Notes for the minicourse on *-algebras

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Last update: September 2025

Contents

1 *-Algebras: definition and first examples

Let us fix some notation first: $\mathbb{F}$ denotes any one of the fields of real or complex numbers, $\mathbb{R}$ or $\mathbb{C}$; the natural numbers are $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If X and Y are two sets, then X^Y denotes the set of maps from Y to X. The empty set is denoted $\emptyset$, and $X^\emptyset$ contains exactly one element, the *empty map*, also denoted by $\emptyset$, so $X^\emptyset = \{\emptyset\}$. For simplicity we write $X^d := X^{\{1,\dots,d\}}$ and $X^{d\times d} := X^{\{1,\dots,d\}\times\{1,\dots,d\}}$, where $d \in \mathbb{N}_0$; in this case the argument of the map is usually put in the index, e.g. $j \mapsto v_j$ for $j \in \{1, \dots, d\}$ and $v \in \mathbb{F}^d$.

1.1 Vector spaces

We all know what an $\mathbb{F}$-*vector space* is, and what a *linear map* $\Phi\colon V \to W$ between two $\mathbb{F}$-vector spaces V and W is. A *linear isomorphism* is an invertible linear map. For example, for every set X, the set $\mathbb{F}^X$ of all maps from X to $\mathbb{F}$ with the pointwise operations is a vector space.

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Exercise 1.1 *Let $\Phi\colon V \to W$ be an invertible map between two $\mathbb{F}$-vector spaces. Show that its inverse $\Phi^{-1}\colon W \to V$ is also linear.*

A *linear subspace* of an $\mathbb{F}$-vector space V is a subset U of V such that $0 \in U$ and $\lambda v + \mu w \in U$ for all $\lambda, \mu \in \mathbb{F}$; $v, w \in U$; so U is again an $\mathbb{F}$-vector space. For example the *kernel*

$$\ker \Phi := \Phi^{-1}(\{0\}) = \left\{ v \in V \;\middle|\; \Phi(v) = 0 \right\} \subseteq V \tag{1.1}$$

and *image*

$$\operatorname{img} \Phi := \Phi(V) = \{ \Phi(v) \mid v \in V \} \subseteq W \tag{1.2}$$

of a linear map $\Phi\colon V \to W$ are linear subspaces (of V and W, respectively). If S is any subset of an $\mathbb{F}$-vector space V, then

$$\langle\!\langle S \rangle\!\rangle_{\mathrm{lin}} := \left\{ \sum\nolimits_{j=1}^{n} \lambda_j v_j \;\middle|\; n \in \mathbb{N}_0;\ \lambda_1, \dots, \lambda_n \in \mathbb{F};\ v_1, \dots, v_n \in S \right\} \tag{1.3}$$

is the *linear subspace generated by* S, which is the smallest linear subspace of V that contains S, i.e.:

i.) $\langle\!\langle S \rangle\!\rangle_{\mathrm{lin}}$ is a linear subspace of V such that $S \subseteq \langle\!\langle S \rangle\!\rangle_{\mathrm{lin}}$,

ii.) $\langle\!\langle S \rangle\!\rangle_{\mathrm{lin}} \subseteq U$ for every linear subspace U of V that fulfils $S \subseteq V$.

For example, consider any set X. For $i \in X$ define $e_i \in \mathbb{F}^X$ as

$$e_i(x) := \begin{cases} 1 & \text{if } x = i \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Then

$$\mathbb{F}^{(X)} := \langle\!\langle \{ e_i \mid i \in X \} \rangle\!\rangle_{\mathrm{lin}} = \left\{ f \in \mathbb{F}^X \;\middle|\; f(x) = 0 \text{ for all but finitely many } x \in X \right\}. \tag{1.5}$$

Note that $\mathbb{F}^{(X)} = \mathbb{F}^X$ if and only if X is finite!

Exercise 1.2 *Let $\Phi\colon V \to W$ be a linear map between two $\mathbb{F}$-vector spaces V and W. Let S be a subset of V such that $\Phi(v) = 0$ for all $v \in S$. Show that $\Phi(v) = 0$ for all $v \in \langle\!\langle S \rangle\!\rangle_{\mathrm{lin}}$, but without using* (1.3) *directly!*

A *basis* of an $\mathbb{F}$-vector space V is a subset B of V such that for every $v \in V$ there is a unique element $\lambda \in \mathbb{F}^{(B)}$, $b \mapsto \lambda_b$, such that

$$v = \sum_{b \in B} \lambda_b b. \tag{1.6}$$

Note that the sum in (1.6) is actually finite, because $\lambda_b = 0$ for all but finitely many $b \in B$! Moreover, the assignment $V \ni v \mapsto \lambda \in \mathbb{F}^{(B)}$ is a linear isomorphism! If the basis B is indexed by a set I, e.g. $B = \{b_1, \dots, b_d\}$ for some $d \in \mathbb{N}_0$, or $B = \{ b_j \mid j \in \mathbb{N} \}$, then we can equivalently write

$$v = \sum_{j=1}^{d} \lambda_j b_j \quad \text{or} \quad v = \sum_{j=1}^{\infty} \lambda_j b_j \quad \text{or} \quad v = \sum_{j \in I} \lambda_j b_j \tag{1.7}$$

with $\lambda \in \mathbb{F}^d$ or $\lambda \in \mathbb{F}^{(\mathbb{N})}$, or, in general, $\lambda \in \mathbb{F}^{(I)}$. For example, the set of monomials $\{x^0, x^1, x^2, \dots\}$ is a basis of the $\mathbb{F}$-vector space of polynomials $\mathbb{F}[x]$. For any set X the maps $\{ e_i \mid i \in X \}$ are the *standard basis* of the vector space $\mathbb{F}^{(X)}$.

If B is a basis of the $\mathbb{F}$-vector space V and $\phi\colon B \to W$ any map into any $\mathbb{F}$-vector space W, then there exists a unique linear map $\Phi\colon V \to W$ such that $\Phi(b) = \phi(b)$ for all $b \in B$: this is the *linear extension* of ϕ,

$$\Phi\Big(\sum\nolimits_{b\in B} \lambda_b b\Big) := \sum_{b\in B} \lambda_b \phi(b) \tag{1.8}$$

for all $\lambda \in \mathbb{F}^{(B)}$.

1.2 Algebras

Having dealt with the most important notions for vector spaces we can move on and endow vector spaces with an additional structure, namely with a product:

Definition 1.1 (algebras and their morphisms) *An (associative)* $\mathbb{F}$-algebra *is an* $\mathbb{F}$*-vector space* $\mathcal{A}$ *together with a* product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ab$, *such that the following properties are fulfilled:*

i.) The product is bilinear, i.e. $(\lambda a + \lambda' a')b = \lambda(ab) + \lambda'(a'b)$ *and* $b(\lambda a + \lambda' a') = \lambda(ba) + \lambda'(ba')$ *holds for all* $\lambda, \lambda' \in \mathbb{F}$ *and* $a, a', b \in \mathcal{A}$.

ii.) The product is associative, i.e. $(ab)c = a(bc)$ *for all* $a, b, c \in \mathcal{A}$.

iii.) There exists a unit $1 \in \mathcal{A}$ *such that* $1a = a = a1$ *for all* $a \in \mathcal{A}$.

An algebra morphism *between* $\mathbb{F}$*-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *is a linear map* $\Phi\colon \mathcal{A} \to \mathcal{B}$ *such that* $\Phi(1) = 1$ *and* $\Phi(aa') = \Phi(a)\Phi(a')$ *for all* $a, a' \in \mathcal{A}$. *An* algebra isomorphism *is an invertible algebra morphism.*

For example, the square matrices $\mathbb{F}^{d\times d}$ with $d \in \mathbb{N}$, the polynomials $\mathbb{F}[x]$, or the functions $\mathbb{F}^X$ with the pointwise product are $\mathbb{F}$-algebras. Evaluating polynomials on a fixed number $t \in \mathbb{F}$ gives an algebra morphism $\mathbb{F}[x] \to \mathbb{F}$, $p \mapsto p(t)$.

Exercise 1.3 *Let* $\mathcal{A}$ *be an* $\mathbb{F}$*-algebra. Show that its unit is uniquely determined, i.e.: If* $1, 1' \in \mathcal{A}$ *fulfil* $1a = a = a1$ *and* $1'a = a = a1'$ *for all* $a \in \mathcal{A}$, *then* $1 = 1'$.

Exercise 1.4 *Let* $\Phi\colon \mathcal{A} \to \mathcal{B}$ *be an invertible algebra morphism between two* $\mathbb{F}$*-algebras* $\mathcal{A}$ *and* $\mathcal{B}$. *Show that its inverse* $\Phi^{-1}\colon \mathcal{B} \to \mathcal{A}$ *is again an algebra morphism!*

Definition 1.2 (subalgebras and ideals) *Let* $\mathcal{A}$ *be an* $\mathbb{F}$*-algebra.*

- *A* subalgebra *of* $\mathcal{A}$ *is a linear subspace* $\mathcal{B}$ *of* $\mathcal{A}$ *such that* $1 \in \mathcal{B}$ *and* $bb' \in \mathcal{B}$ *for all* $b, b' \in \mathcal{B}$.
- *An* ideal *of* $\mathcal{A}$ *is a linear subspace* I *of* $\mathcal{A}$ *such that* $ab \in I$ *and* $ba \in I$ *for all* $a \in \mathcal{A}$, $b \in I$.

A subalgebra of an $\mathbb{F}$-algebra is again an $\mathbb{F}$-algebra. For example, if $\Phi\colon \mathcal{A} \to \mathcal{B}$ is an algebra morphism between two $\mathbb{F}$-algebras $\mathcal{A}$ and $\mathcal{B}$, then its image $\operatorname{img}\Phi$ is a subalgebra of $\mathcal{B}$ and its kernel $\ker\Phi$ is an ideal of $\mathcal{A}$. Let S be a subset of an $\mathbb{F}$-algebra $\mathcal{A}$. Then

$$\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{alg}} := \Big\langle\!\!\Big\langle\, \big\{\, a_1 \cdots a_n \;\big|\; n \in \mathbb{N}_0;\; a_1, \dots, a_n \in S \,\big\} \,\Big\rangle\!\!\Big\rangle_{\mathrm{lin}} \tag{1.9}$$

is the *subalgebra of $\mathcal{A}$ that is generated by S, and*

$$\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{id}} := \langle\!\langle\, \big\{\, bac \;\big|\; a \in S;\; b, c \in \mathcal{A} \,\big\} \,\rangle\!\rangle_{\mathrm{lin}} \tag{1.10}$$

is the *ideal of $\mathcal{A}$ that is generated by S.* Like for the linear subspace generated by a set we note that $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{alg}}$ is the smallest subalgebra of $\mathcal{A}$ that contains S, and $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{id}}$ is the smallest ideal of $\mathcal{A}$ that contains S.

Exercise 1.5 *Let X be any set. Is $\mathbb{F}^{(X)}$ a subalgebra or an ideal of $\mathbb{F}^X$?*

Exercise 1.6 *Show that the only ideals of $\mathbb{F}^{2\times 2}$ are $\{0\}$ and $\mathbb{F}^{2\times 2}$!*

1.3 *-Algebras

Now we can finally give the central definition of this course:

Definition 1.3 (*-algebras and their morphisms) *A* *-algebra *is a $\mathbb{C}$-algebra $\mathcal{A}$ together with a map $\cdot^*\colon \mathcal{A} \to \mathcal{A}$ called* *-involution *such that the following properties are fulfilled:*

i.) *$\cdot^*$ is antilinear, i.e. $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $\lambda, \mu \in \mathbb{C}$ and $a, b \in \mathcal{A}$, where $\overline{\cdot}$ denotes complex conjugation.*

ii.) *$\cdot^*$ is an involution, i.e. $(a^*)^* = a$ for all $a \in \mathcal{A}$.*

iii.) *$(ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$.*

We write $\mathcal{A}_{\mathrm{h}} := \{\, a \in \mathcal{A} \mid a = a^* \,\}$ for the set of hermitian *elements of $\mathcal{A}$. A* *-morphism*, or *-morphism for short, between $\mathbb{F}$-algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra morphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ such that $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$. A* *-isomorphism *is an invertible *-morphism.*

Note that the hermitian elements $\mathcal{A}_{\mathrm{h}}$ of a *-algebra $\mathcal{A}$ are a real linear subspace, i.e. a linear subspace of $\mathcal{A}$ interpreted as a real vector space.

Typical examples of *-algebras are the complex square matrices $\mathbb{C}^{d\times d}$ with $d \in \mathbb{N}$, polynomials $\mathbb{C}[x]$, or functions $\mathbb{C}^X$. Here the *-involutions are defined as follows:

- On $\mathbb{C}^{d\times d}$, the combination of matrix transposition and elementwise complex conjugation: $(a^*)_{i,j} := \overline{a_{i,j}}$ for all $a \in \mathbb{C}^{d\times d}$, $i, j \in \{1, \dots, d\}$.
- On $\mathbb{C}[x]$, complex conjugation of coefficients (so $x^* = x$): $\big(\sum_{n=0}^{\infty} p_n x^n\big)^* := \sum_{n=0}^{\infty} \overline{p_n} x^n$ for all $\sum_{n=0}^{\infty} p_n x^n \in \mathbb{C}[x]$ with coefficients $p_n \in \mathbb{C}$.
- On $\mathbb{C}^X$, pointwise complex conjugation: $f^*(x) := \overline{f(x)}$ for all $f \in \mathbb{F}^X$, $x \in X$.

Exercise 1.7 *Let $\mathcal{A}$ be a *-algebra. Show that its unit 1 is hermitian!*

Exercise 1.8 *Let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be an invertible *-morphism between two *-algebras $\mathcal{A}$ and $\mathcal{B}$. Show that its inverse $\Phi^{-1}\colon \mathcal{B} \to \mathcal{A}$ is again a *-morphism!*

Definition 1.4 (*-subalgebras and *-ideals) *Let $\mathcal{A}$ be a *-algebra.*

- *Consider a subset S of $\mathcal{A}$, then we write $S^* := \{ a^* \mid a \in S \}$, and we say that S is* stable under $\cdot^*$ *if $S = S^*$.*
- *A* *-subalgebra *of $\mathcal{A}$ is a subalgebra that is stable under $\cdot^*$.*
- *A* *-ideal *of $\mathcal{A}$ is an ideal that is stable under $\cdot^*$.*

For example, if $\Phi \colon \mathcal{A} \to \mathcal{B}$ is a *-morphism between two *-algebras $\mathcal{A}$ and $\mathcal{B}$, then its image $\operatorname{img} \Phi$ is a *-subalgebra of $\mathcal{B}$ and its kernel $\ker \Phi$ is a *-ideal of $\mathcal{A}$. Let S be any subset of a *-algebra $\mathcal{A}$. If $S = S^*$, then $\langle\!\langle S \rangle\!\rangle_{\mathrm{alg}}$ is a *-subalgebra of $\mathcal{A}$ and $\langle\!\langle S \rangle\!\rangle_{\mathrm{id}}$ is a *-ideal of $\mathcal{A}$. In the general case we write

$$\langle\!\langle S \rangle\!\rangle_{*\text{-alg}} := \langle\!\langle S \cup S^* \rangle\!\rangle_{\mathrm{alg}} \qquad \text{and} \qquad \langle\!\langle S \rangle\!\rangle_{*\text{-id}} := \langle\!\langle S \cup S^* \rangle\!\rangle_{\mathrm{id}}, \tag{1.11}$$

for the **-subalgebra* and **-ideal*, respectively, of $\mathcal{A}$ *generated by* S. Then $\langle\!\langle S \rangle\!\rangle_{*\text{-alg}}$ is the smallest *-subalgebra of $\mathcal{A}$ that contains S and $\langle\!\langle S \rangle\!\rangle_{*\text{-id}}$ is the smallest *-ideal of $\mathcal{A}$ that contains S.

Exercise 1.9 *What is the *-ideal of $\mathbb{C}^{2\times 2}$ that is generated by the matrix $\left(\begin{smallmatrix} 0 & 3 \\ 2 & 5 \end{smallmatrix}\right)$?*

1.4 Free *-algebras

We want to construct the "most general" *-algebra $\mathbb{C}\langle X \rangle$ that contains a given set X of hermitian elements.

Definition 1.5 (words) *Let X be a set. The set of* words in the alphabet X *is*

$$X^\bullet := \bigcup_{\ell \in \mathbb{N}_0} X^\ell \tag{1.12}$$

and the length $|w| \in \mathbb{N}_0$ *of some $w \in X^\bullet$ is the unique number such that $w \in X^{|w|}$. We compose two words $v, w \in X^\bullet$ in the obvious way: $vw \in X^{|v|+|w|}$ is given by*

$$(vw)_n = \begin{cases} v_n & \text{if } n \in \{1, \dots, |v|\} \\ w_{n-|v|} & \text{if } n \in \{|v|+1, \dots, |v|+|w|\} \end{cases} \tag{1.13}$$

For any word $w \in X^\bullet$ we define its reversal $w^* \in X^{|w|} \subseteq X^\bullet$ in the obvious way, $(w^*)_j := w_{|w|+1-j}$ for all $j \in \{1, \dots, |w|\}$.

The composition of words is associative, $u(vw) = (uv)w$ for all $u, v, w \in X^\bullet$, and the empty map (or *empty word*) $\emptyset \in X^0$ fulfils $\emptyset w = w = w\emptyset$ for all $w \in X^\bullet$. Recall that $X^0 = \{\emptyset\}$, and X^1 is the set of all maps from $\{1\}$ to X, which we can identify with the set X itself. Then every word $w \in X^\bullet$ can be expressed as a composition of $|w|$ single-letter words. E.g. $pizza \in \{a, i, p, z\}^5$ and $(pizza)^* = azzip$.

Proposition 1.6 *Let X be any set and $\{ e_w \mid w \in X^\bullet \}$ the standard basis of $\mathbb{C}^{(X^\bullet)}$. Then $\mathbb{C}^{(X^\bullet)}$ with the bilinear product that is determined by $e_v e_w := e_{vw}$ for all $v, w \in X^\bullet$ is a $\mathbb{C}$-algebra. This*

$\mathbb{C}$-algebra $\mathbb{C}^{(X^\bullet)}$ becomes a *-algebra by defining the *-involution as

$$\Big(\sum\nolimits_{w\in X^\bullet} a_w e_w\Big)^* := \sum_{w\in X^\bullet} \overline{a_w} e_{w^*} \tag{1.14}$$

for all $\sum_{w\in X^\bullet} a_w e_w \in \mathbb{C}^{(X^\bullet)}$ with coefficients $a_w \in \mathbb{C}$.

Proof: Associativity of the product follows from associativity of the composition of words, and the unit is $1 = e_\emptyset \in \mathbb{C}^{(X^\bullet)}$. It is straightforward to check that (1.14) defines a *-involution that turns $\mathbb{C}^{(X^\bullet)}$ into a *-algebra. □

Definition 1.7 (free *-algebra) *Let X be a set, then we write $\mathbb{C}\langle X\rangle$ for the *-algebra $\mathbb{C}^{(X^\bullet)}$ from the previous Proposition 1.6. We call $\mathbb{C}\langle X\rangle$ the* free *-algebra in hermitian variables X*. Moreover, for all $w \in X^\bullet$ we simply write $w := e_w \in \mathbb{C}\langle X\rangle$, this way we identify the set $X^\bullet$ with the standard basis $\{\, e_w \mid w \in X^\bullet \,\}$ of $\mathbb{C}^{(X^\bullet)}$.*

Elements of the free *-algebra in hermitian variables X can simply be expressed as sums of products of elements $x \in X \subseteq \mathbb{C}\langle X\rangle$. Clearly $x^* = x$ for all $x \in X$. For example, $2pizza + wine \in \mathbb{C}\langle\{a, e, i, n, p, w, z\}\rangle$.

Clearly $\mathbb{C}\langle X\rangle$ is generated by X, i.e. $\langle\!\langle\, X \,\rangle\!\rangle_{*\text{-alg}} = \mathbb{C}\langle X\rangle$. The next Proposition 1.8 shows that $\mathbb{C}\langle X\rangle$ is actually *freely* generated by X:

Proposition 1.8 *Let X be a set, $\mathcal{A}$ a *-algebra, and $\phi\colon X \to \mathcal{A}_{\mathrm{h}}$ any map. Then there exists a unique *-morphism $\Phi\colon \mathbb{C}\langle X\rangle \to \mathcal{A}$ such that $\Phi(x) = \phi(x)$ for all $x \in X$.*

Proof: This *-morphism is given by

$$\Phi\Big(\sum\nolimits_{w\in X^\bullet} a_w e_w\Big) := \sum\nolimits_{w\in X^\bullet} a_w \phi(w_1)\ldots\phi(w_{|w|})$$

for all $\sum_{w\in X^\bullet} a_w e_w \in \mathbb{C}\langle X\rangle$ with coefficients $a_w \in \mathbb{C}$. It is straightforward to check that Φ is indeed a *-morphism, and if $\Psi\colon \mathbb{C}\langle X\rangle \to \mathcal{A}$ is any other *-morphism that fulfils $\Psi(x) = \phi(x)$ for all $x \in X$, then $\Psi = \Phi$ because $\{\, a \in \mathbb{C}\langle X\rangle \mid \Psi(a) = \Phi(a) \,\}$ clearly is a *-subalgebra of $\mathbb{C}\langle X\rangle$ that contains X; so $\{\, a \in \mathbb{C}\langle X\rangle \mid \Psi(a) = \Phi(a) \,\} = \mathbb{C}\langle X\rangle$ because $\mathbb{C}\langle X\rangle$ is generated by X. □

Note how similar this property of the free *-algebras is to the property fulfilled by the basis of a vector space: Every map ϕ from X to the hermitian elements of any *-algebra can be extended in a unique way to the whole *-algebra $\mathbb{C}\langle X\rangle$. The existence of such an extension can heuristically be interpreted as follows:

The elements of the subset X of $\mathbb{C}\langle X\rangle$ are not subject to any relations.

In many cases the set X is finite, say $X = \{x_1, \ldots, x_n\}$ with $n \in \mathbb{N}$. Then we write $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ for $\mathbb{C}\langle X\rangle$. One can interpret this *-algebra $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ as the *-algebra of *noncommutative polynomials* in the (noncommutative) variables $x_1, \ldots, x_n$. For any *-algebra $\mathcal{A}$ and $a_1, \ldots, a_n \in \mathcal{A}_{\mathrm{h}}$ the extension $\Phi\colon \mathbb{C}\langle x_1, \ldots, x_n\rangle \to \mathcal{A}$, $p \mapsto \Phi(p)$ like in Proposition 1.8 of the map $\phi\colon \{x_1, \ldots, x_n\} \mapsto \mathcal{A}_{\mathrm{h}}$, $x_j \mapsto \phi(x_j) := a_j$ is then the *evaluation* of the noncommutative polynomials p on $a_1, \ldots, a_n$.

Example 1.9 Consider $p \coloneqq x_1x_2 - x_2x_1 \in \mathbb{C}\langle x_1, x_2\rangle$ and the hermitian 3×3-matrices

$$a_1 \coloneqq \begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad a_1 \coloneqq \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.15}$$

Then evaluating p on a_1, a_2 yields

$$p(a_1, a_2) \coloneqq \Phi(p) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}, \tag{1.16}$$

where $\Phi \colon \mathbb{C}\langle x_1, x_2\rangle \to \mathbb{C}^{3\times 3}$ is the extension of the map $\{x_1, x_2\} \to (\mathbb{C}^{3\times 3})_{\mathrm{h}}$, $x_j \mapsto a_j$ for $j \in \{1, 2\}$ to a *-morphism. In contrast to this, while the *-algebra $\mathbb{C}^{2\times 2}$ is generated by the two hermitian elements $S \coloneqq \big\{\big(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\big), \big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big)\big\}$, it is not freely generated because these generators fulfil additional relations like $\big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big)^2 = 1$. Consequently the map $\phi \colon S \to \mathbb{C}^{3\times 3}$, $\phi\big(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\big) \coloneqq a_1, \phi\big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big) \coloneqq a_2$ cannot be extended to a *-morphism $\mathbb{C}^{2\times 2} \to \mathbb{C}^{3\times 3}$.

Exercise 1.10 *Show that $\mathbb{C}^{2\times 2}$ is indeed generated by $\big\{\big(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\big), \big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big)\big\}$ as claimed in Example 1.9.*

1.5 *-Algebras of generators and relations

By now we know how to construct the "most general" *-algebra generated by a set of elements. However, most problems deal with *-algebras that fulfil additional relations (like commutation or anticommutation relations), so we want to construct the "most general" *-algebra that fulfils such relations. E.g. we might want to construct a *-algebra $\mathcal{A}$ generated by two elements $p, q \in \mathcal{A}_{\mathrm{h}}$ such that $pq - qp = \mathrm{i}$, or equivalently $pq - qp - \mathrm{i} = 0$. Then necessarily $a(pq - qp - \mathrm{i})b = 0$ for all $a, b \in \mathcal{A}$, and actually any element of $\langle\!\langle \{pq - qp - \mathrm{i}\} \rangle\!\rangle_{*\text{-id}}$ is 0. Moreover, if two elements $a, b \in \mathcal{A}$ fulfil $a - b \in \langle\!\langle \{pq - qp - \mathrm{i}\} \rangle\!\rangle_{*\text{-id}}$, then actually $a = b$.

The way to guarantee that such relations hold is via a quotient construction. We first recall quotients of vector spaces:

Proposition 1.10 *Let V be an $\mathbb{F}$-vector space and U a linear subspace of V. For every $v \in V$ let*

$$[v] \coloneqq v + U \coloneqq \{\, v + u \mid u \in U \,\} \subseteq V \tag{1.17}$$

be the corresponding affine subspace through v, and write

$$V/U \coloneqq \big\{\, [v] \;\big|\; v \in V \,\big\} \tag{1.18}$$

for the set of all these affine subspaces. Then the following holds:

i.) Given $v, v' \in V$, then $[v] = [v']$ if and only if $v - v' \in U$.

ii.) The maps $\mathbb{F} \times V/U \to V/U$,

$$\big(\lambda, [v]\big) \mapsto \lambda[v] \coloneqq [\lambda v] \tag{1.19}$$

and $V/U \times V/U \to V/U$,

$$\big([v],[w]\big) \mapsto [v]+[w] := [v+w] \tag{1.20}$$

are welldefined.

iii.) The set V/U with scalar multiplication and addition defined by (1.19) and (1.20) is an $\mathbb{F}$-vector space.

iv.) The map $[\,\cdot\,]\colon V \to V/U$ is linear, surjective, and $\ker[\,\cdot\,] = U$.

v.) For every linear map $\Phi\colon V \to W$ into any $\mathbb{F}$-vector space W that fulfils $U \subseteq \ker\Phi$ there is a unique linear map $\check{\Phi}\colon V/U \to W$ such that $\Phi = \check{\Phi} \circ [\,\cdot\,]$:

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d, "{[\,\cdot\,]}"'] & W \\ V/U \arrow[ur, "\check{\Phi}"'] & \end{tikzcd}$$

Proof: Assume $v, v' \in V$ fulfil $[v] = [v']$, then in particular $v' \in [v]$, so there is $u \in U$ such that $v' = v + u$, i.e. $v - v' = u \in U$. Conversely, if any $v, v' \in V$ fulfil $v - v' \in U$, then a general element $v + u \in [v]$ with $u \in U$ fulfils $v + u = v' + u + (v - v') \in [v']$, so $[v] \subseteq [v']$, and analogously also $[v'] \subseteq [v]$. This proves part *i*.

For part *ii* we have to check that the definitions in (1.19) and (1.20) do not depend on the choice of representative v (and w): Assume $v, v' \in V$ fulfil $[v] = [v']$, i.e. $v - v' \in U$, then for all $\lambda \in \mathbb{F}$ the relation $\lambda v - \lambda v' = \lambda(v - v') \in U$ holds and shows that $[\lambda v] = [\lambda v']$. Similarly, if $v, v', w, w' \in V$ fulfil $[v] = [v']$ and $[w] = [w']$, then $v - v' \in U$ and $w - w' \in U$ imply $(v+w)-(v'+w') = (v-v')+(w-w') \in U$, so $[v + w] = [v' + w']$.

Part *iii* is now straightforward, V/U fulfils all axioms of an $\mathbb{F}$-vectorspace because V does; e.g. $[0]$ is the neutral element of addition because for all $[v] \in V/U$ with representative $v \in V$ the identity $[0] + [v] = [0 + v] = [v]$ holds by the definition of addition on V/U and because the corresponding axiom $0 + v = v$ holds for the vector space V.

Part *iv* is an immediate consequence of the definitions, e.g. $\ker[\,\cdot\,] = U$ because an element $v \in V$ fulfils $[v] = [0]$ if and only if $v - 0 \in U$, i.e. $v \in U$.

Finally, part *v*: Consider an arbitrary linear map $\Phi\colon V \to W$ into any $\mathbb{F}$-vector space W that fulfils $U \subseteq \ker\Phi$. We want to define a map $\check{\Phi}\colon V/U \to W$ as $\check{\Phi}([v]) := \Phi(v)$ for every $[v] \in V/U$ with representative v. In order to do so we have to check that this definition does not depend on the choice of representative $v \in V$ of $[v]$: if $v, v' \in V$ fulfil $[v] = [v']$, then $v - v' \in U$ so that $\Phi(v) = \Phi(v' + (v - v')) = \Phi(v') + \Phi(v - v') = \Phi(v')$. So this map $\check{\Phi}\colon V/U \to W$ is welldefined and one easily checks that $\check{\Phi}$ is linear because Φ is linear. Any map $\check{\Phi}'\colon V/U \to W$ that fulfils $\Phi = \check{\Phi}' \circ [\,\cdot\,]$ necessarily fulfils $\check{\Phi}'([v]) = \Phi(v) = \check{\Phi}([v])$ for all $[v] \in V/U$ with representative $v \in V$, so $\check{\Phi}' = \check{\Phi}$, which shows that $\check{\Phi}$ is uniquely determined by this property. □

Corollary 1.11 *Let $\mathcal{A}$ be an $\mathbb{F}$-algebra and I an ideal of $\mathcal{A}$. Then the quotient vector space $\mathcal{A}/I$ with multiplication defined by $\mathcal{A}/I \times \mathcal{A}/I \to \mathcal{A}/I$,*

$$\big([a],[b]\big) \mapsto [a][b] := [ab] \tag{1.21}$$

becomes an $\mathbb{F}$-algebra. The map $[\,\cdot\,]\colon \mathcal{A} \to \mathcal{A}/I$ then is an algebra morphism, and for every algebra morphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ into any $\mathbb{F}$-algebra $\mathcal{B}$ that fulfils $I \subseteq \ker \Phi$ the corresponding map $\check{\Phi}\colon \mathcal{A}/I \to \mathcal{B}$ is an algebra morphism.

Similarly, if $\mathcal{A}$ is a *-algebra and I a *-ideal, then $\mathcal{A}/I$ with *-involution defined by $[a]^* := [a^*]$ becomes a *-algebra. In this case $[\,\cdot\,]\colon \mathcal{A} \to \mathcal{A}/I$ then is a *-morphism, and $\check{\Phi}\colon \mathcal{A}/I \to \mathcal{B}$ is also a *-morphism provided that $\mathcal{B}$ is a *-algebra and Φ a *-morphism.

Proof: We only have to check that the multiplication and *-involution on $\mathcal{A}/I$ are welldefined, the rest is then straightforward: Assume $a, a', b, b' \in \mathcal{A}$ fulfil $[a] = [a']$ and $[b] = [b']$, i.e. $a - a', b - b' \in I$, then $ab - a'b' = (a - a')b + a'(b - b') \in I$, so $[ab] = [a'b']$. Proving that the *-involution is welldefined is left as an exercise. □

Exercise 1.11 *Complete the proof of Corollary 1.11!*

We can now finally define the most important (for us) type of examples of *-algebras:

Definition 1.12 (*-algebra of generators and relations) *Let X be a set and $R \subseteq \mathbb{C}\langle R\rangle$. The* *-algebra generated by X with relations R *is*

$$\mathbb{C}\langle\, X \mid R \,\rangle := \mathbb{C}\langle X\rangle / \langle\!\langle\, R \,\rangle\!\rangle_{*\text{-id}}. \tag{1.22}$$

Example 1.13 Now it is time for examples. We might use some sloppy notation, e.g. drop some brackets or define relations as equalities "$a = b$" and not as elements $a - b$ that are supposed to be zero.

i.) The Weyl *-algebra

$$\mathcal{W}_1 := \mathbb{C}\Big\langle\, q, p \;\Big|\; qp - pq = \mathrm{i} \,\Big\rangle := \mathbb{C}\Big\langle\, \{q, p\} \;\Big|\; \{qp - pq - \mathrm{i}\rangle\} \,\Big\rangle. \tag{1.23}$$

Higher dimensional Weyl algebras $\mathcal{W}_n$, $n \in \mathbb{N}$ with n position and momentum operators can be defined analogously.

ii.) The *-algebra of $n \in \mathbb{N}$ fermions

$$\mathcal{F}_n := \mathbb{C}\left\langle\, a_{1,r}, a_{1,i}, \dots, a_{n,r}, a_{n,i} \;\middle|\; \begin{array}{ll} a_k a_\ell + a_\ell a_k = 0 & \text{for all } k, \ell \in \{1, \dots, n\} \\ a_k a_\ell^* + a_\ell^* a_k = \delta_{k,\ell} & \text{for all } k, \ell \in \{1, \dots, n\} \end{array} \right\rangle. \tag{1.24}$$

where we write $a_k := a_{k,r} + \mathrm{i} a_{k,i}$.

iii.) The *-algebra of $n \in \mathbb{N}$ commuting spins

$$\mathcal{S}_n := \mathbb{C}\left\langle\, \begin{array}{l} s_{i,j} \text{ with} \\ i \in \{1, \dots, n\}, j \in \{1, 2, 3\} \end{array} \;\middle|\; \begin{array}{l} s_{i,1}s_{i,2}s_{i,3} = \mathrm{i},\, s_{i,j}^2 = 1 \text{ and } s_{i,j}s_{i',j'} = s_{i',j'}s_{i,j} \\ \text{for } i, i' \in \{1, \dots, n\}; j, j' \in \{1, 2, 3\} \text{ with } i \neq i' \end{array} \right\rangle. \tag{1.25}$$

iv.) The *-algebra of a PVM with $n \in \mathbb{N}$ projectors

$$\mathbb{C}\left\langle p_1, \dots, p_n \;\middle|\; p_1 + \dots + p_n = 1 \text{ and } p_i p_j = \delta_{ij} p_j \text{ for all } i, j \in \{1, \dots, n\} \right\rangle, \tag{1.26}$$

where $\delta_{i,j}$ is the Kronecker-δ.

v.) The CHSH *-algebra

$$\mathcal{A}_{CHSH} \coloneqq \mathbb{C}\left\langle a_1, a_2, b_1, b_2 \;\middle|\; a_1^2 = a_2^2 = b_1^2 = b_2^2 = 1 \text{ and } a_i b_j = b_j a_i \text{ for all } i, j \in \{1, 2\} \right\rangle. \tag{1.27}$$

vi.) The Toeplitz *-algebra, i.e. the *-algebra generated by one isometry $a \coloneqq a_r + \mathrm{i} a_i$:

$$\mathbb{C}\left\langle a_r, a_i \;\middle|\; (a_r - \mathrm{i} a_i)(a_r + \mathrm{i} a_i) = 1 \right\rangle. \tag{1.28}$$

vii.) The complexification of the universal enveloping algebra of the $\mathfrak{su}(2)$-Lie algebra:

$$\mathcal{U}^*\big(\mathfrak{su}(2)\big) \coloneqq \mathbb{C}\left\langle e_1, e_2, e_3 \;\middle|\; e_k e_\ell - e_\ell e_k = 2\mathrm{i} \sum\nolimits_{m=1}^{3} \epsilon_{k,\ell,m} e_m \right\rangle \tag{1.29}$$

with $\epsilon_{k,\ell,m} \in \{-1, 0, 1\}$ for $k, \ell, m \in \{1, 2, 3\}$ the Levi-Civita-symbol, i.e. $\epsilon_{k,\ell,m} = 1$ for even permutations (k, ℓ, m) of $(1, 2, 3)$, $\epsilon_{k,\ell,m} = -1$ for odd permutations, and otherwise $\epsilon_{k,\ell,m} = 0$.

viii.) A *-algebra with inconsistent relations

$$\mathbb{C}\left\langle x \;\middle|\; x = 1 \text{ and } x = 2 \right\rangle = \big\{[0]\big\} \tag{1.30}$$

because $1 = (x-1) - (x-2) \in \left\langle\!\left\langle \{x-1, x-2\} \right\rangle\!\right\rangle_{*\text{-id}}$ implies that $\left\langle\!\left\langle \{x-1, x-2\} \right\rangle\!\right\rangle_{*\text{-id}} = \mathbb{C}\langle x \rangle$.

And many, many, more... In order to simplify our notation we will often drop the brackets $[\,\cdot\,]$ and denote the generators of $\mathbb{C}\langle X \mid R \rangle$ by $x \in X \subseteq \mathbb{C}\langle X \mid R \rangle$ instead of $[x]$. This is of course only possible if it is clear from the context that we are not talking about $x \in X \subseteq \mathbb{C}\langle X \rangle$.

Remark 1.14 What can we do at this point with these purely algebraic examples? Here we drop the brackets $[\,\cdot\,]$ for simplicity:

- We can e.g. show by induction that every element of the Weyl algebra can be expressed as a linear combination of $p^k q^\ell$ because by taking any word in p and q, and then moving the p's to the left, we make only a mistake of lower degree. In the case of the *-algebras of fermions we can even express any element as a linear combination of $a_{k_1} \dots a_{k_r} a_{\ell_1}^* \dots a_{\ell_s}^*$ with $1 \leq k_1 < \dots k_r \leq n$ and $1 \leq \ell_1 < \dots \ell_2 \leq n$, by a similar argument like for the Weyl algebra and as $a_k^2 = (a_\ell^*)^2 = 0$ for all $k, \ell \in \{1, \dots, n\}$. In particular, the *-algebra of n fermions $\mathcal{F}_n$ is finite-dimensional. Similarly for the *-algebra of n spins.

- Showing that the above smaller sets of (linear) generators are linearly independent, hence a basis, is more complicated! How can we know that we did not miss some argument that can be used to reduce these sets even more?

- We can derive general algebraic identities, e.g. in $\mathcal{U}^*\big(\mathfrak{su}(2)\big)$ the *Casimir element* $\Omega := \sum_{k=1}^3 e_k^2$ is central. Namely, with $[\,\cdot\,,\,\cdot\,]$ the commutator bracket, $[\,a\,,b\,] := ab - ba$, it holds that

$$[\,\Omega\,,e_\ell\,] = \sum_{k=1}^3 \big(e_k[\,e_k\,,e_\ell\,] + [\,e_k\,,e_\ell\,]e_k\big) = \sum_{k=1}^3\sum_{m=1}^3 2\mathrm{i}\epsilon_{k,\ell,m}(e_k e_m + e_m e_k) = 0 \tag{1.31}$$

by antisymmetry of $\epsilon_{k,\ell,m}$ in k and m; therefore $[\,\Omega\,,a\,] = 0$ for all $a \in \mathcal{U}^*\big(\mathfrak{su}(2)\big)$.

The *-algebras of the form type $\mathbb{C}\langle\, X \mid R\,\rangle$ can be characterized by a "universal property" that essentially goes like this: Whenever $\mathcal{A}$ is a *-algebra that contains a "similar" set of hermitian elements $X' \subseteq \mathcal{A}_\mathrm{h}$ that fulfil the relations R (in the obvious sense), then there exists a unique *-morphism $\check{\Phi}\colon \mathbb{C}\langle\, X \mid R\,\rangle \to \mathcal{A}$ that maps the elements in $[x]$ with $x \in X \subseteq \mathbb{C}\langle X\rangle$ to the corresponding elements of X'. We discuss this in more detail at an easy example:

Proposition 1.15 *Let $\mathcal{A}$ be a *-algebra that contains two hermitian elements $p', q' \in \mathcal{A}_\mathrm{h}$ such that $p'q' - q'p' = \mathrm{i}$. Then there exists a unique *-morphism $\check{\Phi}\colon \mathcal{W}_1 \to \mathcal{A}$ such that $\check{\Phi}([p]) = p'$ and $\check{\Phi}([q]) = q'$.*

Proof: Let $\Phi\colon \mathbb{C}\langle p,q\rangle \to \mathcal{A}$ be the unique *-morphism from Proposition 1.8 that fulfils $\Phi(p) = p'$ and $\Phi(q) = q'$. From $p'q' - q'p' = \mathrm{i}$ it follows that $\Phi(pq - qp - \mathrm{i}) = 0$, and so $\langle\!\langle\, \{pq - qp - \mathrm{i}\}\,\rangle\!\rangle_{*\text{-id}} \subseteq \ker\Phi$. By part *v* of Proposition 1.10 there is a unique linear map $\check{\Phi}\colon \mathcal{W}_1 = \mathbb{C}\langle p,q\rangle / \langle\!\langle\, \{pq - qp - \mathrm{i}\}\,\rangle\!\rangle_{*\text{-id}} \to \mathcal{A}$ such that $\Phi = \check{\Phi} \circ [\,\cdot\,]$, in particular $\check{\Phi}([p]) = \Phi(p) = p'$ and $\check{\Phi}([q]) = \Phi(q) = q'$. By Corollary 1.11, $\check{\Phi}$ is a *-morphism.

Moreover, whenever a *-morphism $\Psi\colon \mathcal{W}_1 \to \mathcal{A}$ fulfils $\Psi([p]) = p'$ and $\Psi([q]) = q'$, then $\Psi \circ [\,\cdot\,] = \Phi$ by the uniqueness statement for Φ, and consequently $\Psi = \check{\Phi}$ by the uniqueness statement for $\check{\Phi}$. □

Exercise 1.12 *Assume $\mathcal{B}$ is another *-algebra that fulfils a property analogous to the one of $\mathcal{W}_1$ from Proposition 1.15, namely, assume that*

i.) there are $\tilde{p}, \tilde{q} \in \mathcal{B}_\mathrm{h}$ that fulfil $\tilde{p}\tilde{q} - \tilde{q}\tilde{p} = \mathrm{i}$ and that

ii.) for any *-algebra $\mathcal{A}$ that contains two hermitian elements $p', q' \in \mathcal{A}_\mathrm{h}$ with $p'q' - q'p' = \mathrm{i}$ there exists a unique *-morphism $\check{\Phi}\colon \mathcal{B} \to \mathcal{A}$ such that $\check{\Phi}(\tilde{p}) = p'$ and $\check{\Phi}(\tilde{q}) = q'$.

Then there are two unique, mutually inverse *-isomorphisms $\Phi\colon \mathcal{W}_1 \to \mathcal{B}$ and $\Psi\colon \mathcal{B} \to \mathcal{W}_1$ that fulfil $\Phi([p]) = \tilde{p}$, $\Phi([q]) = \tilde{q}$, and $\Psi(\tilde{p}) = [p]$, $\Psi(\tilde{q}) = [q]$.

Hint: Show that $\Phi \circ \Psi = \mathrm{id}_\mathcal{B}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathcal{W}_1}$, where $\mathrm{id}_\mathcal{B}\colon \mathcal{B} \to \mathcal{B}$ and $\mathrm{id}_{\mathcal{W}_1}\colon \mathcal{W}_1 \to \mathcal{W}_1$ are the *identity maps* that map any element (of $\mathcal{B}$ or $\mathcal{W}_1$, respectively) to itself. Note that there are 4 *-morphisms involved in these identities, and that the properties of $\mathcal{W}_1$ and $\mathcal{B}$ indeed allow to construct 4 *-morphisms. We also have to make use of the uniqueness parts! □

This is the justification for calling the statement of from Proposition 1.15 the "universal" property of the Weyl *-algebra $\mathcal{W}_1$: it completely determines $\mathcal{W}_1$ up to unique *-isomorphism. Analogous results hold for all similar examples as well.

2 *-Representations and positive functionals

*-Algebras can be used to describe the observables of a physical system. The states of the system are then given by the dual structure: (normalized) positive linear functionals.

2.1 *-Representations

Definition 2.1 (sesquilinear forms, pre-Hilbert spaces, and adjointable maps) *Let V be a $\mathbb{C}$-vector space. A* sesquilinear form *on V is a map $s\colon V \times V \to \mathbb{C}$ that is antilinear in the first argument and linear in the second argument. We say that such a sesquilinear form is*

- hermitian *if $\overline{s(v,w)} = s(w,v)$ for all $v, w \in V$,*
- positive *if s is hermitian and $s(v,v) \geq 0$ for all $v \in V$,*
- non-degenerate *if $s(v,w) = 0$ for one $v \in V$ and all $w \in V$ implies $v = 0$.*

A pre-Hilbert space *is a $\mathbb{C}$-vector space $\mathcal{D}$ equipped with a positive and non-degenerate sesquilinear form $\langle \cdot \,|\, \cdot \rangle\colon \mathcal{D} \times \mathcal{D} \to \mathbb{C}$, which is called the* inner product *on $\mathcal{D}$. We then write $\mathcal{L}^*(\mathcal{D})$ for the space of* adjointable *maps on $\mathcal{D}$, i.e. $a\colon \mathcal{D} \to \mathcal{D}$ is an element of $\mathcal{L}^*(\mathcal{D})$ if there exists an* adjoint map *$a^*\colon \mathcal{D} \to \mathcal{D}$ such that $\langle a^*(\phi) \,|\, \psi \rangle = \langle \phi \,|\, a(\psi) \rangle$ holds for all $\phi, \psi \in \mathcal{D}$.*

The letter $\mathcal{D}$ is for "domain" (of an operator). Note, however, that there also exists a notion of an adjoint operator, which is not equivalent to the above definition of the adjoint map!

For every pre-Hilbert space $\mathcal{D}$, the space $\mathcal{L}^*(\mathcal{D})$ is a *-algebra. Details are left as an exercise:

Exercise 2.1 *Let $\mathcal{D}$ be a pre-Hilbert space.*

i.) Show that every adjointable map $a\colon \mathcal{D} \to \mathcal{D}$ is automatically linear, and that its adjoint a^* is uniquely determined and adjointable!

ii.) Show that the composition of two adjointable maps $a, b\colon \mathcal{D} \to \mathcal{D}$ is adjointable with $(a \circ b)^* = b^* \circ a^*$!

iii.) Show that $\mathcal{L}^*(\mathcal{D})$ is a linear subspace of $\mathcal{D}^{\mathcal{D}}$ with pointwise addition and scalar multiplication!

iv.) Show that $\mathcal{L}^*(\mathcal{D})$ with composition as multiplication, and with the map $a \mapsto a^*$ from the first part as *-involution, is a *-algebra! What is the unit of $\mathcal{L}^*(\mathcal{D})$?

For the product in $\mathcal{L}^*(\mathcal{D})$ we simply write $ab \coloneqq a \circ b$ as usual.

The assumption of positivity of the inner product is actually irrelevant here, but non-degeneracy is crucial! Yet non-degeneracy of a hermitian sesquilinear form is easy to enforce:

Proposition 2.2 *Let V be a $\mathbb{C}$-vector space and $s\colon V \times V \to \mathbb{C}$ a hermitian sesquilinear form on V. Write*

$$\ker s \coloneqq \big\{ v \in V \;\big|\; s(w,v) = 0 \text{ for all } w \in V \big\}. \tag{2.1}$$

Then $\ker s$ *is a linear subspace of* V*, and the map* $\check{s}\colon (V/\ker s)\times(V/\ker s)\to\mathbb{C}$*,*

$$\big([v],[w]\big)\mapsto \check{s}\big([v],[w]\big) := s(v,w) \tag{2.2}$$

is a welldefined non-degenerate hermitian sesquilinear form on the quotient vector space $V/\ker s$*.*

Proof: Consider $v, v', w, w' \in V$ such that $[v]=[v']$ and $[w]=[w']$. We have to show that $s(v,w) = s(v',w')$: indeed, $v-v'\in\ker s$ and $w-w'\in\ker s$ (part *i* of Proposition 1.10), and therefore

$$s(v,w) = s(v',w') + s(v',w-w') + s(v-v',w) = s(v',w') + s(v',w-w') + \overline{s(w,v-v')} = s(v',w').$$

This shows that the map $\check{s}\colon (V/\ker s)\times(V/\ker s)\to\mathbb{C}$ is welldefined. It is straightforward to check that $\check{s}$ is again sesquilinear and hermitian because s is. In addition, $\check{s}$ is non-degenerate: If $[v]\in V/\ker s$ fulfils $\check{s}([v],[w]) = 0$ for all $[w]\in V/\ker s$, then $s(w,v) = \overline{s(v,w)} = 0$ for all $w\in V$, i.e. $v\in\ker s$, which means that $[v]=[0]$. □

Note that for a hermitian sesquilinear forms s on a $\mathbb{C}$-vector space V, its *kernel* $\ker s$ as in (2.1) can equivalently be described as

$$\ker s = \big\{\, v\in V \;\big|\; s(v,w) = 0 \text{ for all } w\in V \,\big\}. \tag{2.3}$$

Corollary 2.3 *Let* V *be a* $\mathbb{C}$*-vector space and* $s\colon V\times V\to\mathbb{C}$ *a positive sesquilinear form on* V*. Then* $\mathcal{D} := V/\ker s$ *with inner product* $\langle\,\cdot\,|\,\cdot\,\rangle := \check{s}\colon \mathcal{D}\times\mathcal{D}\to\mathbb{C}$ *is a pre-Hilbert space.*

Proof: By definition, s is in particular hermitian, so the previous Proposition 2.2 applies. It is straightforward to check that positivity of s implies positivity of $\check{s}$. □

Example 2.4 Write $\mathscr{L}^2(\mathbb{R})$ for the linear subspace of $\mathbb{C}^{\mathbb{R}}$ consisting of all those $\mathbb{C}$-valued functions ψ on $\mathbb{R}$ that are measurable and fulfil $\int_{\mathbb{R}}|\psi(x)|^2\,\mathrm{d}x<\infty$. Then

$$\int_{\mathbb{R}}\big|\overline{\phi(x)}\psi(x)\big|\,\mathrm{d}x \le \int_{\mathbb{R}}\frac{|\phi(x)|^2+|\psi(x)|^2}{2}\,\mathrm{d}x<\infty \tag{2.4}$$

for all $\phi,\psi\in\mathscr{L}^2(\mathbb{R})$, so $\overline{\phi}\psi$ is integrable and therefore the map $s := \mathscr{L}^2(\mathbb{R})\times\mathscr{L}^2(\mathbb{R})\to\mathbb{C}$,

$$(\phi,\psi)\mapsto s(\phi,\psi) := \int_{\mathbb{R}}\overline{\phi(x)}\psi(x)\,\mathrm{d}x \tag{2.5}$$

is welldefined. Clearly s is a positive sesquilinear form on $\mathscr{L}^2(\mathbb{R})$, so the previous Corollary 2.3 applies and shows that the quotient vector space $\mathrm{L}^2(\mathbb{R}) := \mathscr{L}^2(\mathbb{R})/\ker s$ is a pre-Hilbert space (actually even a Hilbert space, but this is irrelevant here).

Exercise 2.2 *Let* $s\colon V\times V\to\mathbb{C}$ *be a sesquilinear map on a* $\mathbb{C}$*-vector space* V*.*

i.) Show that the polarization identiy

$$s(v,w) = \frac{1}{4}\sum_{k=0}^{3}\mathrm{i}^{-k}s\big(v+\mathrm{i}^k w, v+\mathrm{i}^k w\big) \tag{2.6}$$

holds for all $v, w \in V$!

ii.) Show that s is hermitian if and only if $s(v,v) \in \mathbb{R}$ for all $v \in V$!

iii.) Show that s is positive if and only if $s(v,v) \in [0,\infty[$ for all $v \in V$!

The most important property of positive sesquilinear forms is the following:

Proposition 2.5 (Cauchy-Schwarz inequality) *Let V be a $\mathbb{C}$-vector space and $s\colon V \times V \to \mathbb{C}$ a positive sesquilinear form on V. Then for all $v, w \in V$ the inequality*

$$|s(v,w)|^2 \leq s(v,v)\, s(w,w) \tag{2.7}$$

holds. In particular if $s(v,v) = 0$ for one $v \in V$, then $s(v,w) = 0$ for all $w \in V$, i.e. $v \in \ker s$ as in Proposition 2.2. If s is in addition non-degenerate, then this implies $v = 0$.

Proof: For all $\lambda, \mu \in \mathbb{C}$ it holds that

$$\begin{pmatrix} \overline{\lambda} & \overline{\mu} \end{pmatrix} \underbrace{\begin{pmatrix} s(v,v) & s(v,w) \\ s(w,v) & s(w,w) \end{pmatrix}}_{=:M\in\mathbb{C}^{2\times 2}} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = s\big(\lambda v + \mu w, \lambda v + \mu w\big) \geq 0.$$

As the matrix M is hermitian, it is diagonalizable with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, and the above inequality implies that in fact $\lambda_1, \lambda_2 \in [0,\infty[$. Therefore $s(v,v)s(w,w) - |s(v,w)|^2 = \det(M) = \lambda_1\lambda_2 \geq 0$. □

The *-algebras $\mathcal{L}^*(\mathcal{D})$ serve as the prototypical example of *-algebras, and we define:

Definition 2.6 (*-representation) *Let $\mathcal{A}$ be a *-algebra. A* *-representation *of $\mathcal{A}$ is a tuple $(\mathcal{D}, \pi)$ of a pre-Hilbert space $\mathcal{D}$ and a *-morphism $\pi\colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$. In this case we say that π is a* *-representation of $\mathcal{A}$ on $\mathcal{D}$*.*

Exercise 2.3 *Let $\mathcal{D}$ be a pre-Hilbert space and $A_1, A_2, B_1, B_2 \in \mathcal{L}^*(\mathcal{D})$ such that $A_i^2 = B_j^2 = 1$ and $A_i B_j = B_j A_i$ hold for all $\{i,j\} \in \{1,2\}$. Show that there exists a unique *-representation π of the CHSH *-algebra $\mathcal{A}_{\mathrm{CHSH}}$ (part v of Example 1.13) on $\mathcal{D}$ such that $\pi(a_i) = A_i$ and $\pi(b_j) = B_j$ for $i,j \in \{1,2\}$.*

Theorem 2.7 (GNS-construction) *Let $\mathcal{A}$ be a *-algebra and $\omega\colon \mathcal{A} \to \mathbb{C}$ a linear map. Assume that $\omega(a^*) = \overline{\omega(a)}$ and $\omega(a^*a) \geq 0$ hold for all $a \in \mathcal{A}$. Then $s_\omega\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$,*

$$(a,b) \mapsto s_\omega(a,b) := \omega(a^*b) \tag{2.8}$$

is a positive sesquilinear form on $\mathcal{A}$ so that Corollary 2.3 applies, i.e. the quotient vector space $\mathcal{D}_\omega := \mathcal{A}/\ker s_\omega$ with inner product $\langle\,\cdot\,|\,\cdot\,\rangle_\omega := \check{s}_\omega\colon \mathcal{D}\times\mathcal{D} \to \mathbb{C}$ like in Proposition 2.2 is a pre-Hilbert space. Moreover, for every $a \in \mathcal{A}$ the map $\pi_\omega(a)\colon \mathcal{D}_\omega \to \mathcal{D}_\omega$,

$$[b] \mapsto \pi_\omega(a)([b]) := [ab] \tag{2.9}$$

is welldefined, and $\pi_\omega(a) \in \mathcal{L}^*(\mathcal{D}_\omega)$ with $\pi_\omega(a)^* = \pi_\omega(a^*)$. The resulting map $\pi_\omega \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D}_\omega)$, $a \mapsto \pi(a)$ is a *-representation of $\mathcal{A}$ on $\mathcal{D}_\omega$.

Proof: It is straightforward to check that s_ω is a positive sesquilinear map. The construction of the pre-Hilbert space $\mathcal{D}_\omega$ then is just an application of Corollary 2.3. Note that $\langle [a] \,|\, [b] \rangle_\omega = \omega(a^*b)$ for all $[a], [b] \in \mathcal{D}_\omega$ with representatives $a, b \in \mathcal{A}$, respectively.

For arbitrary fixed $a \in \mathcal{A}$ we define the map $\hat{\pi}_\omega(a) \colon \mathcal{A} \to \mathcal{D}_\omega$, $b \mapsto \hat{\pi}_\omega(a)(b) := [ab]$. It is straightforward to check that $\hat{\pi}_\omega(a)$ is linear. If $b \in \mathcal{D}_\omega$, then $s_\omega(c, ab) = \omega(c^*ab) = 0$ for all $c \in \mathcal{A}$, so $ab \in \mathcal{D}_\omega$ and therefore $\pi_\omega(a)(b) = [ab] = [0]$. This shows that $\mathcal{D}_\omega \subseteq \ker \pi_\omega(a)$, so part *v* of Proposition 1.10 applies and shows that the map $\pi_\omega(a)$ is welldefined. A short calculation shows that

$$\langle [b] \,|\, \pi_\omega(a)([c]) \rangle_\omega = \langle [b] \,|\, [ac] \rangle_\omega = s_\omega\big(b, ac\big) = \omega(b^*ac)$$

and

$$\langle \pi_\omega(a^*)([b]) \,|\, [c] \rangle_\omega = \langle [a^*b] \,|\, [c] \rangle_\omega = s_\omega\big(a^*b, c\big) = \omega\big((a^*b)^*c\big) = \omega(b^*ac)$$

for all $[b], [c] \in \mathcal{D}_\omega$ with representatives $b, c \in \mathcal{A}$, so $\pi_\omega(a)$ is adjointable with $\pi_\omega(a)^* = \pi_\omega(a^*)$.

It is straightforward to check that the resulting map $\pi_\omega \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$ is linear. Given $a_1, a_2 \in \mathcal{A}$, then for all $[b] \in \mathcal{D}_\omega$ with representative $b \in \mathcal{A}$ the identity

$$\pi_\omega(a_2)\big(\pi_\omega(a_1)([b])\big) = \pi_\omega(a_2)\big([a_1 b]\big) = [a_2 a_1 b] = \pi_\omega(a_2 a_1)([b])$$

holds, so $\pi_\omega(a_2) \circ \pi_\omega(a_1) = \pi_\omega(a_2 a_1)$. As we have already checked that $\pi_\omega(a)^* = \pi_\omega(a^*)$ holds for any $a \in \mathcal{A}$, this map π_ω is indeed a *-representation of $\mathcal{A}$ on $\mathcal{D}_\omega$. □

Remark 2.8 The *-representations that one obtains by GNS-representation have a noteworthy property: As any *-algebra $\mathcal{A}$ contains a "special" element, namely its unit 1, the representation space $\mathcal{D}_\omega$ from the above Theorem 2.7 contains a "special" vector $\Omega := [1] \in \mathcal{D}_\omega$. This vector essentially represents the functional ω, namely

$$\langle \Omega \,|\, \pi_\omega(a)(\Omega) \rangle_\omega = \langle [1] \,|\, [a] \rangle_\omega = \omega(a) \tag{2.10}$$

holds for all $a \in \mathcal{A}$. Moreover, any element $[a] \in \mathcal{D}_\omega$ can be reached from Ω as $[a] = \pi_\omega(a)(\Omega)$. Such vectors and *-representations are called *cyclic*. In physics one often performs the GNS-representation for a functional ω that represents a state of minimal energy, or without any particles present, etc. In this case one usually refers to Ω as the "vacuum".

Example 2.9 (GNS representation for the Weyl algebra) We discuss the Weyl algebra $\mathcal{W}_1 = \mathbb{C}\langle q, p \mid qp - pq = \mathrm{i} \rangle$. For simplicity we drop brackets indicating equivalence classes and write $p := [p] \in \mathcal{W}_1$ and $q := [q] \in \mathcal{W}_1$. Set $a := (q + \mathrm{i}p)/\sqrt{2}$, then $q = (a + a^*)/\sqrt{2}$ and $p = -\mathrm{i}(a - a^*)/\sqrt{2}$ hold, and therefore $\mathcal{W}_1$ is generated as a *-algebra by a. Moreover,

$$[\, a \,, a^* \,] = \frac{1}{2}[\, q + \mathrm{i}p \,, q - \mathrm{i}p \,] = \frac{1}{2}\big(-\mathrm{i}[\, q \,, p \,] + \mathrm{i}[\, p \,, q \,]\big) = 1, \tag{2.11}$$

which is equivalent to the relation $[\, q \,, p \,] = \mathrm{i}$. Then one can show the following:

i.) The elements $(a^*)^k a^\ell$ with $k, \ell \in \mathbb{N}_0$ are a basis of $\mathcal{W}_1$.

(This is actually quite tricky to prove if one pretends to not already know any *-representation of $\mathcal{W}_1$: Using identity (2.11) one can inductively show that $\{ (a^*)^k a^\ell \mid k, \ell \in \mathbb{N}_0 \}$ generates $\mathcal{W}_1$ as a $\mathbb{C}$-vector space, because reordering factors only introduces commutators that are elements of lower degree. But we also have to show that these elements are linearly independent, which is harder because it requires to show that some elements of $\mathbb{C}\langle q, p\rangle$ are *not* in $\langle\!\langle \{qp - pq - \mathrm{i}\} \rangle\!\rangle_{*\text{-id}}$! The idea is to show that up to any fixed degree, the Weyl algebra $\mathcal{W}_1 = \mathbb{C}\langle\, q, p \mid qp - pq = \mathrm{i}\,\rangle$ has the same dimension as the polynomial algebra $\mathbb{C}[p, q] = \mathbb{C}\langle\, q, p \mid qp - pq = 0\,\rangle$, because the *-ideals $\langle\!\langle \{qp - pq - \mathrm{i}\} \rangle\!\rangle_{*\text{-id}}$ and $\langle\!\langle \{qp - pq\} \rangle\!\rangle_{*\text{-id}}$ contain elements that are identical up to lower degree terms. Alternatively, if one already knows a *-representation $(\mathcal{D}, \pi)$ of $\mathcal{W}_1$ in which the elements $(a^*)^k a^\ell$ with $k, \ell \in \mathbb{N}_0$ are represented by linearly independent operators $\pi\big((a^*)^k a^\ell\big) \in \mathcal{L}^*(\mathcal{D})$, then it follows that the elements $(a^*)^k a^\ell$ also must be linearly independent.)

ii.) Define the linear functional $\omega\colon \mathcal{W}_1 \to \mathbb{C}$ as

$$b \mapsto \omega(b) := \lambda_{0,0} \qquad \text{for all } b = \sum\nolimits_{k,\ell=0}^{\infty} \lambda_{k,\ell} (a^*)^k a^\ell \tag{2.12}$$

with coefficients $\lambda \in \mathbb{C}^{(\mathbb{N}_0 \times \mathbb{N}_0)}$, and let $s_\omega\colon \mathcal{W}_1 \times \mathcal{W}_1 \to \mathbb{C}$ be the sesquilinear form as in (2.8), i.e. $s_\omega(b, c) = \omega(b^* c)$ for all $b, c \in \mathcal{W}_1$. Clearly $\omega(b^*) = \overline{\omega(b)}$ for all $b \in \mathcal{W}_1$, so s_ω is hermitian. A direct calculation shows that

$$s_\omega\big((a^*)^k a^\ell, (a^*)^{k'} a^{\ell'}\big) = \delta_{\ell,0}\delta_{\ell',0}\delta_{k,k'} k!, \tag{2.13}$$

and therefore $\omega(b^* b) = \sum_{k=0}^{\infty} |\lambda_{k,0}|^2 k!$ for all $b = \sum_{k,\ell=0}^{\infty} \lambda_{k,\ell} (a^*)^k a^\ell$ with coefficients $\lambda \in \mathbb{C}^{(\mathbb{N}_0 \times \mathbb{N}_0)}$. In particular ω is positive and $\ker s_\omega = \langle\!\langle \{ (a^*)^k a^\ell \mid k, \ell \in \mathbb{N}_0;\ \ell > 0 \} \rangle\!\rangle_{\mathrm{lin}}$.

iii.) Let $(\mathcal{D}_\omega, \pi_\omega)$ be the *-representation of $\mathcal{W}_1$ from Theorem 2.7 and write $\Omega := [1] \in \mathcal{D}_\omega$. Then the vectors $\pi_\omega\big((a^*)^k\big)(\Omega) = [(a^*)^k]$ with $k \in \mathbb{N}_0$ are a basis of $\mathcal{D}_\omega$, because $\mathcal{D}_\omega = \mathcal{W}_1 / \ker s_\omega$ and because we have seen in parts *i* and *ii* that the elements $(a^*)^k a^\ell$ with $k, \ell \in \mathbb{N}_0$ form a basis of $\mathcal{W}_1$ and those with $\ell > 0$ a basis of $\ker s_\omega$.

This construction simply reproduces the Fock-space representation of the Weyl algebra: The Fock space is generated by the vectors that one obtains by repeatedly acting with the "creation operator" $\pi_\omega(a^*)$ on the "vacuum" Ω. Note, however, that there also exist many other positive functionals on $\mathcal{W}_1$! Some of these would yield *-representations that are equivalent to this Fock-space representation, or sums of Fock space representations. But there are also some that yield completely different *-representations! We might construct an example later on once we have some more powerful techniques at our disposal.

2.2 Quadratic modules

The key ingredient in the GNS-construction is a linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ that is in some sense "positive". In general there can be many different notions of positivity on a given *-algebra, this is the idea behind the following definition:

Definition 2.10 (quadratic modules) *Let $\mathcal{A}$ be a *-algebra. A* quadratic module *of $\mathcal{A}$ is a subset Q of $\mathcal{A}_{\mathrm{h}}$ that fulfils the following conditions for all $q, r \in Q$, $a \in \mathcal{A}$:*

$$q + r \in Q, \qquad a^* q\, a \in Q, \qquad \textit{and} \qquad 1 \in Q. \tag{2.14}$$

In the special case of the *-algebra $\mathcal{L}^*(\mathcal{D})$ on a pre-Hilbert space $\mathcal{D}$ we write

$$\mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}} := \left\{\, a \in \mathcal{L}^*(\mathcal{D})_{\mathrm{h}} \;\middle|\; \langle\, \phi \,|\, a(\phi)\,\rangle \geq 0 \textit{ for all } \phi \in \mathcal{D} \,\right\} \tag{2.15}$$

for its canonical quadratic module.

Note that every quadratic module Q of any *-algebra $\mathcal{A}$ fulfils $\lambda q = \sqrt{\lambda}^* q \sqrt{\lambda} \in Q$ for all $q \in Q$, $\lambda \in [0, \infty[$. It is easy to check that $\mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ is indeed a quadratic module of the *-algebra $\mathcal{L}^*(\mathcal{D})$. Another (trivial) example is the interval $[0, \infty[$, which is a quadratic module of the *-algebra $\mathbb{C}$. The general idea behind this Definition 2.10 is that a quadratic module of a *-algebra $\mathcal{A}$ can be interpreted as the elements of $\mathcal{A}_{\mathrm{h}}$ that we define to be in some sense "positive".

Example 2.11 For any $n \in \mathbb{N}$, the positive semidefinite complex $n \times n$-matrices are a quadratic module of the *-algebra $\mathbb{C}^{n \times n}$. Under the canonical identification $\mathbb{C}^{n \times n} \cong \mathcal{L}^*(\mathbb{C}^n)$ (with the standard inner product on $\mathbb{C}^n$), the positive semidefinite complex $n \times n$-matrices are just the canonical quadratic module $\mathcal{L}^*(\mathbb{C}^n)^+_{\mathrm{h}} \subseteq \mathcal{L}^*(\mathbb{C}^n)$. As discussed in the following Exercise 2.4, this is the only non-trivial quadratic module of $\mathbb{C}^{n \times n}$.

Example 2.12 Let $\mathbb{C}[x]$ be the *-algebra of polynomials in one hermitian variable x. Recall that the hermitian polynomials are those with real coefficients, $\mathbb{C}[x]_{\mathrm{h}} = \mathbb{R}[x]$. Then for every subset $\Xi \subseteq \mathbb{R}$, the set $\{\, p \in \mathbb{R}[x] \mid p(\xi) \geq 0 \text{ for all } \xi \in \Xi \,\}$ of polynomials that are pointwise positive on Ξ is a quadratic module of $\mathbb{C}[x]$.

Exercise 2.4 *Show that for any $n \in \mathbb{N}$, the only quadratic modules of the *-algebra $\mathbb{C}^{n \times n}$ are the quadratic module $\mathcal{L}^*(\mathbb{C}^n)^+_{\mathrm{h}}$ of positive semidefinite matrices and the trivial quadratic module that is given by the set $(\mathbb{C}^{n \times n})_{\mathrm{h}}$ of all hermitian matrices.*

Like for *-ideals (and others) we can generate quadratic modules as we wish: Let $\mathcal{A}$ be a *-algebra and $S \subseteq \mathcal{A}_{\mathrm{h}}$. Then we write

$$\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}} := \left\{\, \sum\nolimits_{j=1}^{n} a_j^* s_j a_j \;\middle|\; n \in \mathbb{N}_0;\, a_1, \dots, a_n \in \mathcal{A};\, s_1, \dots, s_n \in S \cup \{1\} \,\right\} \tag{2.16}$$

for the *quadratic module generated by* S. This is the smallest quadratic module of $\mathcal{A}$ that contains S. In the special case $S = \emptyset$ we write

$$\textstyle\sum \mathcal{A}^*\mathcal{A} := \langle\!\langle\, \emptyset \,\rangle\!\rangle_{\mathrm{qm}} = \left\{\, \sum\nolimits_{j=1}^{n} a_n^* a_n \;\middle|\; n \in \mathbb{N}_0;\, a_1, \dots, a_n \in \mathcal{A} \,\right\} \tag{2.17}$$

for the quadratic module of *sums of hermitian squares of* $\mathcal{A}$. This is the smallest quadratic module of $\mathcal{A}$.

Exercise 2.5 *Let $\mathcal{A}$ be a *-algebra and $S \subseteq \mathcal{A}_{\mathrm{h}}$. Check that $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}}$ is indeed a quadratic module of $\mathcal{A}$ and that $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}} \subseteq Q$ holds for any quadratic module Q of $\mathcal{A}$ that fulfils $S \subseteq Q$!*

Exercise 2.6 *We have already seen that for any *-algebra $\mathcal{A}$ there exists a smallest quadratic module $\sum \mathcal{A}^*\mathcal{A}$. Show that there also is a largest quadratic module of $\mathcal{A}$! (What is it explicitly?)*

Exercise 2.7 *Consider the *-algebra of square matrices $\mathbb{C}^{n\times n}$ and its quadratic module Q of positive semidefinite matrices.*

i.) Show that $Q = \langle\!\langle\, \emptyset \,\rangle\!\rangle_{\mathrm{qm}}$ (so in the following "positive" and "state" means "Q-positive" and "Q-state")!

ii.) Show that for every linear functional $\omega\colon \mathbb{C}^{n\times n} \to \mathbb{C}$ there exists a unique matrix $\rho \in \mathbb{C}^{n\times n}$ such that $\omega(a) = \operatorname{tr}(\rho a)$ for all $a \in \mathcal{A}$, i.e. "$\omega = \operatorname{tr}(\rho\,\cdot\,)$".

iii.) Show that a linear functional $\omega\colon \mathbb{C}^{n\times n} \to \mathbb{C}$ is hermitian if and only if the corresponding matrix $\rho \in \mathbb{C}^{n\times n}$ is hermitian!

iv.) Show that a linear functional $\omega\colon \mathbb{C}^{n\times n} \to \mathbb{C}$ is positive if and only if the corresponding matrix $\rho \in \mathbb{C}^{n\times n}$ is positive semidefinite!

v.) Show that a linear functional $\omega\colon \mathbb{C}^{n\times n} \to \mathbb{C}$ is a state if and only if the corresponding matrix $\rho \in \mathbb{C}^{n\times n}$ is a density matrix (i.e. positive semidefinite with trace 1)!

In particular, the states on a *-algebra $\mathbb{C}^{n\times n}$ can be used to describe the "state" of a (finite) quantum system, as there is a one-to-one correspondence between such states ω and density matrices ρ. Evaluating a state ω on a hermitian matrix $a \in (\mathbb{C}^{n\times n})_{\mathrm{h}}$ results in a real number $\omega(a) = \operatorname{tr}(\rho a)$, which in quantum mechanics describes the expectation value of measuring the observable a on the system in state ω.

2.3 Positive linear functionals

We can now discuss the dual notions of positivity of linear functionals that we obtain for every quadratic module:

Definition 2.13 (Positive linear functionals and states) *Let $\mathcal{A}$ be a *-algebra, Q a quadratic module of $\mathcal{A}$, and $\omega\colon \mathcal{A} \to \mathbb{C}$ a linear functional. We call ω*

- hermitian *if $\omega(a^*) = \overline{\omega(a)}$ for all $a \in \mathcal{A}$,*
- Q-positive *if ω is hermitian and $\omega(q) \geq 0$ for all $q \in Q$,*
- *a* Q-state *if ω is Q-positive and $\omega(1) = 1$.*

Moreover, a positive *linear functional on $\mathcal{A}$ is a $\sum \mathcal{A}^*\mathcal{A}$-positive linear functional, and a* state *on $\mathcal{A}$ is a $\sum \mathcal{A}^*\mathcal{A}$-state.*

Note that the positive linear functionals on a *-algebra $\mathcal{A}$ are precisely those linear functionals ω that fulfil the assumptions of Theorem 2.7, i.e. those that by GNS-construction yield a *-representation π_ω! Moreover, the states are those positive linear functionals for which the corresponding vector $\Omega = [1] \in \mathcal{D}_\omega$ (see Remark 2.8) is normalized, $\langle\, \Omega \,|\, \Omega \,\rangle = 1$, and hence can be used to describe the actual state of a physical system.

There is a Cauchy–Schwarz inequality for positive linear functionals on *-algebras:

Proposition 2.14 *Let $\mathcal{A}$ be a *-algebra and ω a positive linear functional on $\mathcal{A}$. Then*

$$|\omega(a^*b)|^2 \leq \omega(a^*a)\,\omega(b^*b) \qquad \text{for all } a, b \in \mathcal{A}, \tag{2.18}$$

and in particular

$$|\omega(b)|^2 \leq \omega(1)\,\omega(b^*b) \qquad \text{for all } b \in \mathcal{A} \tag{2.19}$$

hold. Consequently, if $\omega(1) = 0$, then $\omega = 0$.

Proof: As ω is positive, the sesquilinear map $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto \omega(a^*b) \in \mathbb{C}$ is positive. The Cauchy–Schwarz inequality from Proposition 2.5 then yields (2.18), and (2.19) is the special case for $a \coloneqq 1$. If $\omega(1) = 0$, then $\omega(b) = 0$ for all $b \in \mathcal{A}$, because $0 \leq |\omega(b)|^2 \leq \omega(1)\,\omega(b^*b) = 0$ by (2.19). □

One consequence of the above is that every non-zero positive linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ can be normalized to a state $\omega(1)^{-1}\omega$.

Not only do positive functionals on a *-algebra give rise to *-representations through the GNS-construction Theorem 2.7, the converse is also true:

Proposition 2.15 (vector states) *Let $\mathcal{A}$ be a *-algebra and $(\mathcal{D}, \pi)$ a *-representation of $\mathcal{A}$. Then for every $\psi \in \mathcal{D}$ the map $\chi_{\pi,\psi}\colon \mathcal{A} \to \mathbb{C}$,*

$$a \mapsto \chi_{\pi,\psi}(a) \coloneqq \langle\, \psi \,|\, \pi(a)(\psi) \,\rangle \tag{2.20}$$

is a positive linear functional on $\mathcal{A}$, and $\chi_{\pi,\psi}$ is a state if and only if $\langle\, \psi \,|\, \psi \,\rangle = 1$.

Proof: Let $\psi \in \mathcal{D}$ be given, then

$$\chi_{\pi,\psi}(a^*) = \langle\, \psi \,|\, \pi(a^*)(\psi) \,\rangle = \langle\, \pi(a)(\psi) \,|\, \psi \,\rangle = \overline{\langle\, \psi \,|\, \pi(a)(\psi) \,\rangle} = \overline{\chi_{\pi,\psi}(a^*)}$$

and

$$\chi_{\pi,\psi}(a^*a) = \langle\, \psi \,|\, \pi(a^*a)(\psi) \,\rangle = \langle\, \pi(a)(\psi) \,|\, \pi(a)(\psi) \,\rangle > 0$$

hold for all $a \in \mathcal{A}$, so $\chi_{\pi,\psi}$ is hermitian and positive. Clearly $\chi_{\pi,\psi}(1) = \langle\, \psi \,|\, \psi \,\rangle$, so $\chi_{\pi,\psi}$ is a state if and only if $\langle\, \psi \,|\, \psi \,\rangle = 1$.

Now assume that $\omega\colon \mathcal{A} \to \mathbb{C}$ is a positive linear functional and that $\pi = \pi_\omega$ is the corresponding GNS-representation from Theorem 2.7. Then $\chi_{\pi,[1]}(a) = \langle\, [1] \,|\, \pi_\omega(a)[1] \,\rangle_\omega = \omega(a)$. □

There is a noteworthy special case of the above Proposition 2.15: If $\pi = \pi_\omega$ is the GNS-representation as in Theorem 2.7 constructed out of a positive linear functional ω on a *-algebra $\mathcal{A}$, then $\chi_{\pi,[1]} = \omega$, because $\chi_{\pi,[1]}(a) = \langle\, [1] \,|\, \pi_\omega(a)[1] \,\rangle_\omega = \omega(a)$ holds for all $a \in \mathcal{A}$.

Remark 2.16 We now have all the tools to describe quantum physics by *-algebras and states instead of operators and (normalized) pre-Hilbert space vectors: Out of *-algebras and states, the GNS-construction allows us to construct operators on pre-Hilbert spaces. Conversely, any normalized vector ψ on a pre-Hilbert space $\mathcal{D}$ describes a vector state $\chi_{\mathrm{id},\psi}$ on $\mathcal{L}^*(\mathcal{D})$ (like in the previous Proposition 2.15 for the identity representation $\mathrm{id}\colon \mathcal{L}^*(\mathcal{D}) \to \mathcal{L}^*(\mathcal{D})$). Here are some first entries of a dictionary that may help translate between the two languages:

$$
\begin{aligned}
&\text{(element of a) }{}^*\text{-algebra} &&\rightsquigarrow&& \text{operator (i.e. adjointable map) on a pre-Hilbert spaces}\\
&\text{state }\omega\text{ on a }{}^*\text{-algebra} &&\rightsquigarrow&& \text{normalized vector }\psi\text{ of a pre-Hilbert space}\\
&f_{0,*\text{-alg}} \coloneqq \min_{\omega\text{ state on }\mathcal{A}} \omega(a) &&\rightsquigarrow&& f_{0,\text{pre-Hilb}} \coloneqq \min_{\substack{(\mathcal{D},\pi)\ {}^*\text{-representation of }\mathcal{A}\\ \psi\in\mathcal{D}\text{ with }\langle\,\psi\,|\,\psi\,\rangle=1}} \langle\,\psi\,|\,\pi(a)(\psi)\,\rangle \qquad (2.21)\\
&\frac{\mathrm{d}}{\mathrm{d}t}\omega_t(a) = \omega_t\Big(\frac{\mathrm{i}}{\hbar}[\,a\,,h\,]\Big) &&\rightsquigarrow&& \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H\psi_t \qquad (2.22)\\
&\vdots &&\rightsquigarrow&& \vdots
\end{aligned}
$$

The left-hand side of (2.22) is known as the Ehrenfest theorem, the right-hand side is of course the Schrödinger equation. But it is not always that easy:

- There are subtleties with pure and mixed states: Under the GNS-representation, every(!) state ω becomes a vector state (of $\Omega \coloneqq [1]$).
- Normalized vectors ψ and ψ' of a pre-Hilbert space that differ only by a phase factor correspond to the same state! But such vectors are usually considered to be physically equivalent.
- What corresponds to the inner product on pre-Hilbert spaces? But what is the physical meaning of the inner product anyways?
- How to describe superpositions of states in the language of *-algebras? Or orthogonality?

Exercise 2.8 *Show that the two optimization problems in* (2.21)*, Remark 2.16, are equivalent, i.e. if $f_{0,*\text{-alg}}$ exists, then also $f_{0,\text{pre-Hilb}}$ exists and $f_{0,*\text{-alg}} = f_{0,\text{pre-Hilb}}$, and vice versa.*

Exercise 2.9 *Prove the equations of motion hinted at in* (2.22)*, Remark 2.16, i.e.: Let $\mathcal{A}$ be a *-algebra and $\pi\colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$ a *-representation of $\mathcal{A}$ on some pre-Hilbert space $\mathcal{D}$. Let $h \in \mathcal{A}_{\mathrm{h}}$ and write $H \coloneqq \pi(h) \in \mathcal{L}^*(\mathcal{D})_{\mathrm{h}}$. Let $(\psi_t)_{t\in\mathbb{R}}$ be a family of normalized vectors of $\mathcal{D}$ that fulfil the Schrödinger equation $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H\psi_t$ for all $t \in \mathbb{R}$ and write $\omega_t \coloneqq \chi_{\pi,\psi_t}$ for the corresponding family of vector states on $\mathcal{A}$. Then show that for all $a \in \mathcal{A}$ the identity $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t(a) = \omega_t\big(\frac{\mathrm{i}}{\hbar}[\,a\,,h\,]\big)$ holds.*

2.4 Q-positive *-representations

One reason for introducing quadratic modules is that this allows us to talk about *-representations that fulfil additional positivity constraints:

Definition 2.17 *Let $\mathcal{A}$ be a *-algebra, Q a quadratic module of $\mathcal{A}$, and $(\mathcal{D},\pi)$ a *-representation of $\mathcal{A}$. Then we say that $(\mathcal{D},\pi)$ is* Q-positive *if $\pi(q) \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ for all $q \in Q$.*

It makes sense to consider *-representations that are positive on quadratic modules (in contrast to positive on more general sets of hermitian elements) because of the following observation:

Proposition 2.18 *Let $\mathcal{A}$ be a *-algebra and $(\mathcal{D}, \pi)$ a *-representation of $\mathcal{A}$. Consider any subset $S \subseteq \mathcal{A}_{\mathrm{h}}$. Then $(\mathcal{D}, \pi)$ is $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}}$-positive if and only if $\pi(s) \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ for all $s \in S$.*

Proof: Left as exercise. □

Exercise 2.10 *Proof the previous Proposition 2.18.*

Example 2.19 We can describe the properties of a POVM (positive operator valued measure) consisting of $n \in \mathbb{N}$ operators abstractly through the *-algebra $\mathcal{A} \coloneqq \mathbb{C}\langle\, x_1, \dots, x_n \mid x_1 + \dots + x_n = 1 \,\rangle$ equipped with the quadratic module

$$Q \coloneqq \langle\!\langle\, \{[x_1], \dots, [x_n]\} \,\rangle\!\rangle_{\mathrm{qm}}. \tag{2.23}$$

Then for every Q-positive *-representation $(\mathcal{D}, \pi)$ of $\mathcal{A}$ the elements $\pi([x_1]), \dots, \pi([x_n])$ form a POVM on $\mathcal{D}$, i.e. $\pi([x_1]), \dots, \pi([x_n]) \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ and $\sum_{i=1}^n \pi([x_i]) = \mathrm{id}_{\mathcal{D}}$ (the identity operator). Conversely, for every POVM $p_1, \dots, p_n$ on any pre-Hilbert space $\mathcal{D}$ there is a unique *-morphism $\check{\Phi} \colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$ that fulfils $\check{\Phi}([x_i]) = p_i$ for all $i \in \{1, \dots, n\}$ (namely the one obtained from $\Phi \colon \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathcal{L}^*(\mathcal{D})$, $x_i \mapsto \Phi(x_i) \coloneqq p_i$ as in part *v* of Proposition 1.10, analogously to Proposition 1.15) and the resulting *-representation $(\mathcal{D}, \check{\Phi})$ is Q-positive by Proposition 2.18.

Remark 2.20 We now know how to deal with two types of relations on *-algebras: equalities and inequalities. Inequalities are encoded by enforcing a set S_1 of hermitian elements to be positive in *-representations by restricting to $\langle\!\langle\, S_1 \,\rangle\!\rangle_{\mathrm{qm}}$-positive *-representations, while equalities are encoded by enforcing a set S_2 of hermitian elements to vanish in *-representations by passing to the quotient *-algebra modulo $\langle\!\langle\, S_2 \,\rangle\!\rangle_{*\text{-id}}$ (in which the elements of S_2 are zero). However, any equality can be described by two inequalities, so in practice we have several options here.

Proposition 2.21 *Let $\mathcal{A}$ be a *-algebra, Q a quadratic module of $\mathcal{A}$, and ω a positive linear functional on $\mathcal{A}$. Let $(\mathcal{D}_\omega, \pi_\omega)$ be the GNS-*-representation from Theorem 2.7. Then $(\mathcal{D}_\omega, \pi_\omega)$ is Q-positive if and only if ω is Q-positive.*

Proof: If ω is Q-positive and $q \in Q$, then $\langle\, [a] \,|\, \pi_\omega(q)([a]) \,\rangle_\omega = \langle\, [a] \,|\, [qa] \,\rangle_\omega = \omega(a^* q a) \geq 0$ for all $a \in \mathcal{A}$, i.e. $\pi_\omega(q) \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$; so $(\mathcal{D}_\omega, \pi_\omega)$ is Q-positive. Conversely, if $(\mathcal{D}_\omega, \pi_\omega)$ is $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}}$-positive, then by the previous Proposition 2.15, the state $\omega = \chi_{\pi,[1]}$ on $\mathcal{A}$ is $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}}$-positive. □

3 The archimedean Positivstellensatz

We now have all the prerequisits together to prove one important theorem about *-algebras, namely the archimedean Positivstellensatz. Let's recall some basics first:

A quadratic modules Q of a *-algebra $\mathcal{A}$ is a subset of the hermitian elements of $\mathcal{A}$ that fulfils some elementary axioms (Definition 2.10) that one generally expects from the set of in some sense “positive” elements of a *-algebra. Such quadratic modules are easy to construct, e.g. every set S of hermitian elements of $\mathcal{A}$ allows to construct a quadratic module $\langle\!\langle\, S \,\rangle\!\rangle_{\mathrm{qm}}$ generated by this set (2.16), and there is a unique smallest quadratic module $\sum \mathcal{A}^*\mathcal{A}$ of $\mathcal{A}$ whose elements are the sums of "hermitian squares" a^*a, (2.17). There also exists a unique largest quadratic module, namely $\mathcal{A}_{\mathrm{h}}$ (this is the solution to Exercise 2.6), but this largest one is "trivial", there is nothing to learn about a *-algebra by declaring all its hermitian elements to be "positive". Moreover, for every pre-Hilbert space $\mathcal{D}$, the *-algebra $\mathcal{L}^*(\mathcal{D})$ of all adjointable maps $\mathcal{D} \to \mathcal{D}$ comes with a canonical quadratic module $\mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ (see again Definition 2.10), e.g. if $\mathcal{D} = \mathbb{C}^n$ with $n \in \mathbb{N}$, then $\mathcal{L}^*(\mathcal{D}) = \mathbb{C}^{n\times n}$ and $\mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ are the positive semidefinite matrices.

For any quadratic module Q on a *-algebra $\mathcal{A}$ we have also defined what Q-positive linear functionals $\omega\colon \mathcal{A} \to \mathbb{C}$ are (Definition 2.13, essentially, $\omega(q) \geq 0$ for all $q \in Q$), and a general positive linear functional is simply defined as a $\sum \mathcal{A}^*\mathcal{A}$-positive linear functional. As $\sum \mathcal{A}^*\mathcal{A}$ is the smallest quadratic module on $\mathcal{A}$, a linear functional that is Q-positive for any given quadratic module Q is automatically also ($\sum \mathcal{A}^*\mathcal{A}$-)positive, i.e. this is the weakest notion of positivity of linear functionals. The GNS-construction from Theorem 2.7 shows how to construct a *-representation $(\mathcal{D}_\omega, \pi_\omega)$ of $\mathcal{A}$ out of any positive linear functional ω on $\mathcal{A}$. If ω is even Q-positive for some given quadratic module Q, then $(\mathcal{D}_\omega, \pi_\omega)$ is also Q-positive: π_ω maps elements of Q to positive operators, i.e. to $\mathcal{L}^*(\mathcal{D}_\omega)^+_{\mathrm{h}}$, see Proposition 2.21.

What we don't understand yet is the this: Given an element $a \in \mathcal{A}_{\mathrm{h}} \setminus Q$, is there some Q-positive *-representation that maps a to a non-positive operator? In short, are there enough Q-positive *-representation so that we can detect whether a hermitian element is in Q or not? In order to prove something like this we would need to find a Q-positive linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ that fulfils $\omega(a) < 0$. But in the general case we don't even know whether there always exists a positive linear functional at all on a given *-algebra! Without having a positive linear functional, the GNS-construction isn't helpful.

The archimedean Positivstellensatz (German for "positive points theorem") solves this problem for a large class of quadratic modules. In order to prove this theorem, we basically need to understand how to construct positive linear functionals. The technique for this is not at all specific to *-algebras, it works in much greater generality.

Definition 3.1 *Let V be a real vector space. A* convex cone *of V is a subset C of V such that*

i.) $0 \in C$ *and*

ii.) $\lambda c + \mu d \in C$ *for all* $c, d \in C$ *and* $\lambda, \mu \in [0, \infty[$.

*Moreover, consider such a convex cone C on V and let $\omega\colon V \to \mathbb{R}$ be a linear functional on V. Then V is called C-*positive *if $\omega(c) \geq 0$ for all $c \in C$.*

For example, if $\mathcal{A}$ is a *-algebra, then every quadratic module Q of $\mathcal{A}$ is a convex cone of $\mathcal{A}_{\mathrm{h}}$, and if $\omega\colon \mathcal{A} \to \mathbb{C}$ is a hermitian linear functional on $\mathcal{A}$, then ω is Q-positive in the sense of Definition 2.13 if and only if the linear functional $\omega_{\mathrm{h}}\colon \mathcal{A}_{\mathrm{h}} \to \mathbb{R}$, $a \mapsto \omega_{\mathrm{h}}(a) := \omega(a)$ on the real vector space $\mathcal{A}_{\mathrm{h}}$ is Q-positive in the sense of the above Definition 3.1.

Before we can start with the construction of positive linear functionals we recall one fundamental property of the real numbers: A *lower bound* of a subset S of $\mathbb{R}$ is a number $\beta \in \mathbb{R}$ such that $\beta \leq \sigma$ for all $\sigma \in S$, and S is called *bounded from below* if there exists a lower bound β of S. For every subset S of $\mathbb{R}$ that is nonempty and bounded from below, the infimum $\inf S \in \mathbb{R}$ exists, which by definition is the greatest lower bound of S (i.e. $\inf S$ itself is a lower bound of S, and every lower bound β of S fulfils $\beta \leq \inf S$).

Lemma 3.2 *Let V be a real vector space, C a convex cone of V, and W a linear subspace of V. Assume that for all $v \in V$ there exist $c \in C$ such that $v + c \in C \cap W$. Let $\rho \colon W \to \mathbb{R}$ be a $C \cap W$–positive linear functional and $x \in V \setminus W$. Set*

$$S := \left\{ \rho(w) \;\middle|\; w \in W \text{ such that } w + x \in C \right\} \subseteq \mathbb{R}. \tag{3.1}$$

Then every element $v \in \langle\!\langle\, W \cup \{x\} \,\rangle\!\rangle_{\mathrm{lin}}$ can be decomposed as $v = w + \lambda x$ with uniquely determined elements $w \in W$ and $\lambda \in \mathbb{R}$, S is a non-empty subset of $\mathbb{R}$ and bounded from below (so $\inf S \in \mathbb{R}$ exists), and the map $\hat{\rho} \colon \langle\!\langle\, W \cup \{x\} \,\rangle\!\rangle_{\mathrm{lin}} \to \mathbb{R}$,

$$w + \lambda x \mapsto \hat{\rho}(w + \lambda x) := \rho(w) - \lambda \inf S \tag{3.2}$$

is linear and $C \cap \langle\!\langle\, W \cup \{x\} \,\rangle\!\rangle_{\mathrm{lin}}$-positive.

Proof: Note that $\langle\!\langle\, W \cup \{x\} \,\rangle\!\rangle_{\mathrm{lin}} = \{\, w + \lambda x \mid w \in W, \lambda \in \mathbb{R} \,\}$, i.e. for every $v \in \langle\!\langle\, W \cup \{x\} \,\rangle\!\rangle_{\mathrm{lin}}$ there exist $w \in W$ and $\lambda \in \mathbb{R}$ such that $v = w + \lambda x$. We show that w and λ are uniquely determined: Indeed, if $w, w' \in W$ and $\lambda, \lambda' \in \mathbb{R}$ fulfil $w + \lambda x = w' + \lambda' x$, then $w - w' = (\lambda' - \lambda)x$; this implies $\lambda' = \lambda$ and consequently $w = w'$, because otherwise $(w - w')/(\lambda - \lambda') = x \notin W$, a contradiction.

By assumption there exist $c_+, c_- \in C$ such that $x + c_+ \in C \cap W$ and $-x + c_- \in C \cap W$. From this it follows that S is nonempty because $-x + c_- \in W$ fulfils $(-x + c_-) + x = c_- \in C$ and therefore $\rho(-x + c_-) \in S$. It also follows that $\rho(x + c_+)$ is a lower bound of S: Namely, consider any $w \in W$ such that $w + x \in C$, then $\rho(w) = \rho\big(w + (x + c_+)\big) - \rho(x + c_+) \geq -\rho(x + c_+)$ by $C \cap W$-positivity of ρ because $w + (x + c_+) = (w + x) + c_+ \in C \cap W$. In total, S is a nonempty subset of $\mathbb{R}$ and bounded from below, therefore $\inf S \in \mathbb{R}$ exists.

It is easy to check that the map $\hat{\rho}$ is linear. Moreover, consider any $c \in C \cap \langle\!\langle\, W \cup \{x\} \,\rangle\!\rangle_{\mathrm{lin}}$, then it only remains to show that $\hat{\rho}(c) \geq 0$: We can express c as $c = w + \lambda x$ with uniquely determined elements $w \in W$ and $\lambda \in \mathbb{R}$, and distinguish three cases: If $\lambda = 0$, then $c = w \in C \cap W$ so that $\hat{\rho}(c) = \rho(c) \geq 0$ by $C \cap W$-positivity of ρ. If $\lambda > 0$, then $\lambda^{-1} w + x = \lambda^{-1} c \in C$ implies that $\rho(\lambda^{-1} w) \in S$ so that

$$\hat{\rho}(c) = \rho(w) - \lambda \inf S = \lambda\big(\rho(\lambda^{-1} w) - \inf S\big) \geq 0$$

because $\inf S$ is a lower bound of S. For the last case $\lambda < 0$, consider any $\tilde{w} \in W$ such that $\tilde{w} + x \in C$. Then $\rho(-\lambda^{-1} w + \tilde{w}) \geq 0$ by $C \cap W$-positivity of ρ because $-\lambda^{-1} w + \tilde{w} = |\lambda|^{-1} c + (\tilde{w} + x) \in C \cap W$. This shows that $\lambda^{-1} \rho(w) \leq \rho(\tilde{w})$ for all $\tilde{w} \in W$ that fulfil $\tilde{w} + x \in C$, i.e. $\lambda^{-1} \rho(w)$ is a lower bound of S. Therefore $\lambda^{-1} \rho(w) \leq \inf S$ and consequently $\rho(w) \geq \lambda \inf S$ (recall that $\lambda < 0$). It follows that $\hat{\rho}(c) = \rho(w) - \lambda \inf S \geq 0$. $\square$

In the proof of Theorem 3.4 we will apply the construction of the above Lemma 3.2 recursively in order to construct $C \cap W$-positive linear functionals on finite dimensional linear subspaces W of V. If V is not too large, then we will even obtain a C-positive linear functional on whole V.

Definition 3.3 *Let V be a vector space. We say that V* has at most countable dimension *if there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in V such that $V = \langle\!\langle \{ v_n \mid n \in \mathbb{N} \} \rangle\!\rangle_{\mathrm{lin}}$.*

Theorem 3.4 (Positivstellensatz for convex cones with interior point) *Let V be a real vector space and assume that V has at most countable dimension. Let C be a convex cone of V, $e \in C$, and assume that for all $v \in V$ there exists $\lambda \in [0, \infty[$ such that $\lambda e - v \in C$. Consider any $x \in V$, then the following are equivalent:*

i.) $x + \epsilon e \in C$ *for all* $\epsilon \in]0, \infty[$,

ii.) $\rho(x) \geq 0$ *for all C-positive linear functionals ρ on V.*

Proof: First assume $x + \epsilon e \in C$ for all $\epsilon \in]0, \infty[$ and let ρ be any C-positive linear functional on V. Then $\rho(x) = \rho(x + \epsilon e) - \epsilon\rho(e) \geq -\epsilon\rho(e)$ for all $\epsilon \in]0, \infty[$, and therefore $\rho(x) \geq 0$. This proves the implication $i \implies ii$.

Conversely, assume that there is $\epsilon \in]0, \infty[$ such that $x + \epsilon e \notin C$. Note that this means that $\lambda e + x \notin C$ for all $\lambda \in]-\infty, \epsilon]$ because $(\lambda e + x) + (\epsilon - \lambda)e = x + \epsilon e \notin C$ with $(\epsilon - \lambda)e \in C$. In particular $x \notin C$. We will construct a C-positive linear functional ρ_∞ on V that fulfils $\rho_\infty(x) < 0$, which then proves the other implication $ii \implies i$. As V has at most countable dimension, there exists a sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ in V such that $V = \langle\!\langle \{ \tilde{v}_n \mid n \in \mathbb{N} \} \rangle\!\rangle_{\mathrm{lin}}$. We use this to define a new sequence $(v_n)_{n\in\mathbb{N}}$ in V as follows:

$$v_n := \begin{cases} e & \text{if } n = 1, \\ x & \text{if } n = 2, \\ \tilde{v}_{n-2} & \text{if } n > 2. \end{cases} \tag{3.3}$$

Then clearly $V = \langle\!\langle \{ v_n \mid n \in \mathbb{N} \} \rangle\!\rangle_{\mathrm{lin}}$ holds as well. For $n \in \mathbb{N}$ we write $W_n := \langle\!\langle \{v_1, \dots, v_n\} \rangle\!\rangle_{\mathrm{lin}}$ and we iteratively construct $C \cap W_n$-positive linear functionals ρ_n on W_n as follows:

First, $\rho_1 \colon \langle\!\langle \{e\} \rangle\!\rangle_{\mathrm{lin}} \to \mathbb{R}$ is defined as $\rho(\lambda e) := \lambda$ for all $\lambda \in \mathbb{R}$. In order to show that ρ_1 is $C \cap W_1$-positive we have to show that $-\lambda e \notin C$ for all $\lambda \in]0, \infty[$: So assume to the contrary that there exists $\lambda \in]0, \infty[$ such that $-\lambda e \in C$. By assumption, for every $v \in V$ there is $\lambda' \in [0, \infty[$ such that $\lambda' e - v \in C$, and then $-v = (\lambda'/\lambda)(-\lambda e) + (\lambda' e - v) \in C$. In particular for $v := -x$ it would follow that $x \in C$, a contradiction.

If $x \in \langle\!\langle \{e\} \rangle\!\rangle_{\mathrm{lin}}$, i.e. $x = \mu e$ for some $\mu \in \mathbb{R}$, then we simply define $\rho_2 := \rho_1 \colon \langle\!\langle \{e\} \rangle\!\rangle_{\mathrm{lin}} \to \mathbb{C}$. Otherwise Lemma 3.2 provides a $C \cap W_2$-positive linear functional $\hat{\rho}_1 \colon \langle\!\langle \{e, x\} \rangle\!\rangle_{\mathrm{lin}} \to \mathbb{R}$ such that $\hat{\rho}_1(x) = -\inf S$ with $S = \{ \lambda \mid \lambda \in \mathbb{R} \text{ such that } \lambda e + x \in C \}$, and we set $\rho_2 := \hat{\rho}_1$. Note that in both cases $\rho_2(x) < 0$: In the first case, $x = \mu e$ for some $\mu \in \mathbb{R}$, we know that $\rho_2(x) = \mu < 0$ because $\mu \geq 0$ would mean that $x \in C$, a contradiction. Otherwise it follows from $\lambda e + x \notin C$ for all $\lambda \in]-\infty, \epsilon]$ that ϵ is a lower bound of S and therefore $\epsilon \leq \inf S$, i.e. $\rho_2(x) = \hat{\rho}_1(x) = -\inf S \leq -\epsilon < 0$.

For $n \in \mathbb{N}$ with $n > 2$ we iteratively define ρ_n in a similar manner: If $v_n \in W_{n-1}$, then $\rho_n := \rho_{n-1}$, and otherwise we use Lemma 3.2 in order to construct the $C \cap W_n$-positive linear functional $\hat{\rho}_{n-1}$ on $W_n = \langle\!\langle W_{n-1} \cup \{v_n\} \rangle\!\rangle_{\mathrm{lin}}$ and set $\rho_n := \hat{\rho}_{n-1}$.

From the above construction of the linear subspaces W_n of V and the linear functionals ρ_n on W_n it is clear that $W_n \subseteq W_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} W_n = V$. Moreover, $\rho_n(w) = \rho_{n+1}(w)$ for all $n \in \mathbb{N}$ and $w \in W_n$, so $\rho_n(w) = \rho_m(w)$ for all $n, m \in \mathbb{N}$ with $n \leq m$ and all $w \in W_n$. Therefore we can define a map $\rho_\infty \colon V \to \mathbb{R}$,

$$\rho_\infty(v) := \rho_n(v) \qquad \text{with any } n \in \mathbb{N} \text{ such that } v \in W_n,$$

and it is easy to check that ρ_∞ is linear and C-positive, and $\rho_\infty(x) = \rho_2(x) < 0$. □

There are two important assumptions made in the above Theorem 3.4: The first is that V has at most countable dimension, the second is the existence of an element $e \in C$ that fulfils a certain technical property (one usually says that e is an *interior point* of C). The first assumption is actually superfluous if one is willing to apply some trickery / black magic (the axiom of choice), the second assumption motivates the following definition:

Definition 3.5 *Let $\mathcal{A}$ be a *-algebra and Q a quadratic module of $\mathcal{A}$. Then Q is called* archimedean *if for all $a \in \mathcal{A}_{\mathrm{h}}$ there exists $\lambda \in [0, \infty[$ such that $\lambda - a \in Q$.*

In other words, Q is archimedean if and only if the unit 1 is an interior point of Q.

Theorem 3.6 (Archimedean Positivstellensatz) *Let $\mathcal{A}$ be a *-algebra and Q an archimedean quadratic module of $\mathcal{A}$. Assume that $\mathcal{A}$ has at most countable dimension. Consider any $b \in \mathcal{A}_{\mathrm{h}}$, then the following are equivalent:*

i.) $b + \epsilon \in Q$ *for all* $\epsilon \in]0, \infty[$,

ii.) $\pi(b) \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$ *for all Q-positive *-representations $(\mathcal{D}, \pi)$ of $\mathcal{A}$.*

Proof: First assume $b + \epsilon \in Q$ for all $\epsilon \in]0, \infty[$ and let $(\mathcal{D}, \pi)$ be any Q-positive *-representation of $\mathcal{A}$. Then

$$\langle \psi \,|\, \pi(b)(\psi) \rangle = \langle \psi \,|\, \pi(b + \epsilon)(\psi) \rangle - \epsilon \langle \psi \,|\, \psi \rangle \geq -\epsilon \langle \psi \,|\, \psi \rangle$$

holds for all $\psi \in \mathcal{D}$, therefore $\langle \psi \,|\, \pi(b)(\psi) \rangle \geq 0$ for all $\psi \in \mathcal{D}$, i.e. $\pi(b) \in \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$. This proves the implication $i \implies ii$.

Conversely, assume that there exists $\epsilon \in]0, \infty[$ such that $b + \epsilon \notin Q$. In order to prove the other implication $ii \implies i$, we will construct a Q-positive *-representation $(\mathcal{D}_\omega, \pi_\omega)$ of $\mathcal{A}$ such that $\pi(b) \notin \mathcal{L}^*(\mathcal{D})^+_{\mathrm{h}}$: Theorem 3.4, applied to the convex cone Q of the real vector space $\mathcal{A}_{\mathrm{h}}$ with $e := 1 \in Q$, shows that for $x := b \in \mathcal{A}_{\mathrm{h}}$ there exists a Q-positive linear functional $\rho \colon \mathcal{A}_{\mathrm{h}} \to \mathbb{R}$ such that $\rho(b) < 0$. Note that $a + a^* \in \mathcal{A}_{\mathrm{h}}$ and $\mathrm{i}(a - a^*) \in \mathcal{A}_{\mathrm{h}}$ for all $a \in \mathcal{A}$, so we can define $\omega \colon \mathcal{A} \to \mathbb{C}$,

$$a \mapsto \omega(a) := \frac{\rho(a + a^*)}{2} + \frac{\rho(\mathrm{i}(a - a^*))}{2\mathrm{i}}$$

and it is easy to check that $\omega(a+b) = \omega(a) + \omega(b)$ and $\omega(\lambda a) = \lambda\omega(a)$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{R}$. Also

$$\omega(\mathrm{i}a) = \frac{\rho(\mathrm{i}a - \mathrm{i}a^*)}{2} + \frac{\rho(-a - a^*)}{2\mathrm{i}} = \mathrm{i}\bigg(\frac{\rho(\mathrm{i}(a - a^*))}{2\mathrm{i}} + \frac{\rho(a + a^*)}{2}\bigg) = \mathrm{i}\omega(a)$$

for all $a \in \mathcal{A}$, so ω is a ($\mathbb{C}$-)linear functional on $\mathcal{A}$. Moreover,

$$\omega(a^*) = \frac{\rho(a^* + a)}{2} + \frac{\rho(\mathrm{i}(a^* - a))}{2\mathrm{i}} = \frac{\rho(a + a^*)}{2} - \frac{\rho(\mathrm{i}(a - a^*))}{2\mathrm{i}} = \overline{\omega(a)}$$

holds for all $a \in \mathcal{A}$, so the linear functional ω on $\mathcal{A}$ is hermitian. Clearly $\omega(a) = \rho(a)$ for all $a \in \mathcal{A}_{\mathrm{h}}$, and in particular $\omega(q) = \rho(q) \geq 0$ for all $q \in Q$, so ω is Q-positive. Let $(\mathcal{D}_\omega, \pi_\omega)$ be the corresponding GNS *-representation from Theorem 2.7. By Proposition 2.21, $(\mathcal{D}_\omega, \pi_\omega)$ is Q-positive. But $\pi_\omega(b) \notin \mathcal{L}^*(\mathcal{D}_\omega)^+_{\mathrm{h}}$ because $\langle\, [1] \,|\, \pi_\omega(b)([1]) \,\rangle_\omega = \langle\, [1] \,|\, [b] \,\rangle_\omega = \omega(b) = \rho(b) < 0$. □

This archimedean Positivstellensatz is a very strong result as it e.g. allows to compare a quadratic module $\langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}$ of $\mathcal{A}$ that is generated by $m \in \mathbb{N}$ elements $s_1, \dots, s_m \in \mathcal{A}_{\mathrm{h}}$ with the typically larger quadratic module of all hermitian elements that are mapped to positive operators in all $\langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}$-positive *-representations (which are those *-representations that map $s_1, \dots, s_m$ to positive operators). The quadratic module $\langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}$ is easy to describe explicitly as

$$\langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}} = \bigg\{ \sum\nolimits_{i=0}^m \sum\nolimits_{j=1}^n a_{i,j}^* s_i a_{i,j} \;\bigg|\; a_{i,j} \in \mathcal{A} \text{ for all } i \in \{0, \dots, m\}, j \in \{1, \dots, n\} \bigg\} \tag{3.4}$$

where $s_0 \coloneqq 1$. There is even an algorithm for checking whether some $a \in \mathcal{A}_{\mathrm{h}}$ is in $\langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}$: Let $A^{(\ell)}$ with $\ell \in \mathbb{N}$ be finite dimensional linear subspaces of $\mathcal{A}$ such that $\mathcal{A}^{(\ell)} \subseteq \mathcal{A}^{(\ell+1)}$ for all $\ell \in \mathbb{N}$ and $\mathcal{A} = \bigcup_{\ell=1}^\infty \mathcal{A}^{(\ell)}$, then membership in

$$\langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}^{(\ell)} = \bigg\{ \sum\nolimits_{i=0}^m \sum\nolimits_{j=1}^n a_{i,j}^* s_i a_{i,j} \;\bigg|\; a_{i,j} \in \mathcal{A}^{(\ell)} \text{ for all } i \in \{0, \dots, m\}, j \in \{1, \dots, n\} \bigg\} \tag{3.5}$$

translates into feasibility of a semidefinite program, and $b \in \langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}$ if and only if $b \in \langle\!\langle\, \{s_1, \dots, s_m\} \,\rangle\!\rangle_{\mathrm{qm}}^{(\ell)}$ for sufficiently large $\ell \in \mathbb{N}$. For example, if $\mathcal{A}$ is a *-algebra given by finitely many generators and some relations like in section 1.5, then $\mathcal{A}$ has at most countable dimension and one can choose $\mathcal{A}^{(\ell)}$ as the linear subspaces generated by words of length at most ℓ. This is the basic idea upon which many algorithms of noncommutative polynomial optimization are built.

The last problem that remains is to prove in specific examples that a given quadratic module is archimedean. This is oftentimes not hard in practice (but note that non-archimedean quadratic modules do exist and are important, in particular when dealing with the Weyl algebra, see part *i* of Example 1.13, which does not admit any non-trivial archimedean quadratic module, see Exercise 3.2):

Exercise 3.1 (About archimedean quadratic modules and C^*-seminorms) *Let $\mathcal{A}$ be a *-algebra and Q a quadratic module of $\mathcal{A}$. Define a relation $\lesssim$ on $\mathcal{A}_{\mathrm{h}}$ as follows: Given $a, b \in \mathcal{A}_{\mathrm{h}}$, then*

$$a \lesssim b \qquad \text{if and only if} \qquad b - a \in Q. \tag{3.6}$$

So in particular $0 \lesssim b$ if and only if $b \in Q$. It is sometimes more convenient to work with this order *relation than with the quadratic module Q. We then define the set of* uniformly bounded *elements of $\mathcal{A}$ (with respect to Q)*

$$\mathcal{A}_{Q\text{-bd}} \coloneqq \left\{ a \in \mathcal{A} \;\middle|\; \textit{there is } \lambda \in \,]0,\infty[\, \textit{ such that } a^*a \lesssim \lambda^2 \right\} \tag{3.7}$$

and the map $\|\cdot\|_Q \colon \mathcal{A}_{Q\text{-bd}} \to [0,\infty[$,

$$a \mapsto \|a\|_Q \coloneqq \inf\left\{ \lambda \;\middle|\; \lambda \in \,]0,\infty[\, \textit{ such that } a^*a \lesssim \lambda^2 \right\}. \tag{3.8}$$

By definition, Q is archimedean if and only if $A = \mathcal{A}_{Q\text{-bd}}$.

- *i.) Show that $\mathcal{A}_{Q\text{-bd}}$ is closed under addition and that $\|a+b\|_Q \leq \|a\|_Q + \|b\|_Q$ for all $a, b \in \mathcal{A}_{Q\text{-bd}}$. Hint: Assume $a^*a \lesssim \mu^2$ and $b^*b \lesssim \sigma^2$ for some $\mu, \sigma \in \,]0,\infty[$, and show that this implies $(a+b)^*(a+b) \lesssim (\mu+\sigma)^2$.*
- *ii.) Show that $\mathcal{A}_{Q\text{-bd}}$ is a linear subspace of $\mathcal{A}$ and that $\|\lambda a\|_Q = |\lambda|\,\|a\|_Q$ for all $a \in \mathcal{A}_{Q\text{-bd}}$, $\lambda \in \mathbb{C}$. Hint: From $\|\lambda a\|_Q \leq |\lambda|\,\|a\|_Q$ for all $\lambda \in \mathbb{C}$ it follows that $\|\lambda a\|_Q \geq |\lambda|\,\|a\|_Q$ for all $\lambda \in \mathbb{C}$.*
- *iii.) Show that $\mathcal{A}_{Q\text{-bd}}$ is a subalgebra of $\mathcal{A}$ and $\|ab\|_Q \leq \|a\|_Q\|b\|_Q$ for all $a, b \in \mathcal{A}_{Q\text{-bd}}$.*
- *iv.) Show that $\mathcal{A}_{Q\text{-bd}}$ is a *-subalgebra of $\mathcal{A}$ and $\|a^*\|_Q = \|a\|_Q$ for all $a \in \mathcal{A}_{Q\text{-bd}}$. Hint: From $\|a^*\|_Q \leq \|a\|_Q$ for all $a \in \mathcal{A}_{Q\text{-bd}}$ it follows that $\|a^*\|_Q \geq \|a\|_Q$ for all $a \in \mathcal{A}_{Q\text{-bd}}$.*
- *v.) Show that $\|a^*a\|_Q = \|a\|_Q^2$ for all $a \in \mathcal{A}_{Q\text{-bd}}$. Hint: Assume $a^*a\,a^*a \lesssim \lambda^2$ for some $\lambda \in \,]0,\infty[$ and show that this implies $a^*a \lesssim \lambda$.*

As $\mathcal{A}_{Q\text{-bd}}$ is a *-subalgebra of $\mathcal{A}$ by part iv it is usually easy to check whether or not $A = \mathcal{A}_{Q\text{-bd}}$ holds (i.e. whether or not Q is archimedean): If $\mathcal{A}$ is generated by some elements $x_1, \dots, x_n \in \mathcal{A}$, then $\mathcal{A} = \mathcal{A}_{Q\text{-bd}}$ if and only if $\{x_1, \dots, x_n\} \subseteq \mathcal{A}_{Q\text{-bd}}$, so one only needs to investigate finitely many elements. For example, this shows that the quadratic module $\sum \mathcal{A}^*\mathcal{A}$ for $\mathcal{A} \coloneqq \mathcal{A}_{CHSH}$ as in part v of Example 1.13 is archimedean.

Properties *i*, *ii*, *iii* and *v* of the map $\|\cdot\|_Q \colon \mathcal{A}_{Q\text{-bd}} \to [0,\infty[$ mean that $\|\cdot\|_Q$ is a C^**-seminorm* on $\mathcal{A}_{Q\text{-bd}}$. Property *iv* can be shown to follow from *iii* and *v*. A C^**-norm* on a general *-algebra $\mathcal{A}$ is a C^*-seminorm $\|\cdot\| \colon \mathcal{A} \to [0,\infty[$ such that $\|a\| = 0$ implies $a = 0$. A *pre-*C^**-algebra* is a *-algebra equipped with a C^*-norm $\|\cdot\|$. One can show that every C^*-(semi-)norm $\|\cdot\|$ on a *-algebra $\mathcal{A}$ is of the form $\|\cdot\| = \|\cdot\|_Q$ as in (3.8) for some archimedean quadratic module Q on $\mathcal{A}$, but different archimedean quadratic modules can give the same C^*-seminorm. Therefore one can learn more about the structure of a *-algebra by investigating its quadratic modules than by investigating its C^*-seminorms. A C^*-algebra finally is a pre-C^*-algebra that is complete with respect to its C^*-norm (i.e. every Cauchy sequence is convergent). In the same way as every pre-Hilbert space can be completed to a Hilbert space, every pre-C^*-algebra can be completed to a C^*-algebra.

Exercise 3.2 *Consider the Weyl algebra $\mathcal{W}_1$ of part i of Example 1.13, i.e. $\mathcal{W}_1$ is generated by two hermitian elements p and q that fulfil the canonical commutation relation $qp - pq = \mathrm{i}$. Define the map*

$\mathrm{ad}_q \colon \mathcal{W}_1 \to \mathcal{W}_1,$

$$a \mapsto \mathrm{ad}_q(a) \coloneqq qa - aq. \tag{3.9}$$

Assume that $\|\cdot\| \colon \mathcal{W}_1 \to \mathcal{W}_1$ *is a* submultiplicative seminorm, *i.e.* $\|a+b\| \leq \|a\|+\|b\|$, $\|\lambda a\| = |\lambda|\,\|a\|$, *and* $\|ab\| \leq \|a\|\,\|b\|$ *for all* $a, b \in \mathcal{W}_1$ *and* $\lambda \in \mathbb{C}$.

i.) Show that $\mathrm{ad}_q(ab) = \mathrm{ad}_q(a)\,b + a\,\mathrm{ad}_q(b)$ *holds for all* $a, b \in \mathcal{W}_1$.

ii.) Show that $(\mathrm{ad}_q)^n(p^n) = n!\,\mathrm{i}^n$ *for all* $n \in \mathbb{N}$.

iii.) Show that $\|(\mathrm{ad}_q)^n(p^n)\| \leq 2^n \|q\|^n \|p\|^n$ *for all* $n \in \mathbb{N}$.

iv.) Conclude that $\|1\| = 0$ *and therefore* $\|a\| = 0$ *for all* $a \in \mathcal{W}_1$*; in particular there is no submultiplicative (or even* C^**-) norm on* $\mathcal{W}_1$.

v.) Apply Exercise 3.1 in order to show that every archimedean quadratic module Q *on* $\mathcal{W}_1$ *fulfils* $-1 \in Q$*, and show that this implies that in fact* $Q = \mathcal{A}_{\mathrm{h}}$*; i.e. there is no non-trivial archimedean quadratic module on* $\mathcal{W}_1$.

Important references

The archimedean Positivstellensatz has first been proven in [2], and the relation between quadratic modules and C^*-algebras as in Exercise 3.1 is due to [1]. A good overview and many more ideas can be found in [3].

References

[1] J. Cimprič. A representation theorem for archimedean quadratic modules on *-rings. *Canadian Mathematical Bulletin*, 52(1):39–52, 2009. doi:10.4153/CMB-2009-005-4.

[2] J. W. Helton and S. A. McCullough. A positivstellensatz for non-commutative polynomials. *Transactions of the American Mathematical Society*, 356(9):3721–3737, 2004. doi:10.1090/S0002-9947-04-03433-6.

[3] K. Schmüdgen. *Noncommutative Real Algebraic Geometry Some Basic Concepts and First Ideas*, pages 325–350. Springer New York, New York, NY, 2009. doi:10.1007/978-0-387-09686-5_9.