

Announcement

Seminar on Deformation Quantization and Geometry

08.05.2026 at 14:00 s.t.

Seminarroom SE 31

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The semianalytic/perturbative Positivstellensatz in commutative and noncommutative real algebraic geometry

It is wellknown that there are pointwise positive polynomials in $d \geq 2$ variables that cannot be represented as a sum of squares of polynomials. However, the cone of sums of squares $\sum \mathbb{R}[x_1, \dots, x_d]^2$ is dense in the cone of all pointwise positive polynomials in a suitable locally convex topology. More precisely, for any polynomial $p \in \mathbb{R}[x_1, \dots, x_d]$, the following are equivalent:

- $p(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$.
- For every $\epsilon \in]0, \infty[$ there is $N \in \mathbb{N}_0$ such that $p + \epsilon \sum_{n=0}^N \frac{(x_1^2 + \dots + x_d^2)^n}{(2n)!} \in \sum \mathbb{R}[x_1, \dots, x_d]^2$.

This is essentially Lasserre's perturbative Positivstellensatz (which was originally proven with applications to nonconvex optimization in mind, because membership in the convex cone $\sum \mathbb{R}[x_1, \dots, x_d]^2$ is rather easy to check numerically).

I will give a short new proof of this theorem, and then discuss how it generalizes to the noncommutative setting, i.e., with the commutative real algebra $\mathbb{R}[x_1, \dots, x_d]$ replaced by a noncommutative $*$ -algebra and with pointwise positivity replaced by positivity in $*$ -representations. To give just one example, for an element $p = p^*$ of the Weyl $*$ -algebra \mathcal{W} (the $*$ -algebra generated by one element a subject to the canonical commutation relation $aa^* - a^*a = 1$), the following are equivalent:

- $\langle \psi | \pi(p) \psi \rangle \geq 0$ for all $\psi \in \mathcal{S}(\mathbb{R})$,
where $\pi: \mathcal{W} \rightarrow \mathcal{L}^*(\mathcal{S}(\mathbb{R}))$ is the Schrödinger representation of \mathcal{W} on the Schwartz space $\mathcal{S}(\mathbb{R})$.
- For every $\epsilon \in]0, \infty[$ there is $N \in \mathbb{N}_0$ such that $p + \epsilon \sum_{n=0}^N \frac{(a^*a)^n}{(2n)!} \in \sum \mathcal{W}^* \mathcal{W}$,
where $\sum \mathcal{W}^* \mathcal{W}$ is the convex cone of \mathcal{W} generated by elements of the form b^*b with $b \in \mathcal{W}$.

This talk is based on joint work (arxiv:2603.13954 and work in progress) with Igor Klep, Victor Magron, and Jurij Volčič.

Invited by Stefan Waldmann