Convergence rates in expectation for Tikhonov-type regularization of Inverse Problems with Poisson data

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Outline

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1. Introduction
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3. Deterministic convergence analysis
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Problem setup

Measurements: Total number N and positions $x_i \in \mathbb{M}$ of photons distributed due to a unknown photon density $g^\dagger$.

Task: Determine the reason $u^\dagger$ of the photon density $g^\dagger$.

Note: The total number N of counted photons depends on the intensity of $g^\dagger$ as well as a parameter t interpreted as exposure time.

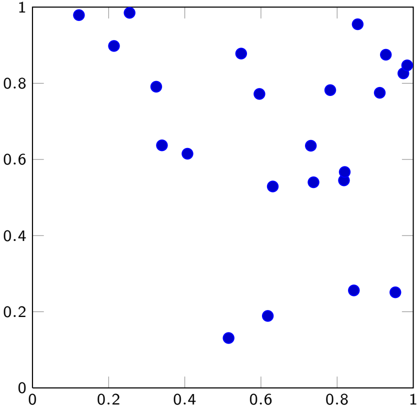
Poisson Processes

Mathematical model:

Poisson Process, i.e.

$$\tilde{G}_t = \sum_{i=1}^{N} \delta_{x_i}$$

with the following properties:



Poisson process - Axiom I

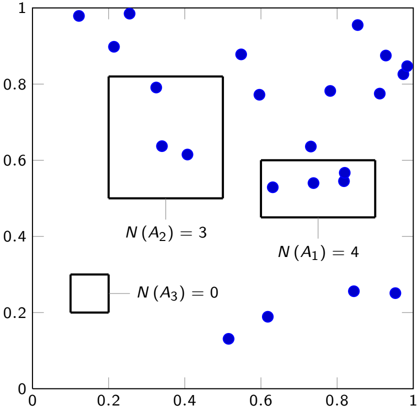
$$N(A) := \#\left\{ i \in \{1, \dots, N\} \;\middle|\; x_i \in A \right\}$$

Independence:

For any choice of $A_1, \dots, A_n \subset \mathbb{M}$ disjoint and measurable, the random variables

$$N(A_1), \dots, N(A_n)$$

are independent.



Poisson process - Axiom II

$$N(A) := \# \left\{ i \in \{1, ..., N\} \;\middle|\; x_i \in A \right\}$$

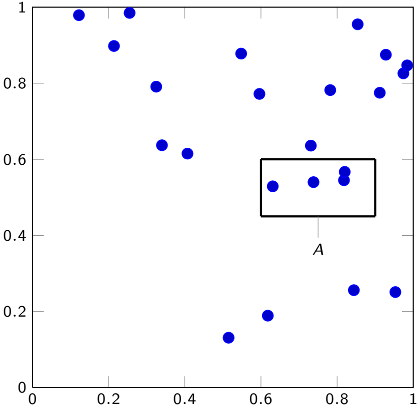
Poisson distribution:

For any measurable $A \subset \mathbb{M}$ the random variable

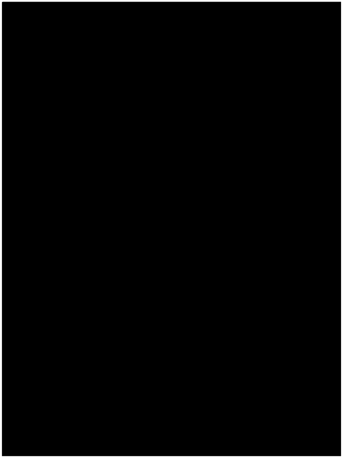
$$N(A)$$

is Poisson distributed with parameter

$$t \int_A g^\dagger \, \mathrm{d}x.$$



Influence of t



We expect 20.000 photons per second

Difficulties I

Model assumption: The imaging process can be descibed by an operator equation

$$F\left(u^{\dagger}\right) = g^{\dagger}$$

where $F : \mathfrak{B} \subset \mathcal{X} \to \mathcal{Y}$ is in general nonlinear and $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces.

The exact right-hand side $g^{\dagger}$ is unknown and in general F^{-1} is not continuous.

$\Rightarrow$ direct reconstruction impossible, regularization necessary!

Difficulties II

Several applications yield only data for small t, i.e.

- positron emission tomography (radiation exposure)
- astronomical imaging (limited observation time, motion artifacts)
- fluorescence microscopy (photobleaching)

$\Rightarrow$ use a negative log-likelihood aproach to use the infomation at hand on the Poisson distribution:

Minimize

$$u \mapsto \mathcal{S}\left(\tilde{G}_t; F(u)\right) := -\ln\left(\mathbf{P}\left[\tilde{G}_t \;\middle|\; \text{the exact photon density is } F(u)\right]\right)$$

over all admissible u.

Approach

We consider a possibly nonlinear,

ill-posed problem with Poisson data.

A. Antoniadis and J. Bigot.
Poisson inverse problems.
Ann. Statist., 34(5):2132–2158, **2006**.

J. M. Bardsley.
A Theoretical Framework for the Regularization of Poisson Likelihood Estimation Problems.
Inverse Probl. Imag., 4:11–17, **2010**.

Tackle the problem with Tikhonov-type regularization:

$$u_\alpha \in \operatorname*{argmin}_{u \in \mathfrak{B}} \left[\mathcal{S}\left(\tilde{G}_t; F(u)\right) + \alpha \mathcal{R}(u) \right]$$

where $\mathcal{R}$ is a convex penalty term and $\alpha > 0$ a regularization parameter.

Outline

Data fidelity terms

Scaled data $G_t = \frac{1}{t} \sum_{i=1}^{N} \delta_{x_i}$, $tG_t = \tilde{G}_t$ Poisson process.

- Negative log-likelihood:

$$\mathcal{S}\left(G_t; g\right) = \int_{\mathbb{M}} g \, \mathrm{d}x - \int_{\mathbb{M}} \ln\left(g\right) \, \mathrm{d}G_t, \qquad g \geq 0 \text{ a.e.}$$

- It holds $\mathbf{E}\left[\mathcal{S}\left(G_t; g\right)\right] = \int_{\mathbb{M}} \left[g - g^\dagger \ln\left(g\right)\right] \, \mathrm{d}x$
- $\rightsquigarrow$ ideal data misfit functional for exact data $g^\dagger$ given by

$$\mathbf{E}\left[\mathcal{S}\left(G_t; g\right)\right] - \mathbf{E}\left[\mathcal{S}\left(G_t; g^\dagger\right)\right] = \int_{\mathbb{M}} \left[g - g^\dagger - g^\dagger \ln\left(\frac{g}{g^\dagger}\right)\right] \, \mathrm{d}x$$

  which is the Kullback-Leibler divergence $\mathbb{KL}\left(g^\dagger; g\right)$.
- Error at g:

$$\left|\mathcal{S}\left(G_t; g\right) - \mathbf{E}\left[\mathcal{S}\left(G_t; g^\dagger\right)\right] - \mathbb{KL}\left(g^\dagger; g\right)\right| = \left|\int_{\mathbb{M}} \ln(g) \left(\mathrm{d}G_t - g^\dagger \, \mathrm{d}x\right)\right|.$$

Controlling the error I

We obtain the following concentration inequality based on

P. Reynaud-Bouret.
Adaptive estimation of the intensity of inhomogeneous Poisson processes via concentration inequalities.
Probab. Theory Rel., 126(1):103–153, 2003.

Uniform concentration inequality (W., Hohage 2012)

- $\mathbb{M} \subset \mathbb{R}^d$ bounded and Lipschitz,
- $B_s(R) := \left\{ \mathfrak{g} \in H^s(\mathbb{M}) \;\middle|\; \|\mathfrak{g}\|_{H^s(\mathbb{M})} \leq R \right\}$ with $s > d/2, R > 1$.

Then there exists $C_{\mathrm{conc}} = C_{\mathrm{conc}}\left(\mathbb{M}, s, g^\dagger\right) \geq 1$ such that

$$\mathbf{P}\left[\sup_{\mathfrak{g} \in B_s(R)} \left| \int_{\mathbb{M}} \mathfrak{g}\left(\mathrm{d}G_t - g^\dagger \,\mathrm{d}x\right)\right| \leq \frac{\rho}{\sqrt{t}}\right] \geq 1 - \exp\left(-\frac{\rho}{RC_{\mathrm{conc}}}\right)$$

for all $t \geq 1$ and $\rho \geq RC_{\mathrm{conc}}$.

Controlling the error II

- Concentration inequality requires $\mathfrak{g} \in H^s(\mathbb{M}) \subset \mathbf{L}^\infty(\mathbb{M})$ due to $s > d/2$
- Error at $g = F(u)$ leads to $\mathfrak{g} = \ln(F(u))$

$\Rightarrow$ Too strong assumption!

$\rightsquigarrow$ Shift by $\sigma > 0$:

$$\mathcal{S}(G_t; g) := \int_{\mathbb{M}} g \,\mathrm{d}x - \int_{\mathbb{M}} \ln(g + \sigma)\, (\mathrm{d}G_t + \sigma \mathrm{d}x)$$

$$\mathcal{T}\left(g^\dagger; g\right) := \mathbb{KL}\left(g^\dagger + \sigma; g + \sigma\right)$$

- Then the error is given by

$$\left| \int_{\mathbb{M}} \ln(g + \sigma) \left(\mathrm{d}G_t - g^\dagger \,\mathrm{d}x\right) \right| .$$

Outline

Noise level I

We have two data fidelity terms:

- $\mathcal{S}$ w.r.t. the measured data g^{obs}
- $\mathcal{T}$ w.r.t. the photon density $g^{\dagger}$

As before: consider the difference between both as noise level!

Noise level

There exist constants **err** ≥ 0 and $C_{\mathrm{err}} \geq 1$ such that

$$\mathcal{S}\left(g^{\mathrm{obs}}; g\right) - \mathcal{S}\left(g^{\mathrm{obs}}; g^{\dagger}\right) \geq \frac{1}{C_{\mathrm{err}}}\mathcal{T}\left(g^{\dagger}; g\right) - \mathbf{err}$$

for all $g \in F(\mathfrak{B})$.

Noise level II

- *Classical deterministic noise model:*
  If $\mathcal{S}(g;\hat{g}) = \mathcal{T}(g;\hat{g}) = \|g-\hat{g}\|_{\mathcal{Y}}^{r}$, then $C_{\mathrm{err}} = 2^{r-1}$ and $\mathbf{err} = 2\left\|g^{\dagger} - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^{r}$.
- *Poisson data:*
  $C_{\mathrm{err}} = 1$ and

$$\mathbf{err} \geq -\int\limits_{\mathbb{M}} \ln\left(g^{\dagger}+\sigma\right)\left(\mathrm{d}G_t - g^{\dagger}\,\mathrm{d}x\right) + \int\limits_{\mathbb{M}} \ln\left(F(u)+\sigma\right)\left(\mathrm{d}G_t - g^{\dagger}\,\mathrm{d}x\right)$$

for all $u \in \mathfrak{B}$.
Uniform concentration inequality: $\mathbf{err} \leq \frac{2\rho}{\sqrt{t}}$ with probability $\geq 1-\exp(-c\rho)$ for some constant $c > 0$.

Source condition I

- Bregman distance:

$$\mathcal{D}_{\mathcal{R}}^{u^*}\left(u, u^\dagger\right) := \mathcal{R}\left(u\right) - \mathcal{R}\left(u^\dagger\right) - \left\langle u^*, u - u^\dagger \right\rangle$$

where $u^* \in \partial\mathcal{R}\left(u^\dagger\right) \subset \mathcal{X}'$.

- Use a variational inequality as source condition:

$$\beta\mathcal{D}_{\mathcal{R}}^{u^*}\left(u, u^\dagger\right) \leq \mathcal{R}\left(u\right) - \mathcal{R}\left(u^\dagger\right) + \varphi\left(\mathcal{T}\left(g^\dagger; F\left(u\right)\right)\right)$$

for all $u \in \mathfrak{B}$ with $\beta > 0$. φ is assumed to fulfill
  - $\varphi\left(0\right) = 0$,
  - $\varphi \nearrow$,
  - φ concave.

Source condition II

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}\left(u, u^\dagger\right) \leq \mathcal{R}(u) - \mathcal{R}\left(u^\dagger\right) + \varphi\left(\mathcal{T}\left(g^\dagger; F(u)\right)\right)$$

- does not depend on the structure of $\mathcal{X}$ and $\mathcal{Y}$
- nonlinear F: combination of source and nonlinearity condition

$\mathcal{X}$, $\mathcal{Y}$ Hilbert spaces, $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$:

- $F(\mathfrak{B}) \subset \mathbf{L}^\infty$ bounded: spectral source + nonlinearity condition imply variational inequality (use $\left\|F(u) - g^\dagger\right\|_{\mathbf{L}^2}^2 \leq C\mathcal{T}\left(g^\dagger; F(u)\right)$)

J. M. Borwein and A. S. Lewis.
Convergence of best entropy estimates.
SIAM J. Optimization, 1:191–205, 1991.

- $\mathcal{T}(g_2; g_1) = \|g_1 - g_2\|_{\mathcal{Y}}^2$: obtained convergence rates are optimal

Deterministic convergence analysis I

Suppose

- the noise assumption is fulfilled with $\mathbf{err} \geq 0$ and
- the variational inequality holds true.

Theorem (error decomposition)

Then

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}\left(u_\alpha, u^\dagger\right) \leq \frac{\mathbf{err}}{\alpha} + (-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

for all $\alpha > 0$.

Fenchel conjugate:

$$(-\varphi)^*(s) = \sup_{\tau \geq 0}\left(s\tau + \varphi(\tau)\right).$$

Deterministic convergence analysis II

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}\left(u_\alpha, u^\dagger\right) \leq \frac{\mathbf{err}}{\alpha} + (-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

Theorem (a priori rates)

The infimum of the right-hand side it attained at $\alpha = \overline{\alpha}$ if and only if

$$\frac{-1}{C_{\mathrm{err}}\overline{\alpha}} \in \partial(-\varphi)(C_{\mathrm{err}}\,\mathbf{err}) \qquad \left[\hat{=} \quad \bar{\alpha} = \frac{1}{C_{\mathrm{err}}\varphi'\left(C_{\mathrm{err}}\,\mathbf{err}\right)}\right]$$

and in that case

$$\beta \mathcal{D}_{\mathcal{R}}^{u^*}\left(u_{\overline{\alpha}}, u^\dagger\right) \leq C_{\mathrm{err}}\varphi\left(\mathbf{err}\right).$$

Deterministic convergence analysis III

Suppose moreover $\mathcal{X}$ Hilbert space, $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$, $\beta \geq \frac{1}{2}$. Set

- $r > 1$
- $\alpha_j := \mathbf{err}\, r^{2j-2}$ for $j = 2, ..., m$ such that $\alpha_{m-1} < 1 \leq \alpha_m$
- $j_{\mathrm{bal}} := \max \left\{ j \leq m \;\middle|\; \left\| u_{\alpha_i} - u_{\alpha_j} \right\|_{\mathcal{X}} \leq 4\sqrt{2} r^{1-i} \text{ for all } i < j \right\}$

Theorem (a posteriori rates)

Then for $\mathbf{err} > 0$ sufficiently small:

$$\left\| u_{\alpha_{j_{\mathrm{bal}}}} - u^\dagger \right\|_{\mathcal{X}}^2 \leq 6r \min_{j=1,\ldots,m} \left[\frac{\mathbf{err}}{\alpha_j} + (-\varphi)^* \left(-\frac{1}{C_{\mathrm{err}} \alpha_j} \right) \right].$$

If $\varphi^{1+\varepsilon}$ is additionally concave ($\varepsilon > 0$), then

$$\left\| u_{\alpha_{j_{\mathrm{bal}}}} - u^\dagger \right\|_{\mathcal{X}}^2 \leq 6r^{1+\frac{1}{\varepsilon}} C_{\mathrm{err}} \varphi(\mathbf{err})$$

as $\mathbf{err} \searrow 0$.

Outline

Convergence rates for known φ

Suppose

- $\mathcal{X}$ Banach space, $u^\dagger \in \mathfrak{B} \subset \mathcal{X}$ bounded, closed and convex
- $\mathbb{M} \subset \mathbb{R}^d$ bounded and Lipschitz
- $F(u) \geq 0$ a.e. for all $u \in \mathfrak{B}$
- there exists a Sobolev index $s > \frac{d}{2}$ such that $F(\mathfrak{B})$ is a bounded subset of $H^s(\mathbb{M})$

A priori convergence rates (W., Hohage 2012)

Then for $\alpha = \alpha(t)$ such that $\frac{1}{\alpha} \in -\partial(-\varphi)\left(\frac{1}{\sqrt{t}}\right)$ we obtain the convergence rate

$$\mathbf{E}\left[\mathcal{D}_{\mathcal{R}}^{u^*}\left(u_\alpha, u^\dagger\right)\right] = \mathcal{O}\left(\varphi\left(\frac{1}{\sqrt{t}}\right)\right), \qquad t \to \infty.$$

Convergence rates for unknown φ

Suppose moreover $\mathcal{X}$ Hilbert space, $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$, $\beta \geq \frac{1}{2}$, $\varphi^{1+\varepsilon}$ concave ($\varepsilon > 0$). Set

- $r > 1$, $\tau > 0$ sufficiently large
- $\alpha_j := \frac{\tau \ln(t)}{\sqrt{t}} r^{2j-2}$ for $j = 2, ..., m$ such that $\alpha_{m-1} < 1 \leq \alpha_m$
- $j_{\mathrm{bal}} := \max \left\{ j \leq m \;\middle|\; \left\|u_{\alpha_i} - u_{\alpha_j}\right\|_{\mathcal{X}} \leq 4\sqrt{2} r^{1-i} \text{ for all } i < j \right\}$

A posteriori convergence rates (W., Hohage 2012)

Then we obtain

$$\mathbf{E}\left[\left\|u_{\alpha_{j_{\mathrm{bal}}}} - u^\dagger\right\|_{\mathcal{X}}^2\right] = \mathcal{O}\left(\varphi\left(\frac{\ln(t)}{\sqrt{t}}\right)\right) \qquad \text{as} \qquad t \to \infty.$$

Adaptivity causes a loss of a logarithmic factor!

A. Tsybakov.
On the best rate of adaptive estimation in some inverse problems.
C. R. Acad. Sci. Paris, 330:835–840, 2000.

Outline

Presented results

- Improvements in the theory of inverse problems with Poisson data:
  - convergence and convergence rates
  - generalized source conditions
  - a priori and a posteriori parameter choice
- regularization theory with general data fidelity terms

F. Werner and T. Hohage.
Convergence rates in expectation for Tikhonov-type regularization of Inverse Problems with Poisson data.
Inverse Problems, to appear, 2012.

Thank you for your attention!