 2$ and arbitrary $p \geq 1$).
- The smoothing Assumption is valid with any exponent $\gamma := 2k/d + 2(p - 1)/p < 6$, we use $\gamma = 6$.
- Discretization: equidistant points $x_1 = \frac{1}{2n}, x_2 = \frac{3}{2n}, \dots, x_n = \frac{2n-1}{2n}$ and composite midpoint rule

$$(Tf)(x) = \int_0^1 k(x, y) f(y) dy \approx \frac{1}{n} \sum_{i=1}^n k(x, x_i) f(x_i).$$

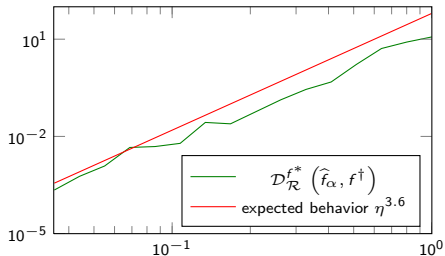
Example 1



(a) Exact solution f^\dagger

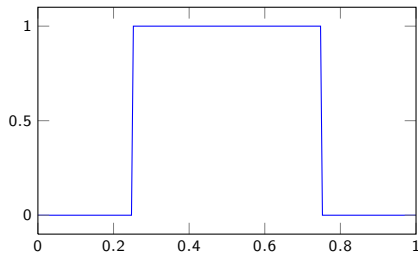
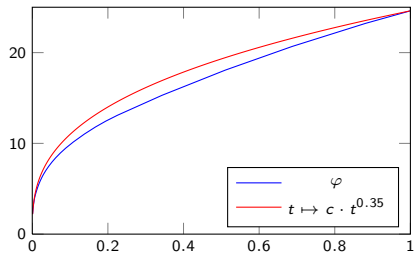
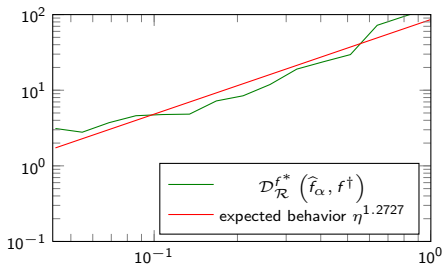


(b) Estimated index function φ



(c) mean convergence in \mathcal{X}

Example 2

(d) Exact solution f^\dagger (e) Estimated index function φ (f) mean convergence in \mathcal{X}

Outline

- ① Impulsive Noise
- ② Analysis of Tikhonov regularization
- ③ Application to Impulsive Noise
- ④ Numerical simulations
- ⑤ Conclusion

Presented results and future work

- Inverse Problems with Impulsive noise
 - continuous model for Impulsive noise
 - improved convergence rates
- numerical examples suggest order optimality
- future work: infinitely smoothing operators!

Thank you for your attention!