Representation Theory of *-Algebras

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Introduction

One main motivation to consider such algebras comes from mathematical physics, in particular from quantum theory: here the observables of a quantum system usually form a *-algebra which in optimal situations is a C^*-algebra.

Since in real life such a nice analytic framework is hard to achieve, one is content with a more algebraic treatment involving only the *-involution but no analytic features. In fact, in constructive approaches to quantum mechanical models one usually starts with an abstract algebra generated by some elements satisfying commutation relations which reflect the physical input for the model. It is then a major task to implement these algebras as algebras of, typically unbounded operators, on a Hilbert space. Only after a sometimes quite sophisticated spectral analysis of the situation is performed, one can pass to the C^*-algebra or the von Neumann algebra generated by the spectral projections of the unbounded operators. However, even if one has now reached a C^*-algebra, the unbounded operators still may carry an important physical interpretation of those observables which have a more direct interpretation.

Beyond these difficulties, in formal deformation quantization the situation is even worse: the observable algebras constructed there are typically defined over the ring of formal power series in $\hbar$ without known convergence properties. Thus one obtains a *-algebra over $\mathbb{C}[[\hbar]]$ instead. From a physical point of view, the convergence of course has to be understood and solved. However, this has been achieved only in the simplest cases while deformation quantization itself works in full generality: by a famous theorem of Kontsevich every Poisson manifold admits a formal star product.

Chapter 1

*-Algebras and Elementary Representation Theory

In this first chapter we introduce the notion of a *-algebra over an ordered ring and establish some first examples of such *-algebras which will be the guiding examples throughout these notes: The main motivation comes from C^*-algebras which serve as quantum mechanical observable algebras in many mathematically oriented approaches to quantum theory. However, also algebras of more unbounded nature are of interest like O^*-algebras of unbounded operators on some Hilbert space. Moreover, the complexified universal enveloping algebra of a real Lie algebra provides another class of examples which controls the representation theory of the Lie algebra, thereby being responsible for various notions of symmetries. The global analogue to this infinitesimal notion of symmetries is then given by a group and its group algebra yielding yet a further important class of *-algebras.

Beside these classical examples also new examples arise when taking deformation theory into account. Here the formal star products are the most important class of *-algebras we want to consider. In fact, this class was one of the main motivations to extend the usual techniques from C^*-algebra theory to the much more general framework since now we even have changed the underlying ring of scalars from the complex numbers to the formal power series $\mathbb{C}[[\lambda]]$.

The aspired representation theory for *-algebras on pre-Hilbert spaces should now capture all relevant features known from the individual representation theories of these examples: a unifying theme is to be found. The analytic theory of C^*-algebras will be a guideline but only those aspects can be used which are algebraic. The perhaps first surprising observation is that *positivity* is not an analytic but an algebraic feature. Indeed, the order of the underlying ring will allow for various notions of positivity. One of the major goals will then be the understanding of the representation theory of a given *-algebra. Based on the concept of a state, the GNS construction yields a systematic way to obtain such *-representations.

We conclude this introductory chapter with a discussion of the aims and expectations one has when investigating the *-representation theory of *-algebras in this generality. It will be clear that only the general algebraic features can be treated, more specific questions and results will typically require either a more specific nature of the underlying *-algebra or some more analytic framework to actually prove the interesting statements.

1.1 First Properties of *-Algebras

In this section we present the basic concepts of *-algebras and collect some first classes of examples. Most of the statements in this section are either simple abstractions from the theory of C^*-algebras or transfers from easy concepts in (linear) algebra.

1.1.1 Ordered Rings and *-Algebras

An ordered ring is a mild generalization of an ordered field. We recall the definition:

Definition 1.1.1 (Ordered ring) *An ordered ring* R *is an associative commutative ring with* $1 \neq 0$ *together with a distinguished subset* $\mathsf{P} \subseteq \mathsf{R}$ *such that*

i.) R *is the disjoint union of* P, $\{0\}$, *and* $-\mathsf{P}$,

ii.) $\mathsf{P} \cdot \mathsf{P} \subseteq \mathsf{P}$ *and* $\mathsf{P} + \mathsf{P} \subseteq \mathsf{P}$.

The elements in P *are called positive, the elements in* $-\mathsf{P}$ *are called negative elements of* R.

Remark 1.1.2 Let R be an ordered ring and $\alpha, \beta \in \mathsf{R}$.

i.) As usual we write $\alpha < \beta$ if $\beta - \alpha \in \mathsf{P}$ and similarly we use the symbols $>$, $\leq$, and $\geq$.

ii.) We have $\alpha^2 \geq 0$ and $\alpha^2 > 0$ iff $\alpha \neq 0$ by a simple case-by-case analysis.

iii.) If $\alpha > 0$ then $\alpha\beta \geq 0$ implies $\beta \geq 0$.

iv.) It follows that $1 = 1^2 > 0$ and hence also $n = 1 + \cdots + 1 > 0$. Thus an ordered ring has necessarily characteristic zero, i.e. $\mathbb{Z} \subseteq \mathsf{R}$ is always a subring. We will sometimes assume that also the rationals $\mathbb{Q} \subseteq \mathsf{R}$ are contained in R.

v.) It is easy to see that $ab = 0$ implies $a = 0$ or $b = 0$, i.e. an ordered ring has no zero divisors. This allows to pass to the quotient field which we denote by $\hat{\mathsf{R}}$. It turns out that this will be an ordered field such that the natural inclusion $\mathsf{R} \longrightarrow \hat{\mathsf{R}}$ is order preserving, see also Exercise 1.4.1.

As for ordered fields we say that an ordered ring R is *Archimedian* if for every positive $\alpha, \beta \in \mathsf{R}$ one finds an $n \in \mathbb{N}$ with $\alpha n > \beta$.

Example 1.1.3 (Ordered rings) The following two basic examples will play a role throughout these notes:

i.) The rings $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are ordered rings (even ordered fields for $\mathbb{Q}$ and $\mathbb{R}$) with respect to the usual notion of positive elements. The order is Archimedian. In fact, Archimedian ordered rings are subrings of $\mathbb{R}$.

ii.) If R is ordered, then the ring of formal power series $\mathsf{R}[[\lambda]]$ is ordered as well via

$$a = \sum_{r=r_0}^{\infty} \lambda^r a_r > 0 \quad \text{if} \quad a_{r_0} > 0. \tag{1.1.1}$$

Now the order is necessarily non-Archimedian. Indeed, the formal parameter λ is positive but $n\lambda < 1$ for all $n \in \mathbb{N}$. This construction can of course be iterated. In formal deformation quantization this is the relevant example with λ playing the role of Planck's constant $\hbar$. Note that the notion of an ordered ring is very suited for formal deformations as one stays in the same framework before and after the deformation. We will come back to this example from time to time and discuss the relevance of formal power series within the context of deformation theory in Chapter 6 in detail.

Given an ordered ring R we can consider the ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$ by a square root i of -1. As usual, we can view R as subring of C. Then every element in C can be uniquely written as $z = x + \mathrm{i}y$ with $x, y \in \mathsf{R}$. The *complex conjugation* is defined as usual by $\overline{z} = x - \mathrm{i}y$ for $z = x + \mathrm{i}y$ and we call x the *real part* and y the *imaginary part* of z. For $z \in \mathsf{C}$ we have $z \in \mathsf{R}$ if and only if $z = \overline{z}$. Moreover, $\overline{z}z > 0$ for $z \neq 0$. Also the ring C has characteristic zero and no zero divisors. Its quotient field is, up to the usual identification, given by $\hat{\mathsf{C}} = \hat{\mathsf{R}}(\mathrm{i})$.

In the following, we always use a fixed choice of R and the corresponding $\mathsf{C} = \mathsf{R}(\mathrm{i})$ as scalars. Then we can define a *-algebra over C as follows:

Definition 1.1.4 (*-Algebra) *An associative algebra* $\mathscr{A}$ *over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$ *is called* **-algebra with* **-involution* $^*\colon \mathscr{A} \longrightarrow \mathscr{A}$ *if the map* * *is a* C*-antilinear involutive anti-automorphism of* $\mathcal{A}$*, i.e.*

i.) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$,

ii.) $(a^*)^* = a$,

iii.) $(ab)^* = b^* a^*$

for all $a, b \in \mathscr{A}$ *and* $\alpha, \beta \in \mathsf{C}$.

In the following, $\mathscr{A}$ will always denote a *-algebra over C. Mainly, we are interested in the *unital* case. In this case, the unit element $\mathbb{1} \in \mathscr{A}$ necessarily satisfies $\mathbb{1}^* = \mathbb{1}$.

A **-homomorphism* $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ from a *-algebra $\mathscr{A}$ to a *-algebra $\mathscr{B}$ is now an algebra homomorphism with $\Phi(a^*) = \Phi(a)^*$. This definition yields the category *-$\mathsf{alg}(\mathsf{C})$ of *-algebras over C. If the reference to C is clear from the context, we simply write *-alg. Moreover, the non-full sub-category of unital *-algebras with unital *-algebra morphisms is denoted by *-Alg. Typically, if we need unital *-homomorphisms, then we will consequently stress this fact.

A **-ideal* $\mathscr{J} \subseteq \mathscr{A}$ is an ideal which is closed under the *-involution. It is necessarily a two-sided ideal. It is now a routine argument that the quotient $\mathscr{A}\big/\mathscr{J}$ becomes a *-algebra again, unital if $\mathscr{A}$ was unital, such that the quotient map $\mathscr{A} \longrightarrow \mathscr{A}\big/\mathscr{J}$ is a (unital) *-homomorphism, see also Exercise 1.4.2.

We recall some standard notions adopted from the well-known situation of operator algebras.

Definition 1.1.5 (Special elements) *Let* $\mathscr{A}$ *be a* **-algebra over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) An element $a \in \mathscr{A}$ *is called Hermitian if* $a^* = a$.

ii.) An element $a \in \mathscr{A}$ *is called anti-Hermitian if* $a^* = -a$.

iii.) An element $a \in \mathscr{A}$ *is called normal if* $a^* a = a a^*$.

iv.) In the unital case, an element $u \in \mathscr{A}$ *is called an isometry if* $u^* u = \mathbb{1}$.

v.) In the unital case, an element $u \in \mathscr{A}$ *is called unitary if* $u^* u = \mathbb{1} = u u^*$.

vi.) An element $p \in \mathscr{A}$ *is called a projection if* $p^2 = p = p^*$.

1.1.2 Positivity

Up to now we have not yet used the fact that R is ordered. The *-involution can already be defined if the ring C of scalars carries a distinguished involution itself. This will now change by considering positive functionals and algebra elements. By convention, a functional will always be *linear* over the scalars C, even if we do not mention this explicitly.

Definition 1.1.6 (Positive functional) *Let* $\mathscr{A}$ *be a* **-algebra over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$*. A linear functional* $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ *is called positive if*

$$\omega(a^* a) \geq 0 \tag{1.1.2}$$

for all $a \in \mathscr{A}$*. The subset of positive linear functionals in the dual of* $\mathscr{A}$ *will be denoted by* $\mathscr{A}^*_+ \subseteq \mathscr{A}^*$*. If* $\mathscr{A}$ *is unital, then a positive functional* ω *is called a state if in addition* $\omega(\mathbb{1}) = 1$.

The basic feature of positive functionals is that they fulfill the Cauchy-Schwarz inequality:

Lemma 1.1.7 (Cauchy-Schwarz inequality) *Let* $\mathscr{A}$ *be a* **-algebra over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$ *and let* $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ *be a positive functional. Then one has the Cauchy-Schwarz inequality*

$$\omega(a^* b)\overline{\omega(a^* b)} \leq \omega(a^* a)\omega(b^* b) \tag{1.1.3}$$

as well as the reality condition

$$\omega(a^* b) = \overline{\omega(b^* a)} \tag{1.1.4}$$

for all $a, b \in \mathscr{A}$.

PROOF: We recall the proof which is entirely standard. For $z, w \in \mathsf{C}$ we consider

$$p(z,w) = \omega((za+wb)^*(za+wb)) = \overline{z}z\omega(a^*a) + \overline{z}w\omega(a^*b) + \overline{w}z\omega(b^*a) + \overline{w}w\omega(b^*b) \geq 0.$$

This is indeed non-negative by the positivity of ω. Evaluating this for $z = 1$ and $w = 1$ as well as $z = \mathrm{i}$ and $w = 1$ gives (1.1.4) at once. Considering the case where $\omega(a^*a) = 0 = \omega(b^*b)$ we get with $w = \omega(b^*a) = \overline{\omega(a^*b)}$ the inequality $(z+\overline{z})\omega(a^*b)\overline{\omega(a^*b)} \geq 0$ for all z. This can only be true if $\omega(a^*b) = 0$, too. For the remaining case we can assume that e.g. $\omega(a^*a) > 0$. Taking $w = \omega(a^*a)$ and $z = -\omega(a^*b)$ gives then $-\omega(a^*a)\omega(a^*b)\overline{\omega(a^*b)} + \omega(a^*a)\omega(a^*a)\omega(b^*b) \geq 0$. Thus (1.1.3) holds also in this case. □

If $\mathscr{A}$ is unital then (1.1.4) implies $\omega(a^*) = \overline{\omega(a)}$. In this case $\omega(\mathbb{1}) = 0$ implies $\omega = 0$ by (1.1.3). Thus one can safely focus on normalized positive functionals where $\omega(\mathbb{1}) = 1$, where one only has to pass to the quotient field if necessary.

Remark 1.1.8 (Physical interpretation) Let us briefly recall the physical interpretation of the positive functionals: while the Hermitian elements of the *-algebra $\mathscr{A}$ are considered to represent the observables of the physical system described by $\mathscr{A}$, the positive functionals play to role of the physical *states*. Without much restriction, we assume that $\mathscr{A}$ is unital and $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ is a state in the sense of Definition 1.1.6. Then for a given Hermitian element $a \in \mathscr{A}$ the scalar $\mathrm{E}_\omega(a) = \omega(a)$ is interpreted as the *expectation value* of a in the physical state represented by ω. This is the average number one gets after (idealized: infinitely) repeated measurements of the quantity a in the state ω, which of course has to be prepared again after each measurement. Note that the possible outcomes of a measurement will be, at least in quantum theory, quite restricted: the available numbers constitute the *physical spectrum* of the observable a. In a next step, one can define the *variance* by

$$\mathrm{Var}_\omega(a) = \omega((a-\omega(a)\mathbb{1})^*(a-\omega(a)\mathbb{1})) = \omega(a^*a) - \omega(a)\overline{\omega(a)} \geq 0. \tag{1.1.5}$$

Now the positivity property of ω ensures that the variance is indeed non-negative. Finally, one can consider the *covariance* for several (Hermitian) observables $a_1, \ldots, a_n \in \mathscr{A}$ defined to be the matrix

$$\mathrm{Cov}_\omega(a_i, a_j) = \omega((a_i - \omega(a_i)\mathbb{1})^*(a_j - \omega(a_j)\mathbb{1})). \tag{1.1.6}$$

Again, one can show that this is a non-negative matrix in sense to be explained below, see also Exercise 1.4.10. This shows that for a physical interpretation the positivity (and the normalization) of the functionals is crucial as variances and covariances should of course be non-negative. However, it is a highly non-trivial question to decide whether there is indeed some probability-theoretic background guaranteeing that these quantities are indeed variances and covariances of a certain probability distribution on the allowed values which the observable can take, i.e. its (physical) *spectrum*. It will need a fair amount of analysis to obtain a reasonable notion of spectrum: One wants to define the (mathematical) spectrum $\mathrm{spec}(a)$ of a normal or Hermitian element a together with a probability distribution, the *spectral measure* μ_ω for each state ω such that

$$\omega(a) = \int_{\lambda \in \mathrm{spec}(a)} \lambda \, \mathrm{d}\mu_\omega, \tag{1.1.7}$$

i.e. the algebraically defined expectation value $\omega(a)$ is the expectation value in a measure-theoretic sense. It is a remarkable and non-trivial result in C^*-algebra theory, the spectral theorem, that this is actually possible. However, beyond C^*-algebras this last aspect needed for a physical application seems to be very hard to get. For O^*-algebras one can still rely on the ambient algebra of bounded operators on a Hilbert space and transfer the spectral theorem to the unbounded situation, provided some technical assumptions are met, like the questions of self-adjointness etc., which becomes much more delicate when dealing with a whole algebra of (unbounded) operators instead of a single operator. Here one can consult e.g. the monographs [104, 105] as well as for some more recent considerations. Needless to say, this becomes even worse in the framework of formal deformation quantization where one works with the rings of formal power series $\mathbb{R}[\![\lambda]\!]$ and $\mathbb{C}[\![\lambda]\!]$ instead of $\mathbb{R}$ and $\mathbb{C}$. Hence we shall ignore these questions in the following and focus on the algebraic part only.

Using positive functionals we can now define positive algebra elements as well:

Definition 1.1.9 (Positive algebra element) *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) An element $a \in \mathscr{A}$ is called positive if for all positive functionals $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ one has

$$\omega(a) \geq 0. \tag{1.1.8}$$

ii.) The subset of positive elements of $\mathscr{A}$ is denoted by $\mathscr{A}^+$.

iii.) The elements in the subset

$$\mathscr{A}^{++} = \Big\{ a \in \mathcal{A} \;\Big|\; a = \sum\nolimits_{i=1}^{n} \beta_i b_i^* b_i \text{ with } b_i \in \mathscr{A}, \beta_i > 0 \Big\}. \tag{1.1.9}$$

are called the algebraically positive elements or sums of squares.

Lemma 1.1.10 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ be a positive functional.*

i.) The positive functionals form a convex cone in the dual $\mathscr{A}^* = \mathrm{Hom}_{\mathsf{C}}(\mathscr{A}, \mathsf{C})$.

ii.) For all $b \in \mathscr{A}$ the functional $a \mapsto \omega_b(a) = \omega(b^* a b)$ is again positive.

iii.) The positive elements $\mathscr{A}^+$ as well as the algebraically positive elements $\mathscr{A}^{++}$ form convex cones in $\mathscr{A}$.

iv.) We have $\mathscr{A}^{++} \subseteq \mathscr{A}^+$.

v.) We have $b^* \mathscr{A}^+ b \subseteq \mathscr{A}^+$ and $b^* \mathscr{A}^{++} b \subseteq \mathscr{A}^{++}$ for all $b \in \mathscr{A}$.

PROOF: The first part is clear and the second follows from $\omega_b(a^* a) = \omega((ab)^*(ab)) \geq 0$. The third part is a consequence of the first. The fourth part is clear by definition and the last part follows from the second. □

The extreme points in the convex set of states are also called the *pure states*: here a state ω is pure if $\omega = \alpha\omega_1 + (1 - \alpha)\omega_2$ with other states ω_1 and ω_2 and $0 < \alpha < 1$ implies that $\omega_1 = \omega_2$. For a commutative *-algebra over $\mathbb{C}$ one would expect from physical considerations that the pure states coincide with the *characters* of the algebra, i.e. the *-homomorphisms to $\mathbb{C}$. Under favorable circumstances and non-trivial usage of analytic techniques one can actually show such statements, see e.g. for a recent approach in the unbounded situation.

In general, it is a highly nontrivial question to decide whether $\mathscr{A}^{++}$ is actually equal to $\mathscr{A}^+$. In fact, for the algebra of complex polynomials in several variables it is one of the famous Hilbert problems. Typically, we expect that $\mathscr{A}^{++}$ is strictly smaller than $\mathscr{A}^+$, see also Exercise 1.4.18. We will have to come back to this difficulty at various places.

Structure preserving maps are in any field of mathematics of major interest. In our situation we have on one hand the *-homomorphisms between *-algebras. However, there is also a slightly weaker notion of maps which preserve only positivity. As preparation we need the following simple observation, see Exercise 1.4.5:

Lemma 1.1.11 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then the matrices $\mathrm{M}_n(\mathscr{A})$ are a *-algebra for all $n \in \mathbb{N}$ with respect to the usual matrix multiplication and the *-involution $(a_{ij})^* = (a_{ji}^*)$ where $(a_{ij}) \in \mathrm{M}_n(\mathscr{A})$.*

Definition 1.1.12 (Positive maps) *A linear map $\phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is called positive if*

$$\phi(\mathscr{A}^+) \subseteq \mathscr{B}^+. \tag{1.1.10}$$

Moreover, ϕ is called n-positive for $n \in \mathbb{N}$ if

$$\phi^{(n)}\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathrm{M}_n(\mathscr{B}) \tag{1.1.11}$$

is positive, where $\phi^{(n)}$ is defined by applying ϕ componentwise. Finally, ϕ is called completely positive if $\phi^{(n)}$ is positive for all $n \in \mathbb{N}$.

The following properties are now obtained analogously to the statements of Lemma 1.1.10:

Remark 1.1.13 (Positive maps) Let $n \in \mathbb{N}$.

i.) The n-positive maps as well as the completely positive maps form convex cones inside all linear maps $\mathrm{Hom}_{\mathsf{C}}(\mathscr{A}, \mathscr{B})$.

ii.) The composition of n-positive (completely positive) maps is again n-positive (completely positive).

iii.) *-Homomorphisms are completely positive. In particular, the inclusions of *-subalgebras $\mathscr{B} \subseteq \mathscr{A}$ turn positive elements of $\mathscr{B}$ into positive elements of $\mathscr{A}$.

iv.) If ϕ is n-positive, then ϕ is also $(n-1)$-positive. This is actually not completely obvious and will be discussed in Exercise 1.4.6.

There are simple examples of positive maps which are not completely positive. In fact, already for the complex 2×2 matrices $\mathrm{M}_2(\mathbb{C})$ it is a standard example that the matrix transposition is positive but *not* 2-positive, see Exercise 1.4.6. The other important observation is that there are completely positive maps which are not *-homomorphisms. We list here the following two examples which will be needed later:

Example 1.1.14 Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then the maps $\mathrm{tr}, \tau\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathscr{A}$ with

$$\mathrm{tr}(A) = \sum_{i=1}^n a_{ii} \quad \text{and} \quad \tau(A) = \sum_{i,j=1}^n a_{ij} \tag{1.1.12}$$

are completely positive maps, see also Exercise 1.4.6.

Remark 1.1.15 (Quantum information) The fact that already for matrices there are positive but not completely positive maps plays a central role in quantum information theory and the theory of open quantum systems. As a first reading one should consult [21, 92].

Remark 1.1.16 (Strong positivity and O^*-algebras) There are other notions of positivity which take care of more specific properties of a *-algebra: for *-algebras over $\mathbb{C}$ one might consider topological properties which allow to ask for continuity of positive functionals. Thus one might specify a *sub-cone* $\mathscr{K}$ of all positive functionals such that $\mathscr{K}$ is still stable under the map $\omega \mapsto \omega_b$ for $b \in \mathscr{A}$. Having *less* positive functionals results in *more* positive elements, now called $\mathscr{K}$-positive elements. If this can be done consistently not only for $\mathscr{A}$ but also for all matrices $\mathrm{M}_n(\mathscr{A})$ then one can speak of completely $\mathscr{K}$-positive maps. Of course, this will all depend crucially on the choice of $\mathscr{K}$ and thus one can not expect the same good functorial behaviour as for the canonical choice of *all* positive functionals. Nevertheless, in the theory of unbounded operator algebras (O^*-algebras) on a pre-Hilbert space $\mathcal{H}$ this plays a central role leading to the notion of *strong positivity*, see [104, Sect. 2.6] for a further reading. Here the basic idea is to specify the positive cone by requiring $\langle \phi, A\phi \rangle \geq 0$ for all $\phi \in \mathcal{H}$ in the domain of the operator A. A more abstract version of O^*-algebras and their cones of positive elements and positive functionals can be found in .

For later use we mention the following result from [48, Lem. 6.5] on positive elements. Note that in general the following positive elements will *not* be sums of squares:

Proposition 1.1.17 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $g \in \mathrm{M}_n(\mathsf{C})^+$ be a positive matrix. Then for all $m \in \mathbb{N}$ and all $a_{k_1 \ldots k_m} \in \mathscr{A}$ with $k_1, \ldots, k_m = 1, \ldots, n$ we have*

$$\sum_{k_1,\ell_1,\ldots,k_m,\ell_m=1}^{n} g^{k_1\ell_1} \cdots g^{k_m\ell_m} a^*_{k_1\ldots k_m} a_{\ell_1\ldots\ell_m} \in \mathscr{A}^+. \tag{1.1.13}$$

Proof: We rely on the characterization of positive matrices from Exercise 1.4.4. First we note that $G = g \otimes \cdots \otimes g \in \mathrm{M}_n(\mathsf{C}) \otimes \cdots \otimes \mathrm{M}_n(\mathsf{C}) = \mathrm{M}_{n^m}(\mathsf{C})$ is still a positive matrix since for all $z^{(1)}, \ldots, z^{(m)} \in \mathsf{C}^m$ we have

$$\begin{aligned}
&\sum_{k_1,\ell_1,\ldots,k_m,\ell_m=1}^{n} g^{k_1\ell_1} \cdots g^{k_m\ell_m} \overline{z^{(1)}_{k_1}} \cdots \overline{z^{(m)}_{k_m}} z^{(1)}_{\ell_1} \cdots z^{(m)}_{\ell_m} \\
&= \left(\sum_{k_1,\ell_1=1}^{n} g^{k_1\ell_1} \overline{z^{(1)}_{k_1}} z^{(1)}_{\ell_1} \right) \cdots \left(\sum_{k_m,\ell_m=1}^{n} g^{k_m\ell_m} \overline{z^{(m)}_{k_m}} z^{(m)}_{\ell_m} \right) \\
&= \left\langle z^{(1)}, g z^{(1)} \right\rangle \cdots \left\langle z^{(m)}, g z^{(m)} \right\rangle \\
&\geq 0.
\end{aligned}$$

Next, we note that for a positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ the matrix $(\omega(a_i^* a_j))_{i,j=1,\ldots,N} \in \mathrm{M}_N(\mathsf{C})$ is positive for arbitrary $a_1, \ldots, a_N \in \mathscr{A}$. Indeed, for $z_1, \ldots, z_N \in \mathsf{C}$ we have

$$\sum_{i,j=1}^{N} \overline{z}_i \omega(a_i^* a_j) z_j = \omega\left(\left(\sum\nolimits_{i=1}^{N} z_i a_i \right)^* \left(\sum\nolimits_{i=1}^{N} z_j a_j \right) \right) \geq 0,$$

which implies the positivity. Thus the matrix $\Omega = (\omega(a^*_{k_1\ldots k_m} a_{\ell_1\ldots\ell_m})) \in \mathrm{M}_{n^m}(\mathsf{C})$ is positive for all positive linear functionals ω. Thus we have

$$\begin{aligned}
\omega\left(\sum_{k_1,\ell_1,\ldots,k_m,\ell_m=1}^{n} g^{k_1\ell_1} \cdots g^{k_m\ell_m} a^*_{k_1\ldots k_m} a_{\ell_1\ldots\ell_m} \right) &= \sum_{k_1,\ell_1,\ldots,k_m,\ell_m=1}^{n} g^{k_1\ell_1} \cdots g^{k_m\ell_m} \omega\left(a^*_{k_1\ldots k_m} a_{\ell_1\ldots\ell_m}\right) \\
&= \operatorname{tr}(G\Omega) \\
&\geq 0,
\end{aligned}$$

since the trace of the product of two positive matrices is still positive, see Exercise 1.4.4. This completes the proof. ☐

Note that the proof would simplify drastically if we knew that the matrix g is diagonalizable with non-negative entries on the diagonal: over C with an ordered ring R this can not be assumed directly. For an ordered field, however, the above proof can be simplified using a basis of eigenvectors.

1.1.3 Examples of *-Algebras

Let us now collect some classes of examples of *-algebras. The first ones are still for the choice $\mathsf{R} = \mathbb{R}$ and hence $\mathsf{C} = \mathbb{C}$.

Example 1.1.18 (Smooth functions) Let M be a smooth manifold and denote by $\mathscr{C}^\infty(M)$ the smooth complex-valued functions. Then $\mathscr{C}^\infty(M)$ forms a unital *-algebra with respect to the pointwise (commutative) multiplication and the pointwise complex conjugation as *-involution. The compactly supported smooth functions $\mathscr{C}^\infty_0(M)$ are then a *-ideal which is proper iff M is non-compact.

This is the class of examples underlying classical geometric mechanics on the one hand and deformation quantization on the other hand. Analogously, other classes of differentiability like $\mathscr{C}^k(M)$ with $k \in \mathbb{N}$ provide *-algebras as well. In general one has $\mathscr{C}^\infty(M)^{++} \neq \mathscr{C}^\infty(M)^+$, see Exercise 1.4.18.

One of the most important class of *-algebras is the class of C^*-algebras. Physically speaking, a C^*-algebra is the prototype of an observable algebra in quantum theory. We recall the definition: a *-algebra $\mathscr{A}$ over $\mathbb{C}$ is called *C^*-algebra* if it is equipped with a complete norm $\|\cdot\|$ satisfying

$$\|ab\| \leq \|a\|\|b\| \tag{1.1.14}$$

and

$$\|a^*a\| = \|a\|^2 \tag{1.1.15}$$

for all $a, b \in \mathscr{A}$. If $\mathscr{A}$ is a unital C^*-algebra, $\|\mathbb{1}\| = 1$ follows from (1.1.15). The first condition simply implies that the product is continuous. The second condition in (1.1.15) is the *C^*-condition*. It follows that

$$\|a^*\| = \|a\| \tag{1.1.16}$$

for all $a \in \mathscr{A}$ and hence the *-involution is continuous as well. If only (1.1.14) and (1.1.16) are satisfied, then $\mathscr{A}$ is called a *Banach *-algebra*.

C^*-algebras have many nice features. The most prominent, among many others, is the continuous spectral calculus which allows us to define continuous functions of Hermitian (more generally, of normal) elements in a consistent way as a consequence. It follows that $\mathscr{A}^+ = \mathscr{A}^{++}$ for a C^*-algebra. In the following, we shall sometimes use C^*-algebras as particular examples to illustrate how the *analytic* techniques provide additional *algebraic* features not present in general. For further reading, consult the extensive literature like e.g. [10, 19, 20, 41, 45, 68, 69, 81].

There are two particular cases of C^*-algebras of interest. In fact, it turns out that up to isomorphism the following two scenarios yield already all C^*-algebras:

Example 1.1.19 (Commutative C^*-algebras) The continuous complex-valued functions $\mathscr{C}(X)$ on a compact Hausdorff space X form a unital commutative C^*-algebra where the C^*-norm is the sup-norm $\|\cdot\|_\infty$. In fact, all commutative C^*-algebras with unit are of this form up to isomorphism with X uniquely determined up to homeomorphism. This classical statement of Gel'fand and Naimark can be formulated as an equivalence of the category of compact Hausdorff spaces and the category of unital commutative C^*-algebras. In this sense, the commutative C^*-algebras serve as model for classical, i.e. commutative geometries.

Example 1.1.20 (Closed subalgebras of $\mathfrak{B}(\mathfrak{H})$) For a complex Hilbert space $\mathfrak{H}$ the continuous (i.e. bounded) operators $\mathfrak{B}(\mathfrak{H})$ form a C^*-algebra with respect to the operator product and the operator norm. More generally, any closed *-subalgebra of $\mathfrak{B}(\mathfrak{H})$ is again a C^*-algebra. It is one of the remarkable aspects of the theory of C^*-algebras that *any* C^*-algebra can be obtained like this up to isomorphism. In view of the commutative situation, one considers noncommutative C^*-algebras as being the continuous functions on a *noncommutative* topological space, which of course only exists by means of its function algebra, i.e. the noncommutative C^*-algebra. This is the starting point of noncommutative geometry in the sense of Connes [38], see also the various textbooks and monographs on noncommutative geometry [3, 35, 39, 59, 74, 75, 80, 86, 111]. Beside being mathematically extremely fruitful, such noncommutative geometries are considered to play a central role in fundamental theories of physics at very small length scales: here quantum effects should become relevant for the very notion of geometry itself, thereby leading to some sort of noncommutativity. Of course, at the moment this is very much speculation as there are by no means experiments possible which could probe the relevant scales.

While the above examples work over the complex numbers the next class requires to pass from $\mathbb{R}$ to $\mathbb{R}[\![\lambda]\!]$. In Chapter 6 we will put this into a much larger context when discussing formal deformations of *-algebras in general.

Example 1.1.21 (Star products) For a smooth manifold M one considers a $\mathbb{C}[\![\lambda]\!]$-bilinear multiplication $\star$ for $\mathscr{C}^\infty(M)[\![\lambda]\!]$. The required bilinearity over $\mathbb{C}[\![\lambda]\!]$ implies that for $f, g \in \mathscr{C}^\infty(M)$ we can write $f \star g$ as

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g) \tag{1.1.17}$$

with $\mathbb{C}$-bilinear operators C_r, which we extend $\mathbb{C}[\![\lambda]\!]$-bilinearly to all of $\mathscr{C}^\infty(M)[\![\lambda]\!]$. Such a product is called a *star product* if C_0 is just the usual pointwise multiplication of functions [5]. In this case the antisymmetric part of C_1 defines a *Poisson bracket* for $\mathscr{C}^\infty(M)$ and M becomes a *Poisson manifold*. Conversely, for a given Poisson manifold, one tries to find and classify such deformations $\star$. For technical reasons one requires the operators C_r to be bidifferential operators. Moreover, for higher $r \geq 1$ they should vanish on constants which is equivalent to $1 \star f = f = f \star 1$. Up to now, no reality assumption has been made. One considers

$$\{f, g\} = \frac{1}{\mathrm{i}}(C_1(f, g) - C_1(g, f)) \tag{1.1.18}$$

as Poisson bracket and requires this to be a *real* Poisson bracket, i.e. $\overline{\{f, g\}} = \{\overline{f}, \overline{g}\}$. Then the star product is called *Hermitian* if

$$\overline{f \star g} = \overline{g} \star \overline{f}, \tag{1.1.19}$$

i.e. the complex conjugation is a *-involution for $\star$. Note that the rescaling (1.1.18) is necessary as $\lambda \in \mathbb{R}[\![\lambda]\!]$ is considered to be a *real* and in fact positive element in the ordered ring $\mathbb{R}[\![\lambda]\!]$ of scalars. It is a celebrated theorem of Kontsevich [77] that gives both the general existence of star products on Poisson manifolds as well as a full classification of them. Much more can be said on this class of *-algebras: details on star products and deformation quantization can be found e.g. in the introductory textbook [116] as well as in [47]. It is this class of examples which provides the core motivation to extend the notions of representation theory from C^*-algebras to our more general framework. Note that the usage of formal power series and hence the usage of a *non-Archimedian* ordered ring is inevitable as in general not much can be said about the convergence of star products. Of course, $\lambda > 0$ points already into the correct direction which for a yet to be shown convergence of the formal series will be relevant: once we have convergence established, the formal parameter becomes the positive Planck constant $\hbar$. Thus another way of interpreting the ordering of $\mathbb{R}[\![\lambda]\!]$ is that λ is infinitesimally small but yet positive. We will come back to this example at many occasions and consider deformation theory in general in Chapter 6.

The next two examples are more algebraic. They allow to encode symmetries of various kinds:

Example 1.1.22 (Group algebra) Let G be a group. Then the complex group algebra $\mathbb{C}[G]$ is defined as the vector space with basis vectors being the group elements $g \in G$. The multiplication is just the group multiplication extended $\mathbb{C}$-bilinearly. It is then an easy check that $g^* = g^{-1}$ extends to a *-involution for $\mathbb{C}[G]$ such that the group elements are unitary. In addition, the group algebra is also a Hopf algebra where the coproduct is determined by $\Delta(g) = g \otimes g$, the counit is determined by $\epsilon(g) = 1$, and the antipode is determined by $S(g) = g^{-1}$. Then the group can be recovered as the set of group-like elements in $\mathbb{C}[G]$. Finally, the *-involution is compatible with the Hopf algebra structure leading to a Hopf *-algebra. More information on the general theory of Hopf *-algebras and their representation theory can be found e.g. in the textbooks [37, 73, 76]. Needless to say, we can replace $\mathbb{C}$ by any ring $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and obtain a group algebra $\mathsf{C}[G]$ as well.

While the group algebra is used to describe symmetries in various contexts, the infinitesimal versions of symmetries are usually formulated via Lie algebras.

Example 1.1.23 (Universal enveloping algebra) Let $\mathfrak{g}$ be a real Lie algebra. Then the *universal enveloping algebra* of $\mathfrak{g}$ is

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}^\bullet(\mathfrak{g}) \big/ \big\langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \bigm| \xi, \eta \in \mathfrak{g} \big\rangle, \tag{1.1.20}$$

where $\mathrm{T}^\bullet(\mathfrak{g})$ denotes the tensor algebra over $\mathfrak{g}$ and $\langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \rangle$ denotes the two-sided ideal generated by the elements of the form $\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{g}$. The universal enveloping algebra is generated by $\mathbb{1}$ and $\mathfrak{g}$ and is universal for the relations $\xi \cdot \eta - \eta \cdot \xi = [\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$. Then the complexification

$$\mathrm{U}_{\mathbb{C}}(\mathfrak{g}) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C} \tag{1.1.21}$$

becomes a *-algebra over $\mathbb{C}$ by requiring

$$\xi^* = -\xi \tag{1.1.22}$$

for all $\xi \in \mathfrak{g}$, i.e. the Lie algebra elements become *anti-Hermitian* elements. Also in this case we have a Hopf *-algebra structure where $\Delta(\xi) = \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi$, $\epsilon(\xi) = 0$, and $S(\xi) = -\xi$. Finally, we can replace $\mathbb{R}$ by any ordered ring R and $\mathbb{C}$ by $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

In order to obtain more examples we first have to recall the definition of a pre-Hilbert space over C. This is the direct translation of the usual definition of a complex pre-Hilbert space.

Definition 1.1.24 (Pre-Hilbert space and adjointable maps) *A pre-Hilbert space $\mathcal{H}$ over C is a C-module with an inner product*

$$\langle \cdot\,, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \longrightarrow \mathsf{C} \tag{1.1.23}$$

such that

i.) $\langle \cdot\,, \cdot \rangle$ *is C-linear in the second argument,*

ii.) $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$ *for all $\phi, \psi \in \mathcal{H}$,*

iii.) $\langle \phi, \phi \rangle > 0$ *for $\phi \neq 0$.*

A map $A\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ between pre-Hilbert spaces is called adjointable if there exists an adjoint map $A^*\colon \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

$$\langle \phi, A\psi \rangle_2 = \langle A^*\phi, \psi \rangle_1 \tag{1.1.24}$$

for all $\phi \in \mathcal{H}_2$ and $\psi \in \mathcal{H}_1$. We denote the set of all such adjointable operators by $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$.

Note that the notion "space" does *not* imply that the module $\mathcal{H}$ behaves like a vector space, i.e. $\mathcal{H}$ needs *not* to be free at all. In fact, later on we shall meet pre-Hilbert modules over other *-algebras, so the usage of "space" should only imply that the module is over the scalars C.

Note also, that the adjoint A^* of an adjointable operator is necessarily unique and both A and A^* are C-linear. We will come back to these properties in a much more general context, see also Exercise 1.4.8 for a first approach.

Example 1.1.25 (Pre-Hilbert space) The standard example of a pre-Hilbert space is given by the free module C^n for $n \in \mathbb{N}$. The canonical inner product is given by

$$\langle z, w \rangle = \sum_{k=1}^{n} \overline{z}^k w^k, \tag{1.1.25}$$

where $z = \sum_{k=1}^{n} z^k e_k$ with the standard basis vectors $e_1, \ldots, e_n$ of C^n. More generally, for any set I we can consider the direct sum $\mathsf{C}^{(I)}$ of I copies of C and define the canonical inner product the same way, as now only finitely many coefficients z^i are different from zero in $z = \sum_{i \in I} z^i e_i \in \mathsf{C}^{(I)}$. Hence

$$\langle z, w \rangle = \sum_{i \in I} \overline{z}^i w^i \tag{1.1.26}$$

is well-defined and yields a pre-Hilbert space structure for $\mathsf{C}^{(I)}$. Note that the same formula would *not* work for the Cartesian product C^I if I is infinite.

Example 1.1.26 (Adjointable maps) Let $\mathcal{H}$ be a pre-Hilbert space. The adjointable operators

$$\mathfrak{B}(\mathcal{H}) = \mathfrak{B}(\mathcal{H}, \mathcal{H}) \tag{1.1.27}$$

form a subalgebra of $\mathrm{End}_{\mathsf{C}}(\mathcal{H})$. It is easy to see that $\mathfrak{B}(\mathcal{H})$ becomes a *-algebra with unit $\mathbb{1} = \mathrm{id}_{\mathcal{H}}$ and with the *-involution $^*\colon A \mapsto A^*$. In fact, the composition of adjointable operators is again adjointable in general. Note that for the case of a complex *Hilbert space* $\mathfrak{H}$ instead of $\mathcal{H}$, this definition reproduces the bounded linear operators by the Hellinger-Toeplitz theorem, see e.g. [102, p. 117]. However, already for complex pre-Hilbert spaces the operators $\mathfrak{B}(\mathcal{H})$ typically contain many *unbounded* operators, see e.g. the monograph [104] for a detailed discussion of such unbounded operator algebras. As a first remark on the structure of the *-algebra $\mathfrak{B}(\mathcal{H})$ we note that it is *torsion-free*: if $A \in \mathfrak{B}(\mathcal{H})$ is non-zero and $z \in \mathsf{C}$ satisfies $zA = 0$ then $z = 0$: indeed, choose a vector $\phi \in \mathcal{H}$ with $A\phi \neq 0$. Then $0 < \langle A\phi, A\phi \rangle$ and $0 = \langle zA\phi, zA\phi \rangle = \overline{z}z\langle A\phi, A\phi \rangle$. This implies $z = 0$. It follows that also all *-subalgebras of $\mathfrak{B}(\mathcal{H})$ are torsion-free, a feature which is of course trivially fulfilled if R and hence C are fields. With the same argument one shows that $\mathcal{H}$ is torsion-free as well. Moreover, if $A \in \mathfrak{B}(\mathcal{H})$ satisfies $A^*A = 0$ then we already have $A = 0$. Finally, if A is normal and nilpotent then $A = 0$ follows, see also Exercise 1.4.8.

Example 1.1.27 (Finite-rank operators) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be pre-Hilbert spaces. For $\phi \in \mathcal{H}_2$ and $\psi \in \mathcal{H}_1$ we define the *rank-one operator* $\Theta_{\phi,\psi}\colon \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ by

$$\Theta_{\phi,\psi}(\chi) = \phi\langle \psi, \chi \rangle_1. \tag{1.1.28}$$

In Dirac's *bra-ket notation* this would be $\Theta_{\phi,\psi} = |\phi\rangle\langle\psi|$. However, we will prefer the notation (1.1.28). The *finite-rank operators* are then defined by

$$\mathfrak{F}(\mathcal{H}) = \left\{ A\colon \mathcal{H} \longrightarrow \mathcal{H} \;\middle|\; A = \sum\nolimits_i \Theta_{\phi_i,\psi_i} \text{ for some } \phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_n \in \mathcal{H} \right\} \subseteq \mathfrak{B}(\mathcal{H}). \tag{1.1.29}$$

They form a *-ideal in $\mathfrak{B}(\mathcal{H})$. Note that

$$\Theta^*_{\phi,\psi} = \Theta_{\psi,\phi}. \tag{1.1.30}$$

In the case of a complex Hilbert space $\mathfrak{H}$, the completion of $\mathfrak{F}(\mathfrak{H})$ with respect to the operator norm will give the *compact operators* $\mathfrak{K}(\mathfrak{H})$, which form a norm-closed *-ideal in $\mathfrak{B}(\mathfrak{H})$. Hence $\mathfrak{K}(\mathfrak{H})$ is a (in general non-unital) C^*-algebra. For an infinite-dimensional separable Hilbert space, the quotient C^*-algebra $\mathfrak{C}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{H})/\mathfrak{K}(\mathfrak{H})$ is called the *Calkin algebra*, see e.g. [69, Sect. 10.4] or [10, Sect. I.8.2].

Example 1.1.28 For the pre-Hilbert space C^n it is easy to see that the finite-rank operators $\mathfrak{F}(\mathsf{C}^n)$ just coincide with all linear endomorphisms of C^n. Hence we get

$$\mathfrak{F}(\mathsf{C}^n) = \mathfrak{B}(\mathsf{C}^n) = \mathrm{End}(\mathsf{C}^n). \tag{1.1.31}$$

If we pass to the infinite-dimensional version of *countably many* copies of C, we denote the corresponding pre-Hilbert space simply by C^∞. In this case, it is easy to see that the finite-rank operators are given by the infinite matrices with at most finitely many non-trivial entries, denoted by $\mathrm{M}_\infty(\mathsf{C})$. Clearly, such matrices provide a (non-unital) *-algebra with respect to the matrix multiplication and the usual adjoint of matrices. Identifying these matrices with the operators acting on C^∞ we get

$$\mathfrak{F}(\mathsf{C}^\infty) = \mathrm{M}_\infty(\mathsf{C}). \tag{1.1.32}$$

The last example we would like to mention is based on the passage from a *-algebra to matrices with entries in a *-algebra:

Example 1.1.29 (The algebras $\mathrm{M}_n(\mathscr{A})$ and $\mathrm{M}_\infty(\mathscr{A})$) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and $n \in \mathbb{N}$. Then the $n \times n$ matrices $\mathrm{M}_n(\mathscr{A})$ with entries in $\mathscr{A}$ form a *-algebra with respect to the usual matrix multiplication based on the product of $\mathscr{A}$ and the usual matrix adjoint using the *-involution of $\mathscr{A}$ instead of complex conjugation as we have seen in Lemma 1.1.11. Clearly, $\mathrm{M}_n(\mathscr{A})$ is unital iff $\mathscr{A}$ is unital. We can extend this also to infinite matrices of arbitrary size (indexed by pairs (i, j) with $i, j \in I$), provided we have at most finitely many non-zero entries. The case of $I = \mathbb{N}$, i.e. countably infinite matrices will simply be denoted by $\mathrm{M}_\infty(\mathscr{A})$. Now $\mathrm{M}_\infty(\mathscr{A})$ is no longer unital.

1.2 *-Representations and the GNS Construction

It is a general theme in algebra to study a ring by means of its modules. In our case we would like to establish a module category which captures the structure of a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ as good as possible. To this end, we use the situation of C^*-algebras as guideline.

The reasons are at least two-fold. First, for C^*-algebras there is a very well-developed theory of *-representations on Hilbert spaces which provides deep insight into the structure of C^*-algebras. Since any C^*-algebra can, by means of a *-representation, be identified as a closed *-subalgebra of the bounded operators on a suitable Hilbert space, this seems to be the right choice.

The second motivation comes from quantum physics: the observables of a quantum system will be described by a C^*-algebra (or just a *-algebra) while the physical states are encoded as positive normalized functionals, i.e. the states in the sense of Definition 1.1.6. This will be enough to define expectation values for measurements. However, one crucial ingredient for quantum physics is still missing: two pure states ω_1 and ω_2 of quantum systems can be *superposed*. Here we encounter a serious problem since we can form convex combination of positive functionals but clearly not arbitrary linear combinations. The way out in quantum physics is that one has to pass to a representation π on a (pre-) Hilbert space $\mathcal{H}$ and then try to find vectors $\phi_1, \phi_2 \in \mathcal{H}$ such that ω_1 and ω_2 are realized as *vector states*, i.e.

$$\omega_k(a) = \langle \phi_k, \pi(a)\phi_k \rangle \tag{1.2.1}$$

for all $a \in \mathscr{A}$. In this situation one can form arbitrary linear combinations $\phi = z_1\phi_1 + z_2\phi_2$ of the *state vectors* to get a new state vector ϕ. After normalizing properly, this will give a new state $\omega(a) = \langle \phi, \pi(a)\phi \rangle$. It is clear that the additional linear structure of the representation space is needed for this construction. The crucial questions are now, which *-representation of $\mathscr{A}$ should be used and whether and how a given state ω can be realized as vector state.

1.2.1 *-Representations on Pre-Hilbert Spaces

The following definition of a *-representation is now the direct translation of the C^*-algebraic situation. The main point is that $\mathfrak{B}(\mathcal{H})$ is a *-algebra itself. Hence we can give yet another motivation for *-representations, namely we want to compare a "complicated" *-algebra $\mathscr{A}$ with the "standard" *-algebra like $\mathfrak{B}(\mathcal{H})$.

Definition 1.2.1 (*-Representation) *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) A *-representation $(\mathfrak{H}, \pi)$ of $\mathcal{A}$ is a *-homomorphism

$$\pi \colon \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H}), \tag{1.2.2}$$

where $\mathfrak{H}$ is a pre-Hilbert space over C.

ii.) An intertwiner $T \colon (\mathfrak{H}_1, \pi_1) \longrightarrow (\mathfrak{H}_2, \pi_2)$ from one *-representation of $\mathcal{A}$ to another is an adjointable map $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ such that for all $a \in \mathcal{A}$ one has

$$T\pi_1(a) = \pi_2(a)T. \tag{1.2.3}$$

iii.) The representation theory of $\mathcal{A}$ is the category *-$\mathsf{rep}(\mathcal{A})$ with *-representations of $\mathcal{A}$ as objects and intertwiners as morphisms.

From what we said on the composition of adjointable operators it is clear that the intertwiners indeed form morphisms of a category. This justifies the third part of the definition.

Remark 1.2.2 (Different conventions for intertwiners) There is yet another possibility of introducing the notion of intertwiners. Equally reasonable would be to consider *isometric* C-linear maps $T \colon (\mathfrak{H}_1, \pi_1) \longrightarrow (\mathfrak{H}_2, \pi_2)$ satisfying the relation (1.2.3). Again, the composition of such isometric intertwiners is an isometric intertwiner and we obtain a category. The isomorphisms in this category would be *unitary* intertwiner and not adjointable bijective intertwiners as in our above version. In case of Hilbert spaces and hence representations by bounded operators, the polar decomposition then shows that an adjointable bijective intertwiner can be factored into a unitary and a positive intertwiner. Hence in this case, both notions of *equivalent* representations lead to the same equivalence classes. In our more general setting, however, these choices will lead to different notions. The reason that we favour adjointable over isometric intertwiners is that there might be isometric maps not allowing for an adjoint. The set of adjointable intertwiners also carries the additional feature of being a module over C, since (1.2.3) is clearly a linear condition for T. Finally, if $T \colon (\mathfrak{H}_1, \pi_1) \longrightarrow (\mathfrak{H}_2, \pi_2)$ is an intertwiner then its adjoint is an intertwiner in the opposite direction $T^* \colon (\mathfrak{H}_2, \pi_2) \longrightarrow (\mathfrak{H}_1, \pi_1)$ since π_1 and π_2 are *-representations and thus one can just take the adjoint of the defining condition (1.2.3). We will later use this fact to formulate the unitary equivalence of representations in a more categorical language. Nevertheless, even with our notion of intertwiner we will mainly be interested in the situation of having unitary intertwiners and unitary equivalence of representations.

Up to now, a *-representation of $\mathcal{A}$ can be quite trivial. To enforce some non-triviality one has several options. It turns out that the strongly non-degenerate *-representations will play a particular role: Here a *-representation $(\mathfrak{H}, \pi)$ is called *strongly non-degenerate* if

$$\pi(\mathcal{A})\mathfrak{H} = \mathfrak{H}, \tag{1.2.4}$$

i.e the C-linear span of vectors of the form $\pi(a)\phi$ is the whole representation space, see also Exercise 1.4.9. If $\mathcal{A}$ has a unit element $\mathbb{1} \in \mathcal{A}$ then $\pi(\mathbb{1}) \in \mathfrak{B}(\mathfrak{H})$ is a projection. In fact, with respect to the orthogonal decomposition $\mathfrak{H} = \pi(\mathbb{1})\mathfrak{H} \oplus (\mathrm{id} - \pi(\mathbb{1}))\mathfrak{H}$, the representation π becomes block-diagonal such that π is identically zero on $(\mathrm{id} - \pi(\mathbb{1}))\mathfrak{H}$ and strongly non-degenerate on the block $\pi(\mathbb{1})\mathfrak{H}$. Hence we can just forget about the trivial part and focus on the strongly non-degenerate block. Also in the non-unital case, we shall mainly be interested in strongly non-degenerate *-representations. In the unital case they can simply be characterized by the condition $\pi(\mathbb{1}) = \mathrm{id}$. We denote the full subcategory of strongly non-degenerate *-representations of $\mathcal{A}$ by *-$\mathsf{Rep}(\mathcal{A})$.

We list some basic examples. The first class of examples shows that the *-representations of group algebras correspond precisely to the unitary representations of the underlying group.

Example 1.2.3 (Group representation) A *-representation $(\mathcal{H}, \pi)$ of the group algebra $\mathsf{C}[G]$ gives a unitary representation of the group G on the pre-Hilbert space $\mathcal{H}$ by $g \mapsto \pi(g)$, i.e. the maps $\pi(g)$ are unitary for all $g \in G$. Conversely, any unitary representation of G canonically extends to a *-representation of $\mathsf{C}[G]$. If $\mathsf{C} = \mathbb{C}$, it is a standard argument that a unitary map on a pre-Hilbert space is actually bounded and thus extends to the Hilbert space completion while staying unitary. Thus we get a unitary representation of G on a Hilbert space. The functoriality is slightly more tricky as our notion of intertwiner does not provide any continuity assumption: so in general the intertwiners between pre-Hilbert space representations will not extend to intertwiners between the completions. However, if the intertwiners happen to be unitary themselves then they extend. Thus the notions of *unitarily equivalent* *-representations of G in both contexts coincide.

Also for C^*-algebras one has an automatic continuity statement:

Example 1.2.4 (*-Representation of C^*-algebra) Let $\mathscr{A}$ be a C^*-algebra and let $(\mathcal{H}, \pi)$ be a *-representation of $\mathscr{A}$ on a pre-Hilbert space $\mathcal{H}$ over $\mathbb{C}$. Then the operators $\pi(a)$ turn out to be bounded and we get a canonical extension $\widehat{\pi}$ to the Hilbert space completion $\widehat{\mathcal{H}}$. For the functoriality we have the same situation as in the group algebra case.

The case of the universal enveloping algebra of a real Lie algebra is closely related to the group algebra case. However, the continuity properties are now much more delicate:

Example 1.2.5 (Lie algebra representation) Let $\mathfrak{g}$ be a Lie algebra over R. A *-representation $(\mathcal{H}, \pi)$ of $\mathrm{U}_{\mathsf{C}}(\mathfrak{g})$ restricts to an *anti-Hermitian* representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}$. Conversely, an anti-Hermitian Lie algebra representation of $\mathfrak{g}$ canonically extends to a *-representation of $\mathrm{U}_{\mathsf{C}}(\mathfrak{g})$ by the universal property of the universal enveloping algebra. In the case $\mathsf{C} = \mathbb{C}$, we do not have any reasonable continuity of the operators $\pi(\xi)$. Instead, for finite-dimensional $\mathfrak{g}$, the interesting question is whether the Lie algebra representation is actually coming from a unitary (strongly continuous) representation of the corresponding (connected and simply connected) Lie group G such that the vectors in $\mathcal{H}$ are smooth vectors of the representation of G. In general, this is a highly nontrivial question, see e.g. [104, Chap. 10] for more details.

We conclude this subsection with the following remark: given an index set I and a family of *-representations $\{(\mathcal{H}_i, \pi_i)\}_{i \in I}$ of $\mathscr{A}$ we can consider their *direct orthogonal sum*. The representation space is defined by

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i, \tag{1.2.5}$$

with the inner product given by

$$\langle (\phi_i)_{i \in I}, (\psi_j)_{j \in J} \rangle = \sum_{i \in I} \langle \phi_i, \psi_i \rangle_i. \tag{1.2.6}$$

Note that this is indeed again a well-defined inner product as thanks to the direct sum we only have finitely many non-zero contributions for two given vectors. The *-representation is then the block-diagonal sum of the single π_i, i.e. we set

$$\pi(a) = \bigoplus_{i \in I} \pi_i(a) \tag{1.2.7}$$

for $a \in \mathscr{A}$. It is now a routine check to see that this defines indeed a *-representation of $\mathscr{A}$. Moreover, let $P_i \in \mathfrak{B}(\mathcal{H})$ be the projection operator onto the i-th component $\mathcal{H}_i$ of the direct sum. Note that we have $P_i^* = P_i$ for all $i \in I$. Then we have

$$P_i \pi(a) = \pi(a) P_i = \pi_i(a). \tag{1.2.8}$$

Hence P_i is a self-intertwiner of the *-representation π. This shows that we can recover the original *-representations form the direct sum by means of these projections. Moreover, it shows that the subspaces $\mathcal{H}_i \subseteq \mathcal{H}$ are invariant subspaces under the representations since $\mathcal{H}_i = \operatorname{im} P_i$.

Conversely, for a given *-representation $(\mathcal{H}, \pi)$ of $\mathscr{A}$ it would be very desirable to decompose it into a direct sum $(\mathcal{H}, \pi) = \bigoplus_{i \in I} (\mathcal{H}_i, \pi_i)$ such that each component $(\mathcal{H}_i, \pi_i)$ is no longer decomposable in a non-trivial way. Such "atoms" of the representation theory are usually called *irreducible*. A complete understanding of the representation theory would be a full list of all irreducible *-representations up to equivalence together with a statement how a given *-representation can be decomposed.

However, beside very particular situations for $\mathsf{C} = \mathbb{C}$ and nice algebras, this seems to be *completely unrealistic*: there might be *-representations which contain invariant subspaces but which do not decompose into a direct sum, as there might be no invariant complement. Moreover, finding a complete list of irreducible *-representations will not be achievable due to the complexity of the problem. Finally, even in the very nice situations for $\mathsf{C} = \mathbb{C}$ and Hilbert spaces instead of pre-Hilbert spaces, an understanding of the decomposition requires heavy analytic machinery: the projections commuting with the *-representation are in the commutant of π, a von Neumann algebra, the type of which might prevent us from a reasonable decomposition.

1.2.2 The GNS Construction

Let us now discuss one of the standard ways to construct *-representations, the GNS construction. We start with a positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$. Then

$$\mathscr{J}_\omega = \left\{ a \in \mathcal{A} \;\middle|\; \omega(a^*a) = 0 \right\} \tag{1.2.9}$$

is a left ideal in $\mathscr{A}$, the so-called *Gel'fand ideal* since

$$\mathscr{J}_\omega = \left\{ a \in \mathcal{A} \;\middle|\; \omega(b^*a) = 0 \text{ for all } b \in \mathscr{A} \right\} = \left\{ a \in \mathcal{A} \;\middle|\; \omega(a^*b) = 0 \text{ for all } b \in \mathscr{A} \right\}. \tag{1.2.10}$$

This follows immediately from the Cauchy-Schwarz inequality (1.1.3). Thus the quotient

$$\mathcal{H}_\omega = \mathcal{A} / \mathcal{J}_\omega \tag{1.2.11}$$

becomes a left $\mathscr{A}$-module in the canonical way, namely by setting

$$\pi_\omega(a)\psi_b = \psi_{ab}, \tag{1.2.12}$$

where $\psi_b \in \mathcal{H}_\omega$ denotes the equivalence class of b. Moreover, we have a well-defined inner product

$$\langle \psi_a, \psi_b \rangle_\omega = \omega(a^*b), \tag{1.2.13}$$

which turns $\mathcal{H}_\omega$ into a pre-Hilbert space. With respect to this inner product, π_ω is a *-representation, the *GNS representation* of $\mathscr{A}$ with respect to ω, named after Gel'fand, Naimark and Segal who considered mainly the case of a C^*-algebra. In this case one can complete the pre-Hilbert space to a Hilbert space and extend the *-representation π to the completion thanks to the automatic continuity of the operators $\pi(a)$ mentioned in Example 1.2.4. This will of course not be available in our general algebraic situation.

Finally, we assume that in addition $\mathscr{A}$ is unital. This gives a distinguished vector $\psi_{\mathbb{1}} \in \mathcal{H}_\omega$ in the GNS representation space. This vector has two features: first it is *cyclic* in the sense that

$$\psi_a = \pi_\omega(a)\psi_{\mathbb{1}}, \tag{1.2.14}$$

i.e. any other vector can be obtained by applying an algebra element to $\psi_{\mathbb{1}}$. In particular, the GNS representation is always strongly non-degenerate. Second, we can recover the positive functional ω as a vector state since

$$\omega(a) = \langle \psi_{\mathbb{1}}, \pi_\omega(a)\psi_{\mathbb{1}} \rangle_\omega \tag{1.2.15}$$

for all $a \in \mathscr{A}$. It is now an easy check that these features characterize the GNS representation up to unitary equivalence:

Proposition 1.2.6 *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ be a positive functional. If $(\mathcal{H}, \pi, \Omega)$ is a cyclic *-representation with cyclic vector Ω such that $\omega(a) = \langle \Omega, \pi(a)\Omega\rangle_{\mathcal{H}}$, then this *-representation is unitarily equivalent to the GNS representation with respect to ω via the unitary intertwiner explicitly given by*

$$U\colon \mathcal{H}_\omega \ni \psi_a \mapsto \pi(a)\Omega \in \mathcal{H}. \tag{1.2.16}$$

PROOF: First we show that U is well-defined at all: let $a \in \mathscr{J}_\omega$ be in the Gel'fand ideal. Then $\langle \pi(a)\Omega, \pi(a)\Omega\rangle = \langle \Omega, \pi(a^*a)\Omega\rangle = \omega(a^*a) = 0$ shows that $\pi(a)\Omega = 0$. Thus U is well-defined and injective. Moreover, U is surjective since we assume that Ω is a cyclic vector for π. Finally, U is isometric since $\langle U\psi_a, U\psi_b\rangle = \langle \pi(a)\Omega, \pi(b)\Omega\rangle = \langle \Omega, \pi(a^*b)\Omega\rangle = \omega(a^*b) = \langle \psi_a, \psi_b\rangle_\omega$ for all $a, b \in \mathscr{A}$. Since U is bijective, this is all we need to conclude that it is unitary. The intertwiner property is clear by construction. □

There are numerous examples where the GNS representation can be determined explicitly for a given positive functional. The first is the defining representation of $\mathfrak{B}(\mathcal{H})$:

Example 1.2.7 Let $\mathcal{H}$ be a pre-Hilbert space. For a fixed unit vector $\Omega \in \mathcal{H}$ we consider the functional

$$\omega\colon \mathfrak{B}(\mathcal{H}) \ni A \mapsto \omega(A) = \langle \Omega, A\Omega\rangle \in \mathsf{C}. \tag{1.2.17}$$

We claim that the GNS representation of this obviously positive functional is the defining representation of $\mathfrak{B}(\mathcal{H})$ on $\mathcal{H}$. Indeed, all we have to show is that Ω is cyclic: but this is clear since for every other vector $\phi \in \mathcal{H}$ we have $\phi = \Theta_{\phi,\Omega}\Omega$ with $\Theta_{\phi,\Omega} \in \mathfrak{F}(\mathcal{H}) \subseteq \mathfrak{B}(\mathcal{H})$ since Ω is assumed to be a unit vector. Thus we can apply Proposition 1.2.6. However, note that in ring-theoretic framework it might well happen that $\mathcal{H}$ does not contain any unit vector.

The next example is from deformation quantization: there one has many other examples of positive functionals leading to physically interesting GNS representations, we just mention one of them without presenting the details:

Example 1.2.8 (Schrödinger representation) Consider the *-algebra of formal power series $C^\infty(\mathbb{R}^{2n})[\![\lambda]\!]$ with smooth functions as coefficients equipped with the usual Weyl-Moyal star product

$$f \star_{\mathrm{Weyl}} g = \mu \circ \mathrm{e}^{\frac{\mathrm{i}\lambda}{2}\sum_k \left(\frac{\partial}{\partial q^k} \otimes \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q^k}\right)} f \otimes g, \tag{1.2.18}$$

where $\mu(f \otimes g) = fg$ is the undeformed product. This star product is Hermitian and thus a *-algebra over
$mathbbC[\![\lambda]\!]$ as claimed. The functions $\mathscr{C}^\infty_0(\mathbb{R}^{2n})[\![\lambda]\!]$ are now easily shown to form a *-ideal. On this *-ideal, the functional

$$\omega(f) = \int_{\mathbb{R}^n} f(q, p = 0)\, \mathrm{d}^n q \tag{1.2.19}$$

is well-defined. In fact, it would suffice to require compact support in every order of λ in q-direction only. By some elementary integration by parts, this functional turns out to be positive. Moreover, the GNS pre-Hilbert space is, up to a simple identification, the space of wave functions $C^\infty_0(\mathbb{R}^n)[\![\lambda]\!]$ with the canonical $\mathbb{C}[\![\lambda]\!]$-valued L^2-inner product [18, Sect. 8]. From Exercise 1.4.12 we know that the GNS representation extends from the *-ideal to the whole algebra $\mathscr{C}^\infty(\mathbb{R}^{2n})[\![\lambda]\!]$. Then the GNS representation becomes the Schrödinger representation, i.e. for polynomials in p's and q's we have the usual action by momentum and position operators on wave functions together with the Weyl symmetrization rule. Analogous results hold for any cotangent bundle [13–15], see also [116, Sect. 7.2] for further details and explicit proofs.

On consequence of the GNS construction is now that we can find *faithful* *-representations of a *-algebra provided it has many positive functionals. Here the following definition turns out to guarantee this feature [26, Def. 2.7]:

Definition 1.2.9 (Sufficiently many positive functionals) *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then $\mathscr{A}$ has sufficiently many positive linear functionals if for every non-zero Hermitian element $a = a^* \in \mathscr{A}$ there is a positive functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ with $\omega(a) \neq 0$.*

Proposition 1.2.10 *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then the following statements are equivalent:*

i.) The *-algebra $\mathscr{A}$ has sufficiently many positive linear functionals.

ii.) For every non-zero Hermitian element $a = a^* \neq 0$ there is a *-representation π with $\pi(a) \neq 0$.

iii.) The *-algebra has a faithful *-representation.

In this case, the *-algebra $\mathscr{A}$ satisfies the following additional properties:

i.) If $a \in \mathscr{A}$ satisfies $a^*a = 0$ then $a = 0$.

ii.) If $a \in \mathscr{A}$ is normal and nilpotent then $a = 0$.

iii.) The *-algebra $\mathscr{A}$ is torsion-free, i.e. for $a \in \mathscr{A}$ and $z \in \mathsf{C}$ with $z \neq 0$ and $za = 0$ one has $a = 0$.

Proof: We consider the set of all positive linear functionals $\mathscr{A}^*_+ \subseteq \mathscr{A}^*$ and define the *-representation π to be the direct sum of all GNS representations, i.e.

$$\pi = \bigoplus_{\omega \in \mathscr{A}^*_+} \pi_\omega \quad \text{on the pre-Hilbert space} \quad \mathfrak{H} = \bigoplus_{\omega \in \mathscr{A}^*_+} \mathfrak{H}_\omega. \tag{$*$}$$

Suppose *i.)*. Then we claim that π is faithful. Indeed, let first $a \in \mathscr{A}$ be Hermitian. Then we find a positive linear functional ω with $\omega(a) \neq 0$. Hence $\omega(a) = \langle \psi_{\mathbb{1}}, \pi_\omega(a)\psi_{\mathbb{1}} \rangle_\omega$ shows that $\pi_\omega(a) \neq 0$. Being a direct summand of π shows then $\pi(a) \neq 0$, too. Thus *ii.)* follows. Now assume *ii.)*, let π be a *-representation which is non-trivial on non-zero Hermitian elements, and let $a \in \mathscr{A}$ be arbitrary. Then clearly π is also non-trivial on non-zero anti-Hermitian elements. Now $a + a^*$ is Hermitian and $a - a^*$ is anti-Hermitian. Suppose that $\pi(a) = 0$ then $\pi(a + a^*) = 0 = \pi(a - a^*)$, too. By assumption, this implies $a + a^* = 0 = a - a^*$ and thus $a = a^*$ is Hermitian. But then $\pi(a) = 0$ implies $a = 0$. This shows that π is non-trivial on all non-zero elements of $\mathscr{A}$, hence *iii.)* follows. In particular, $(*)$ is faithful. Finally, if we have a faithful *-representation π of $\mathscr{A}$ on $\mathfrak{H}$ then the functionals $\omega_\phi(a) = \langle \phi, \pi(a)\phi \rangle$ are positive and will detect all non-zero Hermitian elements, see also Exercise 1.4.8, *v.)*. This shows the equivalence of the first three statements. If $\mathscr{A}$ satisfies these conditions then $\mathscr{A}$ is *-isomorphic to a *-subalgebra of $\mathfrak{B}(\mathfrak{H})$ for some suitable pre-Hilbert space $\mathfrak{H}$. Then $\mathscr{A}$ inherits the remaining properties from the adjointable operators, see also Example 1.1.26. □

Example 1.2.11 The existence of a unit can be weakened [26, Prop. 2.8] but not completely abandoned: consider the one-dimensional non-unital *-algebra $\mathscr{A} = \mathsf{C}x$ with the *-algebra structure defined by $x^2 = 0$ and $x^* = x$. In this case, every linear functional is positive for trivial reasons and hence $\mathscr{A}$ has sufficiently many positive functionals. However, a faithful *-representation of $\mathscr{A}$ would yield a Hermitian operator $\pi(x) \neq 0$ with $\pi(x)^2 = 0$ which is not possible.

If $\mathscr{A}$ has sufficiently many positive functionals, the *-ideal structure of $\mathscr{A}$ allows for some nice characterization [25], see also Exercise 1.4.14 and Exercise 1.4.15. In the case of $\mathsf{C} = \mathbb{C}$ such *-algebras were treated in detail in [104, Sect. 6.4], where these algebras are called **-semisimple*.

Physically speaking, the condition of having sufficiently many positive linear functionals means that we can detect an observable a being non-zero by investigating all the expectation values $\mathrm{E}_\omega(a) = \omega(a)$. This is clearly desirable and anything else would not qualify as a physically reasonable observable algebra.

1.3 Aims and Expectations

Let us now list several aims and expectation which should be addressed in the course of these notes.

- First, we would like to see how *-representations can be constructed beyond the fundamental construction out of positive functionals as in the GNS approach. Here suitable generalizations will be discussed.
- We shall focus on the *algebraic* aspects of *-representation theory. This is motivated by the generality of the examples which we would like to discuss on a common ground: for deformation quantization it will be important to deal with formal star products as well as with $\mathscr{C}^\infty(M)$ and, ultimately, with more analytic situations like C^*-algebras. This requires the usage of general ordered rings in favour of the real numbers $\mathbb{R}$. Note, however, in the case of *-algebras over $\mathbb{C}$, and in particular for C^*-algebras and O^*-algebras, much more sophisticated results can be obtained after appropriate use of (heavy) analytic techniques.
- We will need generalizations of *-representations on spaces beyond pre-Hilbert spaces. Such generalizations will naturally occur when we discuss the question of *functors*
$$^*\text{-}\mathsf{Rep}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}(\mathscr{B}) \tag{1.3.1}$$
between categories of representations for different *-algebras. Particular and very important examples of functors like (1.3.1) will be obtained from certain tensor product constructions. The reason to consider such functors is to *compare* the representation theories of different *-algebras *without* knowing them explicitly: we have seen that it is very unrealistic to have a complete understanding of $^*\text{-}\mathsf{Rep}(\mathscr{A})$. However, it will turn out that under certain circumstances it will be possible to say that $\mathscr{A}$ and $\mathscr{B}$ have the *same* representation theory even *without* knowing the representation theories individually.
- This is closely related to the question how much information about $\mathscr{A}$ is contained in the representation theory $^*\text{-}\mathsf{Rep}(\mathscr{A})$: Can one reconstruct $\mathscr{A}$ from of $^*\text{-}\mathsf{Rep}(\mathscr{A})$? For rings and their module categories this is the classical task of Morita theory. We shall discuss this question for several adapted versions of Morita equivalence taking care of the additional structure present for *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.
- Since the concept of an ordered ring works well together with formal power series we are able to study the behaviour of *-algebras and their *-representations under *formal deformations* in the sense of Gerstenhaber [53–56]. The main class of examples of such deformations are of course the star products from deformation quantization. In fact, deformation quantization will be one of the main guidelines to develop the relevant notions in the sequel. As applications, representation-theoretical techniques can be used to understand e.g. the Dirac monopole and the behavior of representations under reduction with respect to symmetries. Without being able to go into the details, we mention in particular the recent works [27, 33, 62, 67]. Note, however, that we will focus on the general deformation theory of *-algebras and their *-representations which makes our concepts applicable also beyond star products.

1.4 Exercises

Exercise 1.4.1 (From ordered rings to ordered fields) Let R be an ordered ring.

i.) Show that R has no zero divisors. Hence one obtains a quotient field $\hat{\mathsf{R}}$ of R.

ii.) Show that $\hat{\mathsf{R}}$ inherits the ordering of R in a unique way such that the canonical inclusion $\mathsf{R} \subseteq \hat{\mathsf{R}}$ is order-preserving.

iii.) Show that the ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$ by a square root i of -1 has no zero divisors, too. Conclude that the quotient field $\hat{\mathsf{C}}$ of C is canonically isomorphic to $\hat{\mathsf{R}}(\mathrm{i})$.

Exercise 1.4.2 (Quotients of *-algebras) Show that the quotient of a (unital) *-algebra by a *-ideal becomes in a unique way a (unital) *-algebra again, such that the quotient map is a *-homomorphism. Formulate and prove the universal property of this quotient procedure.

Exercise 1.4.3 (Polarization identity) Let V and W be two modules over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with an ordered ring R. Moreover, let $S\colon V \times V \longrightarrow W$ be a *sesquilinear* map, i.e. assume that S is antilinear

$$S(\alpha u + \beta v, w) = \overline{\alpha} S(u, w) + \overline{\beta} S(v, w) \tag{1.4.1}$$

in the first argument and linear in the second $S(u, \alpha v + \beta w) = \alpha S(u, v) + \beta S(u, w)$, where $\alpha, \beta \in \mathsf{C}$ and $u, v, w \in V$.

i.) Show that the *polarization identity*

$$S(v, w) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k S\big(v + \mathrm{i}^{-k} w, v + \mathrm{i}^{-k} w\big) \tag{1.4.2}$$

holds for all $v, w \in V$ and conclude that S is the constant 0-map if and only if $S(v, v) = 0$ for all $v \in V$.

ii.) Now assume that $W = \mathsf{C}$. A sesquilinear map $S\colon V \times V \longrightarrow \mathsf{C}$ is usually called a sesquilinear form. Such a sesquilinear form is said to be *Hermitian* if $\overline{S(v, w)} = S(w, v)$ holds for all $v, w \in V$. Show that a sesquilinear form S on V is Hermitian iff $S(v, v) \in \mathsf{R}$ holds for all $v \in V$.

iii.) Finally, let $\mathscr{A}$ be a unital *-algebra over C. Show that for every $a \in \mathscr{A}$ there exist 4 algebraically positive elements $b_0, b_1, b_2, b_3 \in \mathscr{A}^{++}$ such that $a = \sum_{k=0}^{3} \mathrm{i}^k b_k$ holds.

Hint: Show first that for all $a, b \in \mathscr{A}$ one has

$$ab = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k (a + \mathrm{i}^k) b (a + \mathrm{i}^k)^* \tag{1.4.3}$$

and

$$ba = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k (a + \mathrm{i}^k)^* b (a + \mathrm{i}^k). \tag{1.4.4}$$

iv.) Discuss under which assumptions the above statement also holds for (particular) non-unital *-algebras and give examples where it does not hold.

Exercise 1.4.4 (Positive matrices) Let $n \in \mathbb{N}$ and consider $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with an ordered ring R. The aim of this exercise is to prove Lemma 2.1.10. As a tool we will need the quotient fields $\hat{\mathsf{R}}$ and $\hat{\mathsf{C}}$ of R and C. For further reading see [26, App. A].

i.) Show that every C-linear functional $\omega\colon \mathrm{M}_n(\mathsf{C}) \longrightarrow \mathsf{C}$ is of the form

$$\omega(A) = \operatorname{tr}(\varrho A) \tag{1.4.5}$$

with a uniquely determined matrix $\varrho \in \mathrm{M}_n(\mathsf{C})$. Show that ω is a real functional, i.e. $\omega(A^*) = \overline{\omega(A)}$, iff $\varrho = \varrho^*$. Thus, since $\mathrm{M}_n(\mathsf{C})$ is unital, for positive functionals it will be sufficient to consider a Hermitian ϱ in the following.

ii.) Let $\varrho = \varrho^* \in \mathrm{M}_n(\mathsf{C})$. Show that the corresponding linear functional ω is positive iff ϱ satisfies

$$\langle v, \varrho v \rangle \geq 0 \tag{1.4.6}$$

for all $v \in \mathsf{C}^n$.

Hint: For a given matrix $A \in \mathrm{M}_n(\mathsf{C})$ consider the vectors $v^{(k)} = (v_i^{(k)})_{i=1,\ldots,n} \in \mathsf{C}^n$ with entries $v_i^{(k)} = \overline{A_{ki}}$. Conclude that for ϱ with (1.4.6) we then have $\operatorname{tr}(\varrho A^* A) \geq 0$. For the converse, let $v \in \mathsf{C}^n$ be given and consider the matrix $A \in \mathrm{M}_n(\mathsf{C})$ with entries $A_{ki} = \overline{v_i}$ for all $k = 1, \ldots, n$. Conclude that $\operatorname{tr}(\varrho A^* A)$ implies $\langle v, \varrho v \rangle \geq 0$.

iii.) Let $\varrho = \varrho^* \in \mathrm{M}_n(\mathsf{C})$. Show that ϱ satisfies (1.4.6) iff ϱ viewed as matrix $\varrho \in \mathrm{M}_n(\hat{\mathsf{C}})$ satisfies (1.4.6) for all $v \in \hat{\mathsf{C}}^n$ by choosing suitable common denominators.

iv.) Let $\varrho \in \mathrm{M}_n(\hat{\mathsf{C}})$ satisfy (1.4.6). Show that there exists a basis $v_1, \ldots, v_n \in \hat{\mathsf{C}}^n$ and non-negative numbers $p_1, \ldots, p_n \in \hat{\mathsf{R}}$ such that $\langle v_i, \varrho v_j \rangle = \delta_{ij} p_i$ for all $i, j = 1, \ldots, n$. Without further assumptions on $\hat{\mathsf{R}}$ we can not assume that the $v_1, \ldots, v_n$ are *orthonormal*, nevertheless they define idempotents P_i projecting onto v_i according the the direct sum decomposition induced by the basis. Use this to conclude that for the matrix $U \in \mathrm{M}_n(\hat{\mathsf{C}})$ defined by $\mathsf{e}_i = U v_i$ we have

$$\varrho = \sum_{i=1}^{n} p_i U^* P_i^* P_i U \in \mathrm{M}_n(\hat{\mathsf{C}})^{++}, \tag{1.4.7}$$

even though U is *not* a unitary matrix in general and the P_i are *not* Hermitian in general.

Hint: Use e.g. [65, Thm. 6.19].

v.) Show that $\varrho \in \mathrm{M}_n(\mathsf{C})$ is positive iff ϱ satisfies (1.4.7) when viewed as element of $\mathrm{M}_n(\hat{\mathsf{C}})$.

vi.) Let $A, B \in \mathrm{M}_n(\mathsf{C})^+$. Show that $\operatorname{tr}(AB) \geq 0$.

vii.) Let $A \in \mathrm{M}_n(\mathsf{C})^+$. Show that $A \in \mathrm{M}_n(\hat{\mathsf{C}})^{++}$.

Exercise 1.4.5 (Matrix algebras) Prove Lemma 1.1.11.

Exercise 1.4.6 (Positive maps) Let $\mathscr{A}$ and $\mathscr{B}$ be a *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) Show that a linear map $\phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is positive iff for every $a \in \mathscr{A}$ and every positive linear functional $\omega\colon \mathscr{B} \longrightarrow \mathsf{C}$ one has $\omega(\phi(a^*a)) \geq 0$, i.e. the linear functional $\phi^*\omega = \omega \circ \phi\colon \mathscr{A} \longrightarrow \mathsf{C}$ is positive.

ii.) Show that a linear map $\phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is positive iff $\phi(\mathscr{A}^{++}) \subseteq \mathscr{B}^+$.

iii.) Let $\omega\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a positive linear functional. Show that ω is completely positive.

Hint: Use Exercise 1.4.4.

iv.) Solve [68, Exercise 11.5.15] to show that there are positive maps which are not completely positive.

Exercise 1.4.7 (Complete positivity of tr and τ) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) Show that $\mathrm{M}_n(\mathrm{M}_m(\mathscr{A})) \cong \mathrm{M}_{nm}(\mathscr{A})$ as *-algebras for all $n, m \in \mathbb{N}$.

ii.) Show that $\operatorname{tr}(A^*A) \in \mathscr{A}^{++}$ for all $A \in \mathrm{M}_n(\mathscr{A})$. Conclude that $\operatorname{tr}\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathscr{A}$ is a completely positive linear map.

iii.) Show analogously that the map $\tau\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathscr{A}$ from (1.1.12) is completely positive.

Exercise 1.4.8 (Adjointable operators) Let $\mathcal{H}, \mathcal{H}'$ be pre-Hilbert spaces over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) Let $A\colon \mathcal{H} \longrightarrow \mathcal{H}'$ be an adjointable map. Show that A is C-linear and that the adjoint A^* is uniquely determined. Show that A^* is adjointable, too, and compute its adjoint.

ii.) Show that linear combinations of adjointable maps are again adjointable and determine their adjoints. Also, show that the composition of adjointable maps (into some further pre-Hilbert space $\mathcal{H}''$) is adjointable and compute the adjoint.

iii.) Conclude that $\mathfrak{B}(\mathcal{H})$ is a unital *-algebra over C with respect to the usual composition of linear maps and $A \mapsto A^*$ as *-involution.

iv.) Let $A \in \mathfrak{B}(\mathcal{H})$ satisfy $A^*A = 0$. Show that this implies $A = 0$. Moreover, show that the only nilpotent and normal element $A \in \mathfrak{B}(\mathcal{H})$ is $A = 0$.

v.) Show by a suitable polarization that if $\langle \phi, A\phi \rangle = 0$ for $A \in \mathfrak{B}(\mathcal{H})$ and all $\phi \in \mathcal{H}$ then $A = 0$.

Exercise 1.4.9 (Non-degenerate *-representations) Let $\mathscr{A}$ be a *-algebra over C and let $(\mathcal{H}, \pi)$ be a *-representation of $\mathscr{A}$ on a pre-Hilbert space $\mathcal{H}$. Then $(\mathcal{H}, \pi)$ is called *non-degenerate* if $\pi(a)\phi = 0$ for all $a \in \mathscr{A}$ implies $\phi = 0$.

i.) Show that $(\mathcal{H}, \pi)$ is non-degenerate if $(\mathcal{H}, \pi)$ is strongly non-degenerate.

ii.) Show that for a non-degenerate *-representation $(\mathcal{H}, \pi)$ the orthogonal space of $\pi(\mathscr{A})\mathcal{H}$ is trivial, i.e.

$$(\pi(\mathscr{A})\mathcal{H})^\perp = \{0\}. \tag{1.4.8}$$

iii.) Show that $(\mathcal{H}, \pi)$ is strongly non-degenerate if $(\mathcal{H}, \pi)$ is the direct orthogonal sum of cyclic *-representations.

iv.) Assume that $\mathscr{A}$ is unital. Show that a *-representation is strongly non-degenerate iff it is non-degenerate.

Exercise 1.4.10 (Variance, covariance, and uncertainty) Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ be a state.

i.) Show that for all $a_1, \ldots, a_n \in \mathscr{A}$ the covariance matrix $(\mathrm{Cov}_\omega(a_i, a_j)) \in \mathrm{M}_n(\mathsf{C})$ is positive.

ii.) Show that for all $a, b \in \mathscr{A}$ one has

$$\overline{(\omega(a^*b) - \omega(a^*)\omega(b))}(\omega(a^*b) - \omega(a^*)\omega(b)) \le \mathrm{Var}_\omega(a)\,\mathrm{Var}_\omega(b). \tag{1.4.9}$$

Conclude that a linear functional ω is a unital *-homomorphism iff it is a state and the variances $\mathrm{Var}_\omega(a)$ vanish for all algebra elements $a \in \mathscr{A}$.

iii.) Let $a, b \in \mathscr{A}$ be Hermitian. Prove that one has *Heisenberg's uncertainty relations*

$$4\,\mathrm{Var}_\omega(a)\,\mathrm{Var}_\omega(b) \ge \omega([a,b])\overline{\omega([a,b])} \tag{1.4.10}$$

for the variances of a and b.

Exercise 1.4.11 (Positive elements of a subalgebra) Let $\mathscr{A}$ be a *-algebra over C with a *-subalgebra $\mathscr{B} \subseteq \mathscr{A}$.

i.) Show that the restriction of a positive functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ is a positive functional on $\mathscr{B}$.

ii.) Show that a positive element $b \in \mathscr{B}$ is also a positive element in $\mathscr{A}$.

iii.) Give easy examples that the two reverse implications are not true in general: not all positive linear functionals on $\mathscr{B}$ are restrictions of positive linear functionals on $\mathscr{A}$ and not all elements $b \in \mathscr{B}$ which are positive when viewed as elements of $\mathscr{A}$ are also positive in $\mathscr{B}$.

Hint: Consider $\mathbb{C}[x] \subseteq \mathscr{C}([0,1])$.

This is the mechanism used in O^*-algebra theory: the strong positivity of an O^*-algebra is inherited from the ambient *-algebra of *all* adjointable operators on a pre-Hilbert space.

Exercise 1.4.12 (The GNS construction for a *-ideal) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\mathscr{B} \subseteq \mathscr{A}$ be a *-ideal. Consider a positive linear functional $\omega\colon \mathscr{B} \longrightarrow \mathsf{C}$ and denote the Gel'fand ideal of ω by $\mathscr{J}_\omega \subseteq \mathscr{B}$ as usual.

i.) Use the Cauchy-Schwarz inequality to show that $\mathscr{J}_\omega \subseteq \mathscr{A}$ is still a left ideal.

ii.) Show that the GNS representation π_ω of $\mathscr{B}$ on $\mathcal{H}_\omega$ extends to a *-representation π_ω of $\mathscr{A}$ by setting $\pi_\omega(a)\psi_b = \psi_{ab}$. Why is this well-defined at all?

Exercise 1.4.13 (Inner-product spaces and indefinite GNS construction) Analogously to a pre-Hilbert space one defines an *inner-product space* $\mathcal{H}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ to be a C-module endowed with a non-degenerate inner product as in Definition 1.1.24 except for the requirement of positivity. Instead, the non-degeneracy is explicitly required, i.e. $\langle \phi, \psi \rangle = 0$ for all $\phi \in \mathcal{H}$ implies $\psi = 0$.

i.) Show that the analogous definition of adjointable and finite-rank operators between inner-product spaces gives $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathfrak{F}(\mathcal{H}_1, \mathcal{H}_2)$ enjoying the properties as in the pre-Hilbert case: the composition of adjointable operators is again adjointable, the finite-rank operators form a *-ideal, etc.

ii.) Show that an inner-product space $\mathcal{H}$ has no torsion, i.e. $z\phi = 0$ for $\phi \neq 0$ implies $z = 0$, where $z \in \mathsf{C}$ and $\phi \in \mathcal{H}$. Show that also $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathfrak{F}(\mathcal{H}_1, \mathcal{H}_2)$ have no torsion.

A *-representation π of a *-algebra on an inner-product space $\mathcal{H}$ is now defined to be a *-homomorphism $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathcal{H})$ as in the case of a pre-Hilbert space. We consider now a *real* linear functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$, i.e. for all $a, b \in \mathscr{A}$ we have $\omega(a^*b) = \overline{\omega(b^*a)}$.

iii.) Show that for a unital *-algebra a functional ω is real iff $\omega(a^*) = \overline{\omega(a)}$ for all $a \in \mathscr{A}$.

iv.) Show that $\mathscr{J}_\omega = \{a \in \mathscr{A} \mid \omega(b^*a) = 0 \text{ for all } b \in \mathscr{A}\}$ is a left ideal in $\mathscr{A}$, called again the *Gel'fand ideal*.

v.) Show that on the quotient $\mathcal{H}_\omega = \mathscr{A} \big/ \mathscr{J}_\omega$ the definition $\langle \psi_a, \psi_b \rangle = \omega(a^*b)$ yields a well-defined inner product, making $\mathcal{H}_\omega$ an inner-product space.

vi.) Show that the canonical left $\mathscr{A}$-module structure $\pi_\omega(a)\psi_b = \psi_{ab}$ on $\mathcal{H}$ is a *-representation, again called the GNS representation of ω.

Exercise 1.4.14 (The kernel of a *-representation) Consider a unital *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) Let $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ be a positive linear functional with GNS representation $(\mathcal{H}_\omega, \pi_\omega)$. For $b \in \mathscr{A}$ we define ω_b as in Lemma 1.1.10, *ii.)*. Show that

$$\ker \pi_\omega = \bigcap_{b \in \mathscr{A}} \mathscr{J}_{\omega_b} = \bigcap_{b \in \mathscr{A}} \ker \omega_b. \tag{1.4.11}$$

Note that even though neither $\mathscr{J}_{\omega_b}$ nor $\ker \omega_b$ is a *-ideal in general, their intersection turns out to be a *-ideal.

Hint: For the second equation use polarization to get $\omega(c^*ab) = 0$ for all $c, b \in \mathscr{A}$ and a in the kernel of all ω_b.

ii.) Use the unit element of $\mathscr{A}$ to show that $\ker \pi_\omega \subseteq \mathscr{J}_\omega \subseteq \ker \omega$.

iii.) Now let $(\pi, \mathcal{H})$ be an arbitrary *-representation of $\mathscr{A}$ and denote by ω_ϕ the vector state $\omega_\phi(a) = \langle \phi, \pi(a)\phi \rangle$ where $\phi \in \mathcal{H}$. Show that

$$\ker \pi = \bigcap_{\phi \in \mathcal{H}} \ker \pi_{\omega_\phi} = \bigcap_{\phi \in \mathcal{H}} \mathscr{J}_{\omega_\phi} = \bigcap_{\phi \in \mathcal{H}} \ker \omega_\phi, \tag{1.4.12}$$

using again a suitable polarization.

Exercise 1.4.15 (The minimal ideal: scalar case) Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Following [25], we call a *-ideal $\mathscr{J} \subseteq \mathscr{A}$ *closed* if it is the kernel of a *-representation $(\mathcal{H}, \pi)$ of $\mathscr{A}$. We will later put this into a much larger context in Section 5.3.4.

i.) Show that an arbitrary intersection of closed *-ideals is again closed. In particular, the intersection of all closed *-ideals of $\mathscr{A}$ is a closed *-ideal, called the *minimal *-ideal* $\mathscr{J}_{\min}(\mathscr{A})$. This can also be viewed as a *-algebra version of the *Jacobson radical*.

ii.) Show that

$$\mathscr{J}_{\min}(\mathscr{A}) = \bigcap_{\omega} \ker \pi_\omega = \bigcap_{\omega} \mathscr{J}_\omega = \bigcap_{\omega} \ker \omega, \tag{1.4.13}$$

where the intersections are taken over all positive linear functionals. Note that the Gel'fand ideals $\mathscr{J}_\omega$ are only left ideals, the kernels of positive linear functions $\ker \omega$ have no ideal property at all.

iii.) Determine the minimal *-ideal of $\mathfrak{B}(\mathcal{H})$ for a pre-Hilbert space $\mathcal{H}$ over C.

iv.) Let $\mathscr{B}$ be another unital *-algebra and $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ a *-homomorphism. Show that one has $\Phi(\mathscr{J}_{\min}(\mathscr{A})) \subseteq \mathscr{J}_{\min}(\mathscr{B})$.

v.) Show that $\mathscr{A}$ has a faithful *-representation on a pre-Hilbert space iff $\mathscr{J}_{\min}(\mathscr{A}) = \{0\}$.

vi.) Let $a \in \mathscr{A}$ satisfy $a^*a = 0$. Show that $a \in \mathscr{J}_{\min}(\mathscr{A})$. Let $b \in \mathscr{A}$ be a normal element with $b^k = 0$ for some $k \in \mathbb{N}$. Show that $b \in \mathscr{J}_{\min}(\mathscr{A})$.

vii.) Show that passing to the quotient *-algebra $\mathscr{A}/\mathscr{J}_{\min}(\mathscr{A})$ is functorial.

viii.) Let $\mathscr{J} \subseteq \mathscr{J}_{\min}(\mathscr{A})$ be a *-ideal contained in the minimal ideal. Show that the representation theories *-$\mathsf{Rep}(\mathscr{A})$ and *-$\mathsf{Rep}(\mathscr{A}/\mathscr{J})$ are equivalent.

Hint: Any *-representation of the quotient *-algebra $\mathscr{A}/\mathscr{J}$ can be pulled back to $\mathscr{A}$. More nontrivial is the fact that every *-representation π of $\mathscr{A}$ can be pushed forward to a *-representation of the quotient by setting $\pi([a]) = \pi(a)$. These two procedures are functorial and implement an equivalence (in fact even an isomorphism).

ix.) Show that $\mathscr{J}_{\min}(\mathscr{A}/\mathscr{J}_{\min}) = \{0\}$.

The idea is that the minimal *-ideal contains all the unpleasant elements of $\mathscr{A}$ concerning representation theory on pre-Hilbert spaces. Passing to the quotient $\mathscr{A}/\mathscr{J}_{\min}(\mathscr{A})$ allows to get rid of them in a functorial way without changing the representation theory, see also [25].

Exercise 1.4.16 (The *-algebra $\mathbb{Z}_2$) Consider $\mathsf{R} = \mathbb{Z}$ and hence $\mathsf{C} = \mathbb{Z}(\mathrm{i})$. Let $\mathscr{A} = \mathbb{Z}_2$.

i.) Show that $\mathbb{Z}_2$ with its usual ring structure becomes a unital *-algebra over C in a unique way. How has $\mathrm{i} \cdot \mathbb{1}$ to be defined?

ii.) Show that $\mathscr{A}$ has no non-zero positive functional.

iii.) Show that every *-representation π of $\mathscr{A}$ is trivial, i.e. $\pi(\mathbb{1}) = 0$.

iv.) Show that $\mathscr{A} = \mathscr{A}^{++}$, in particular, every element in $\mathscr{A}$ is positive.

Clearly, such a *-algebra is quite far away from any reasonable physical observable algebra.

Exercise 1.4.17 (Positive elements and positive functionals of $\mathscr{C}^\infty(M)$) Consider a smooth manifold M and the *-algebra $\mathscr{C}^\infty(M)$ of complex-valued smooth function on it with the *-ideal $\mathscr{C}^\infty_0(M)$ of compactly supported such functions. Much of the following can also be done in slightly different settings as well, like e.g. for not too badly behaved topological spaces and continuous functions etc.

i.) Prove that $f \in \mathscr{C}^\infty(M)$ is positive iff $f(p) \geq 0$ for all points $p \in M$.

Hint: One direction is trivial. For the other, consider first the function $f + \epsilon$ with some $\epsilon > 0$ and show that it has a smooth square root. Compute now $\omega(f + \epsilon)$ for a positive linear functional $\omega\colon \mathscr{C}^\infty(M) \longrightarrow \mathbb{C}$. How does this argument simplify if you only work with continuous functions instead of smooth ones?

ii.) Show that for the *-algebra $\mathscr{C}^\infty_0(M)$ the same conclusion still holds: $f \in \mathscr{C}^\infty_0(M)$ is positive iff $f(p) \geq 0$ for all points $p \in M$.

Hint: Again, one direction is trivial. For the other, let $\chi \in \mathscr{C}^\infty_0(M)$ be a cut-off function with $\chi \geq 0$ and $\chi\big|_{\mathrm{supp}(f)} = 1$. For a positive linear functional $\omega\colon \mathscr{C}^\infty_0(M) \longrightarrow \mathbb{C}$ consider $\tilde{\omega}(g) = \omega(\overline{\chi} g \chi)$ for $g \in \mathscr{C}^\infty(M)$. Show that $\tilde{\omega}$ is positive and use *i.)*.

We continue now first with the compactly supported functions and their positive linear functionals.

iii.) Let $\omega\colon \mathscr{C}^\infty_0(M) \longrightarrow \mathbb{C}$ be a positive linear functional. Then ω is continuous in the $\mathscr{C}_0$-topology: for every compact subset $K \subseteq M$ there exists a $c_K > 0$ with

$$|\omega(f)| \leq c_K \|f\|_\infty \tag{1.4.14}$$

for all $f \in \mathscr{C}^\infty_0(M)$ with $\mathrm{supp}(f) \subseteq K$.

Hint: Let again $\chi \in \mathscr{C}^\infty_0(M)$ with $\chi\big|_K = 1$ be a cut-off function. Consider $\tilde{\omega}(g) = \omega(\overline{\chi} g \chi)$ as before and evaluate $\tilde{\omega}$ on the functions $\|f\|_\infty \pm \mathrm{Re}(f)$ and $\|f\|_\infty \pm \mathrm{Im}(f)$ using *i.)*. Prove that $c_K = 2\omega(\overline{\chi}\chi)$ will do the job.

iv.) Use the density of $\mathscr{C}_0^\infty(M)$ inside $\mathscr{C}_0(M)$ with respect to the $\mathscr{C}_0$-topology to extend a positive linear functional $\omega\colon \mathscr{C}_0^\infty(M) \longrightarrow \mathbb{C}$ to a positive linear functional on $\mathscr{C}_0(M)$ by continuity. Prove that this extension (still denoted by ω) is positive.

v.) Use Riesz' representation theorem, see e.g. [101, Thm. 2.14], to conclude that for a positive linear functional $\omega\colon \mathscr{C}_0^\infty(M) \longrightarrow \mathbb{C}$ there exists a σ-algebra containing the topology of M and a uniquely determined, positive, complete, and regular Borel measure μ on it such that

$$\omega(f) = \int_M f\mu \tag{1.4.15}$$

for all $f \in \mathscr{C}_0^\infty(M)$. This determines the positive linear functionals of $\mathscr{C}_0^\infty(M)$ completely.

In a last step one wants to extend this to arbitrary smooth functions without restriction on the supports. The main idea is now that we can have arbitrarily fast growth of smooth functions which forces a positive functional to have compact support.

vi.) Let $\omega\colon \mathscr{C}^\infty(M) \longrightarrow \mathbb{C}$ be a positive linear functional. Show that there exists a $\chi \in \mathscr{C}_0^\infty(M)$ with

$$\omega(f) = \omega_\chi(f) = \omega(\overline{\chi} f \chi) \tag{1.4.16}$$

for all $f \in \mathscr{C}^\infty(M)$.

Hint: Prove the statement by contradiction. Use an exhausting sequence $K_n \subseteq M$ of compact subsets, i.e. compact subsets with $K_n \subseteq K_{n+1}^\circ$ and $M = \bigcup_{n\in\mathbb{N}} K_n$. Let $f_n \in \mathscr{C}^\infty(M)$ be non-negative functions with $\operatorname{supp} f_n \subseteq M \setminus K_n$ and $\omega(f_n) = 1$. Show that $F_N = 1 + \sum_{n=N}^\infty f_n$ is a well-defined smooth positive function for all $N \in \mathbb{N}$. Conclude that $\omega(F_1) \geq N$ and arrive at a contradiction. Why does this prove the claim?

vii.) Show that for a positive linear functional $\omega\colon \mathscr{C}^\infty(M) \longrightarrow \mathbb{C}$ there exists a σ-algebra containing the topology of M and a uniquely determined, positive, complete, and regular Borel measure μ on it with compact support such that

$$\omega(f) = \int_M f\mu \tag{1.4.17}$$

for all $f \in \mathscr{C}^\infty(M)$.

Hint: Let $\chi \in \mathscr{C}_0^\infty(M)$ be given as in *vi.)*. Apply *v.)* to conclude that

$$\omega_\chi(f) = \omega(\overline{\chi} f \chi) = \int_M \overline{\chi}\chi f \nu$$

for some positive Borel measure ν and all $f \in \mathscr{C}^\infty(M)$. Define μ by $\mu(A) = \int_A \overline{\chi}\chi\nu$ for measurable sets A.

viii.) Extend the above statements to matrix-valued functions $\mathrm{M}_n(\mathscr{C}^\infty(M)) = \mathscr{C}^\infty(M, \mathrm{M}_n(\mathbb{C}))$ and determine the positive elements and the positive functionals explicitly.

Exercise 1.4.18 (Algebraically positive elements of $\mathscr{C}^\infty(M)$) While $\mathscr{C}^\infty(M)^+$ consists of the non-negative smooth functions on M according to Exercise 1.4.17, the algebraically positive elements $\mathscr{C}^\infty(M)^{++}$ form typically a strict subset:

i.) Show that the polynomial

$$p(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \tag{1.4.18}$$

is non-negative on $\mathbb{R}^3$ but not a sum of squares. This an explicit example to the 17th Hilbert problem due to Motzkin [89].

ii.) Consider now the case of smooth functions and show that p can not be written as sum of squares of smooth functions, i.e. $p \in \mathscr{C}^\infty(\mathbb{R}^3)^+$ but $p \neq \mathscr{C}^\infty(\mathbb{R}^3)^{++}$.

Hint: Use the Taylor expansion of the smooth functions around 0 to achieve a contradiction based on *i.)*, see [108].

Thus in general, non-negative smooth functions on a manifold M are not sums of squares. For more details on the regularity required to write non-negative functions as (sums of) squares, see [11,12].

Exercise 1.4.19 (Positive linear functionals of $\mathbb{C}[\![z,\overline{z}]\!]$) Consider the complex formal power series $\mathscr{A} = \mathbb{C}[\![z,\overline{z}]\!]$ in two variables.

i.) Show that $\mathscr{A}$ becomes a complex commutative *-algebra with respect to the usual product of formal series and the *-involution

$$a^* = \left(\sum_{k,\ell=0}^{\infty} a_{k\ell} z^k \overline{z}^\ell\right)^* = \sum_{k,\ell=0}^{\infty} \overline{a_{k\ell}} z^\ell \overline{z}^k. \tag{1.4.19}$$

ii.) Let $\delta\colon \mathscr{A} \longrightarrow \mathbb{C}$ be the δ-functional at $z = 0 = \overline{z}$, i.e. $a \mapsto a_{00}$ for a given as in (1.4.19). Show that δ is a positive linear functional.

Consider now a positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ and let $k \geq 1$. Define $\alpha = \omega(z^k)$ and set $H = \overline{z}z$.

iii.) Use the Cauchy-Schwarz inequality to show $\overline{\alpha}\alpha \leq \omega(H^k)$. Show by induction $(\overline{\alpha}\alpha)^{2^n} \leq \omega(H^k)^{2^n} \leq \omega((H^k)^{2^n})$ for all $n \in \mathbb{N}$.

iv.) Assume $\alpha \neq 0$ and define for a fixed $N \in \mathbb{N}_0 \cup \{+\infty\}$ the new algebra element

$$a_N = \sum_{n=0}^{N} \frac{1}{(\overline{\alpha}\alpha)^{2^n}} (H^k)^{2^n}. \tag{1.4.20}$$

Write $a_\infty = a_N + b_N$ and show that $a_N \in \mathscr{A}^{++}$ for $N \in \mathbb{N}_0$. Show that $a_\infty \in \mathscr{A}$ is indeed a well-defined formal series. Show that b_N can be written as a square of some Hermitian element c_N, by constructing c_N recursively order by order in the z and $\overline{z}$ variables. Conclude that $a_\infty \in \mathscr{A}^{++}$.

v.) Conclude that $\omega(a_\infty) \geq N$ for all $N \in \mathbb{N}_0$. Hence we arrive at a contradiction showing that in fact $\alpha = 0$.

vi.) Show that $\omega(H^k)$ and $\omega(H) = 0$ as well as $\omega(z^k) = 0$ for all $k \geq 1$. Conclude that $\omega(z^k\overline{z}^\ell) = 0$ for all $k + \ell \geq 1$.

vii.) Now let $a \in \mathscr{A}$ with $\delta(a) = 0$ be given. Write $a = zb + \overline{z}c$ with some $b, c \in \mathscr{A}$ and use the Cauchy-Schwarz inequality to get $\omega(a) = 0$.

viii.) Show that δ is the only state of $\mathscr{A}$.

ix.) Show that $\mathbb{C}[z,\overline{z}] \subseteq \mathscr{A}$ is a unital *-subalgebra which has, quite contrary to $\mathscr{A}$, for every non-zero Hermitian element $a \in \mathbb{C}[z,\overline{z}]$ a positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ with $\omega(a) \neq 0$.

Exercise 1.4.20 (A scary Banach *-algebra) Consider the complex-valued continuous functions $\mathscr{A} = \mathscr{C}(\mathbb{S}^2)$ on the 2-sphere $\mathbb{S}^2$ with the involution $f \mapsto f^\dagger$ defined by

$$f^\dagger(x) = \overline{f(-x)} \tag{1.4.21}$$

for $f \in \mathscr{C}(\mathbb{S}^2)$ and $x \in \mathbb{S}^2$.

i.) Show that $f \mapsto f^\dagger$ is a *-involution on $\mathscr{A}$.

ii.) Show that $\mathscr{A}$ becomes a Banach *-algebra with respect to the sup-norm $\|\cdot\|_\infty$.

iii.) Let $f \in \mathscr{A}$ and $\lambda \in \mathbb{C}$. Show that $f - \lambda$ is invertible iff λ is in the range of f: the spectrum of an algebra element is not referring to the *-involution but only to the associative structure.

iv.) Show that there are non-zero Hermitian elements $h \in \mathscr{A}$ with purely imaginary spectrum.

v.) Let $\lambda \in \mathbb{R} \setminus \{0\}$. Show that there are unitary elements $u \in \mathscr{A}$ with spectrum containing λ. In particular, spectral values of unitary need not to have absolute value 1.

vi.) Show that there is a positive element $a \in \mathscr{A}$ with spectrum in $(-\infty, 0]$.

vii.) Show that the δ-functionals δ_x for $x \in \mathbb{S}^2$ are not positive.

Chapter 2

Pre-Hilbert Modules

While pre-Hilbert spaces over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ are already a valuable class of representation spaces, we have to enlarge the framework drastically: instead of scalar products taking their values in the scalars C we are now looking for a replacement of C by an arbitrary *-algebra $\mathscr{A}$. This way, the pre-Hilbert spaces will be replaced by pre-Hilbert modules over $\mathscr{A}$ which now play the role of the new representation spaces on which another *-algebra $\mathscr{B}$ will be represented.

In this chapter we give a detailed study of such inner-product modules over a *-algebra and explain the necessary positivity requirements which turn an inner-product module into a pre-Hilbert module. Many examples of such pre-Hilbert modules will be given, in particular from differential geometry. An important class of pre-Hilbert modules will arise from projective modules, leading ultimately to the study of various types of K_0-theories. The guiding class of examples will be the Hilbert modules over C^*-algebras as discussed e.g. in [79, 87]. As before, we will abandon the continuity and completeness aspects and focus solely on the algebraic aspects.

2.1 Module Categories

In this section we establish the fundamental notions of the module categories which we want to study: after a short reminder on the purely algebraic situation we introduce algebra-valued inner products. Modules with non-degenerate algebra-valued inner products will constitute the first category of interest, the inner-product modules. Requesting an additional positivity requirement takes care of both structures, the *-involution and the notions of positivity. This leads to the category of pre-Hilbert modules.

2.1.1 Ring-Theoretic Module Categories

As a warming-up we recall the basic definitions from a ring-theoretic approach to module categories. We fix a ring $\mathscr{A}$ and denote by $\mathsf{mod}(\mathscr{A})$ the category of left $\mathscr{A}$-modules as objects where the morphisms are given by left $\mathscr{A}$-linear maps between the modules. We write ${}_{\mathscr{A}}\mathscr{M}$ for a left $\mathscr{A}$-module $\mathscr{M}$ in order to emphasize that $\mathscr{A}$ acts from the left. Analogously, we write $\mathscr{M}_{\mathscr{A}}$ for a right $\mathscr{A}$-module. For a detailed study of modules over rings and algebras one can consult any standard textbook in algebra as e.g. [78].

Since in the following we are mainly interested in the case where $\mathscr{A}$ is an algebra over a fixed commutative unital ring C of scalars (later it will be even of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$), we require all modules to carry a compatible C-module structure in addition, even though we do not emphasize this in our notation.

In general, the way $\mathscr{A}$ acts on a module can be quite trivial: recall that even in the unital situation we do not require algebra homomorphisms to be unital. Thus $a \cdot m = 0$ will always define a module structure for $a \in \mathscr{A}$ and $m \in {}_{\mathscr{A}}\mathscr{M}$. Like for *-representations also here we can impose the condition

$$\mathscr{A} \cdot {}_{\mathscr{A}}\mathscr{M} = {}_{\mathscr{A}}\mathscr{M} \tag{2.1.1}$$

and call such a left $\mathscr{A}$-module *strongly non-degenerate*. This gives a full sub-category $\mathsf{Mod}(\mathcal{A})$ of strongly non-degenerate left $\mathscr{A}$-modules. Again, if $\mathscr{A}$ is unital then ${}_{\mathscr{A}}\mathscr{M}$ is strongly non-degenerate iff it is *unital*, i.e. $\mathbb{1} \cdot m = m$ for all $m \in \mathscr{M}$.

As a last step, we can consider two algebras $\mathscr{A}$ and $\mathscr{B}$ where one plays the role of *coefficients* and the other one is considered as *being represented*: we consider a $(\mathscr{B}, \mathscr{A})$-bimodule ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ where $\mathscr{B}$ acts from the left and $\mathscr{A}$ acts from the right. Recall that *bimodule* means that the two actions commute. So far the situation is completely symmetric in $\mathscr{A}$ and $\mathscr{B}$. But anticipating later applications we denote the corresponding bimodule category by $\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ and speak of the representation theory of $\mathscr{B}$ on right $\mathscr{A}$-modules. We also have the corresponding full sub-category of strongly non-degenerate representations of $\mathscr{B}$ on right $\mathscr{A}$-modules which we denote by $\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$. Note that we do not require any non-degeneracy of ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \in \mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ with respect to the coefficient algebra $\mathscr{A}$.

2.1.2 Algebra-Valued Inner Products

The reason to treat the two algebras in $\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ in an asymmetric way becomes clearer when we consider inner products. From now on, $\mathsf{C} = \mathsf{R}(\mathrm{i})$ will be again the ring extension of an ordered ring R by a square root i of -1 and all algebras over C will be *-algebras. In this section we will focus on the coefficient algebra and describe how we can get an algebra-valued inner product for it. Note that $\mathscr{A}$ may well be non-commutative. The following definitions are motivated by the well-known situation of Hilbert modules over C^*-algebras, see e.g. the textbooks [79, 81, 95]. We follow in our presentation mainly the approach of [26, 29].

The central definition of this subsection is the algebra-valued inner product:

Definition 2.1.1 (Algebra-valued inner product) *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module. An $\mathscr{A}$-valued inner product on $\mathscr{E}_{\mathscr{A}}$ is a map*

$$\langle \cdot\,, \cdot \rangle_{\mathscr{A}} \colon \mathscr{E}_{\mathscr{A}} \times \mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{A} \tag{2.1.2}$$

with the following properties:

i.) $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ *is C-linear in the second argument.*

ii.) $\langle \phi, \psi \cdot a \rangle_{\mathscr{A}} = \langle \phi, \psi \rangle_{\mathscr{A}} a$ *for all $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$ and $a \in \mathscr{A}$.*

iii.) $\langle \phi, \psi \rangle_{\mathscr{A}} = (\langle \psi, \phi \rangle_{\mathscr{A}})^*$ *for all $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$.*

If we have in addition

iv.) $\langle \phi, \psi \rangle_{\mathscr{A}} = 0$ *for all ψ implies $\phi = 0$,*

then $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is called non-degenerate and $(\mathscr{E}_{\mathscr{A}}, \langle \cdot, \cdot \rangle_{\mathscr{A}})$ is called an inner-product module over $\mathscr{A}$.

From the first and third requirement we see that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is C-antilinear in the first argument. Moreover, for all $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$ and $a \in \mathscr{A}$ we have

$$\langle \phi \cdot a, \psi \rangle_{\mathscr{A}} = a^* \langle \phi, \psi \rangle_{\mathscr{A}}. \tag{2.1.3}$$

Thanks to the symmetry property *iii.)* it is enough to require non-degeneracy in one of the two arguments: non-degeneracy in the second argument is then a consequence of non-degenerate in the first. As already for modules, the position of the subscript $\mathscr{A}$ indicates the algebra where the inner product takes its values and to which direction we have $\mathscr{A}$-linearity.

The definition matches best to right modules. For left modules we can state an analogous definition of inner products with the only change that they are required to be C- and $\mathscr{A}$-linear to the *left* in the *first* argument. Consequently, we write ${}_{\mathscr{A}}\langle \cdot\,, \cdot \rangle$ for such an inner product. The following construction shows that we can obtain a left $\mathscr{A}$-module from a right $\mathscr{A}$-module including the inner product by complex conjugation.

Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Then we consider the complex conjugate left $\mathscr{A}$-module ${}_{\mathscr{A}}\overline{\mathscr{E}}$ of $\mathscr{E}_{\mathscr{A}}$ which is defined as follows: as additive group we set ${}_{\mathscr{A}}\overline{\mathscr{E}} = \mathscr{E}_{\mathscr{A}}$, where the identity map is denoted by $\mathscr{E}_{\mathscr{A}} \ni \phi \mapsto \overline{\phi} \in {}_{\mathscr{A}}\overline{\mathscr{E}}$. The multiplication with scalars $\alpha \in \mathsf{C}$ is defined by

$$\alpha\overline{\phi} = \overline{\overline{\alpha}\phi} \tag{2.1.4}$$

for $\phi \in \mathscr{E}_{\mathscr{A}}$. This makes ${}_{\mathscr{A}}\overline{\mathscr{E}}$ a C-module. Now for $a \in \mathscr{A}$ we define the left multiplication

$$a \cdot \overline{\phi} = \overline{\phi \cdot a^*} \tag{2.1.5}$$

for $\phi \in \mathscr{E}_{\mathscr{A}}$, which is easily shown to be a C-bilinear *left* $\mathscr{A}$-module structure. Thus ${}_{\mathscr{A}}\overline{\mathscr{E}}$ becomes a left $\mathscr{A}$-module. Finally, we set

$${}_{\mathscr{A}}\langle \overline{\phi}, \overline{\psi} \rangle = \langle \phi, \psi \rangle_{\mathscr{A}} \tag{2.1.6}$$

for $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$. Then a straightforward computation shows that this is now C-linear and left $\mathscr{A}$-linear in the first argument and hence an inner product on the left $\mathscr{A}$-module ${}_{\mathscr{A}}\overline{\mathscr{E}}$. Conversely, one can pass from a left $\mathscr{A}$-module with inner product to a right $\mathscr{A}$-module with inner product by complex conjugation and the two operations are inverse to each other, see also Exercise 2.4.3. We summarize this construction in the following proposition:

Proposition 2.1.2 (Complex conjugate module) *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$.*

i.) The complex conjugate module ${}_{\mathscr{A}}\overline{\mathscr{E}}$ is a left $\mathscr{A}$-module with inner product ${}_{\mathscr{A}}\langle \cdot\,, \cdot \rangle$.

ii.) The inner product ${}_{\mathscr{A}}\langle \cdot\,, \cdot \rangle$ is non-degenerate iff $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is non-degenerate.

iii.) The complex conjugate of $({}_{\mathscr{A}}\overline{\mathscr{E}}, {}_{\mathscr{A}}\langle \cdot\,, \cdot \rangle)$ is again $(\mathscr{E}_{\mathscr{A}}, \langle \cdot\,, \cdot \rangle_{\mathscr{A}})$.

Thus we can pass freely from the left to the right and back. Note that the *-involution of $\mathscr{A}$ is crucial: for a general ring there is no canonical way to construct a left module out of a given right module as above.

To allow degenerate inner products is convenient as an intermediate step in many constructions. However, at the end we want to get rid of the degeneracy spaces. This can always be done, see Exercise 2.4.4:

Proposition 2.1.3 *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$.*

i.) The left and right degeneracy spaces of $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ coincide, i.e.

$$\left\{\phi \in \mathscr{E}_{\mathscr{A}} \;\middle|\; \langle \psi, \phi \rangle_{\mathscr{A}} = 0 \textit{ for all } \psi \in \mathscr{E}_{\mathscr{A}}\right\} = \left\{\phi \in \mathscr{E}_{\mathscr{A}} \;\middle|\; \langle \phi, \psi \rangle_{\mathscr{A}} = 0 \textit{ for all } \psi \in \mathscr{E}_{\mathscr{A}}\right\}. \tag{2.1.7}$$

ii.) The degeneracy space, denoted by $\mathscr{E}_{\mathscr{A}}^{\perp}$, is a right $\mathscr{A}$-submodule of $\mathscr{E}_{\mathscr{A}}$.

iii.) The inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ passes to the quotient right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}\big/\mathscr{E}_{\mathscr{A}}^{\perp}$ which becomes an inner-product right $\mathscr{A}$-module.

Thus we can always pass from an arbitrary right $\mathscr{A}$-module with inner product to an inner-product module in a canonical way.

In a next step we consider the maps compatible with the structures of an inner-product module. The following definition is the direct analogue of Definition 1.1.24 of adjointable maps for pre-Hilbert spaces.

Definition 2.1.4 (Adjointable maps) *Let $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}'_{\mathcal{A}}$ be right $\mathcal{A}$-modules with $\mathcal{A}$-valued inner products $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ and $\langle \cdot\,, \cdot \rangle'_{\mathcal{A}}$, respectively. A right $\mathcal{A}$-linear map $T\colon \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}'_{\mathcal{A}}$ is called adjointable if there exists a right $\mathcal{A}$-linear map $T^*\colon \mathcal{E}'_{\mathcal{A}} \longrightarrow \mathcal{E}_{\mathcal{A}}$ with*

$$\langle \phi, T\psi \rangle'_{\mathcal{A}} = \langle T^*\phi, \psi \rangle_{\mathcal{A}} \tag{2.1.8}$$

for all $\phi \in \mathcal{E}'_{\mathcal{A}}$ and $\psi \in \mathcal{E}_{\mathcal{A}}$. The set of all adjointable maps is denoted by $\mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$.

If the coefficient algebra $\mathcal{A}$ is clear from the context, we shall sometimes simply write $\mathfrak{B}(\mathcal{E}, \mathcal{E}')$. Using this definition we can speak of isometries and unitary isomorphisms of inner products as usual. Note that an isometric map needs not to be adjointable. Nevertheless, an isometric map is necessarily injective once the inner product of the domain is non-degenerate. If in addition it is surjective, then it is necessarily adjointable with adjoint given by the inverse, i.e. it is unitary.

Lemma 2.1.5 (Adjointable maps) *Let $\mathcal{E}_{\mathcal{A}}$, $\mathcal{E}'_{\mathcal{A}}$, and $\mathcal{E}''_{\mathcal{A}}$ be right $\mathcal{A}$-modules with inner products.*

i.) Compositions and C-linear combinations of adjointable maps are again adjointable.

ii.) If the inner products are non-degenerate then adjointable maps are necessarily right $\mathcal{A}$-linear and the adjoint T^* of T is uniquely determined. The map $T \mapsto T^*$ is C-antilinear.

iii.) For inner-product modules and $T \in \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$ and $S \in \mathfrak{B}_{\mathcal{A}}(\mathcal{E}'_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}})$ the composition $ST \in \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}})$ as well as the adjoint $T^* \in \mathfrak{B}_{\mathcal{A}}(\mathcal{E}'_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}})$ are again adjointable, where

$$(ST)^* = T^*S^* \quad \textit{and} \quad (T^*)^* = T. \tag{2.1.9}$$

iv.) For an inner-product module $\mathcal{E}_{\mathcal{A}}$ the adjointable maps $\mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ form a *-algebra over C with unit element $\mathrm{id}_{\mathcal{E}}$. Moreover, $\mathcal{E}_{\mathcal{A}}$ is a $(\mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}), \mathcal{A})$-bimodule.

PROOF: The proof is completely analogous to the case of a pre-Hilbert space over C and can safely be done as an exercise, see also Exercise 1.4.8. ☐

Note that for a Hilbert module over a C^*-algebra one can speak of continuous endomorphisms. However, this notion is not quite as useful as it seems at a first glance. The reason is that there is no analogue of the Hellinger-Toeplitz Theorem available: continuous endomorphisms need not to have an adjoint at all. Nevertheless, it can be shown that the adjointable operators on a Hilbert module are necessarily continuous, by the same closed graph argument as in the usual Hellinger-Toeplitz Theorem, see Exercise 2.4.2. A similar difficulty arises also for the compact operators on a Hilbert module: in general the closure of the finite-rank operators needs not to be equal to the compact operators, this is a feature present for Hilbert spaces but fails for general Banach spaces. Thus one defines the "compact" operators for Hilbert modules as the norm closure of the finite-rank operators. Note, however, that as already in the Hilbert module case there is no analogue of Riesz' Representation Theorem for the dual module. Thus we make explicit use of the inner product instead of requiring (continuous) right $\mathcal{A}$-linear functionals in the definition of "finite-rank" operators:

Definition 2.1.6 (Finite-rank operators) *For right $\mathcal{A}$-modules $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}'_{\mathcal{A}}$ with $\mathcal{A}$-valued inner products one defines the map $\Theta_{\phi,\psi}\colon \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}'_{\mathcal{A}}$ for $\phi \in \mathcal{E}'_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{A}}$ by*

$$\Theta_{\phi,\psi}(\chi) = \phi \cdot \langle \psi, \chi \rangle_{\mathcal{A}} \tag{2.1.10}$$

for $\chi \in \mathcal{E}_{\mathcal{A}}$. The C-linear span of these maps is denoted by $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$ and its elements are called the finite-rank operators.

Lemma 2.1.7 *Let $\mathcal{E}_{\mathcal{A}}$, $\mathcal{E}'_{\mathcal{A}}$, and $\mathcal{E}''_{\mathcal{A}}$ be right $\mathcal{A}$-modules with inner products. Then $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}}) \subseteq \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$ and an adjoint (the unique adjoint for non-degenerate inner products) for $\Theta_{\phi,\psi}$ is $\Theta_{\psi,\phi}$. Moreover,*

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}'_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}}) \circ \mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}}) \subseteq \mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}}) \quad \text{and} \quad \mathfrak{F}_{\mathcal{A}}(\mathcal{E}'_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}}) \circ \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}}) \subseteq \mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}}). \tag{2.1.11}$$

In particular, $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}) \subseteq \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ is a *-ideal for an inner-product module $\mathcal{E}_{\mathcal{A}}$.

PROOF: Again, this is an elementary verification analogously to the case $\mathcal{A} = \mathsf{C}$. For (2.1.11) one notes that for $A \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}'_{\mathcal{A}}, \mathcal{E}''_{\mathcal{A}})$ and $B \in \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$ one has

$$A \circ \Theta_{\phi,\psi} = \Theta_{A\phi,\psi} \quad \text{and} \quad \Theta_{\phi,\psi} \circ B = \Theta_{\phi,B^*\psi}. \qquad \square$$

As already for pre-Hilbert spaces, an important construction with inner-product modules is the direct orthogonal sum:

Lemma 2.1.8 *Let $\{\mathcal{E}^{(i)}\}_{i\in I}$ be right $\mathcal{A}$-modules with $\mathcal{A}$-valued inner products $\langle \cdot\,, \cdot \rangle^{(i)}_{\mathcal{A}}$. The direct sum*

$$\mathcal{E} = \bigoplus_{i\in I} \mathcal{E}^{(i)} \tag{2.1.12}$$

becomes a right $\mathcal{A}$-module with $\mathcal{A}$-valued inner product via

$$(\phi_i)_{i\in I} \cdot a = (\phi_i \cdot a)_{i\in I} \tag{2.1.13}$$

and

$$\big\langle (\phi_i)_{i\in I}, (\psi_j)_{j\in I} \big\rangle_{\mathcal{A}} = \sum_{i\in I} \langle \phi_i, \psi_i \rangle^{(i)}_{\mathcal{A}}, \tag{2.1.14}$$

such that $\langle \mathcal{E}^{(i)}, \mathcal{E}^{(j)} \rangle = 0$ for $i \neq j$. The inner product $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is non-degenerate if and only if $\langle \cdot\,, \cdot \rangle^{(i)}_{A}$ is non-degenerate for all $i \in I$.

PROOF: Clearly, (2.1.13) gives a right $\mathcal{A}$-module structure and (2.1.14) is well-defined as in the direct sum only finitely many entries in $(\phi_i)_{i\in I}$ are non-zero. The remaining statements follow easily. $\square$

2.1.3 Complete Positivity and Pre-Hilbert Modules

Up to now, the notion of an algebra-valued inner product and an inner-product module only makes use of the *-involution but not of the concepts of positivity. We have to find an appropriate notion of positivity for an $\mathcal{A}$-valued inner product which generalizes the positive-definite inner product of a pre-Hilbert space as good as possible. Guided by the notion of completely positive maps we state the following definition:

Definition 2.1.9 (Completely positive inner product) *Let $(\mathcal{E}_{\mathcal{A}}, \langle \cdot\,, \cdot \rangle_{\mathcal{A}})$ be a right $\mathcal{A}$-module with $\mathcal{A}$-valued inner product.*

i.) $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is called positive if $\langle \phi, \phi \rangle_{\mathcal{A}} \in \mathcal{A}^+$ for all $\phi \in \mathcal{E}_{\mathcal{A}}$.

ii.) $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is called n-positive if $(\langle \phi_i, \phi_j \rangle_{\mathcal{A}}) \in \mathrm{M}_n(\mathcal{A})^+$ for all $\phi_1, \ldots, \phi_n \in \mathcal{E}_{\mathcal{A}}$.

iii.) $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is called completely positive if $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is n-positive for all $n \in \mathbb{N}$.

iv.) $\langle \cdot\,, \cdot \rangle_{\mathcal{A}}$ is called positive definite if it is positive and $\langle \phi, \phi \rangle_{\mathcal{A}} \neq 0$ for $\phi \neq 0$.

In order to learn something about the relations between these different notions we first recall the following result on positive matrices, see e.g. [116, Example 7.1.27] and Exercise 1.4.4.

Lemma 2.1.10 *Let $n \in \mathbb{N}$.*

i.) The positive linear functionals $\omega\colon \mathrm{M}_n(\mathsf{C}) \longrightarrow \mathsf{C}$ are of the form

$$\omega(A) = \operatorname{tr}(\varrho A) \tag{2.1.15}$$

with a uniquely determined matrix $\varrho \in \mathrm{M}_n(\mathsf{C})$ satisfying

$$\langle z, \varrho z\rangle \geq 0 \tag{2.1.16}$$

for all $z \in \mathsf{C}^n$.

ii.) A matrix $A \in \mathrm{M}_n(\mathsf{C})$ is positive if and only if it satisfies (2.1.16)*.*

Corollary 2.1.11 *Let $\mathcal{H}$ be a pre-Hilbert space over C. Then the scalar product $\langle \cdot\,, \cdot \rangle$ is completely positive in the sense of Definition 2.1.9.*

Proof: Let $\phi_1, \ldots, \phi_n \in \mathcal{H}$ be given and let $z \in \mathsf{C}^n$. Then we have

$$\langle z, (\langle \phi_i, \phi_j\rangle) \cdot z\rangle = \sum\nolimits_{i,j} \overline{z}_i \langle \phi_i, \phi_j\rangle z_j = \left\langle \sum\nolimits_i z_i\phi_i, \sum\nolimits_j z_j\phi_j \right\rangle \geq 0,$$

thanks to the positivity of the scalar product $\langle \cdot\,, \cdot \rangle$ of $\mathcal{H}$. With the criterion of Lemma 2.1.10 we conclude $(\langle \phi_i, \phi_j\rangle) \in \mathrm{M}_n(\mathsf{C})^+$. □

Lemma 2.1.12 *Let $n \in \mathbb{N}$.*

i.) For all $a_1, \ldots, a_n \in \mathscr{A}$ one has $a^*a^{\mathrm{T}} = (a_i^* a_j) \in \mathrm{M}_n(\mathscr{A})^{++}$.

ii.) For a positive linear functional $\Omega\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathsf{C}$ also the functional $\tilde{\Omega}\colon \mathrm{M}_{n+1}(\mathscr{A}) \longrightarrow \mathsf{C}$, defined by

$$\tilde{\Omega}\begin{pmatrix} a & b^{\mathrm{T}} \\ c & D \end{pmatrix} = \Omega(D), \tag{2.1.17}$$

is positive, where $a \in \mathscr{A}$, $b, c \in \mathscr{A}^n$ and $D \in \mathrm{M}_n(\mathscr{A})$.

iii.) An $(n+1)$-positive $\mathscr{A}$-valued inner product is n-positive, too.

Proof: For the first part we consider the matrix

$$B = \begin{pmatrix} a_1 & \ldots & a_n \\ & 0 & \end{pmatrix} \in \mathrm{M}_n(\mathscr{A}),$$

for which we have $B^*B = a^*a^{\mathrm{T}}$. For the second part we compute

$$\tilde{\Omega}\left(\begin{pmatrix} a & b^{\mathrm{T}} \\ c & D \end{pmatrix}^* \begin{pmatrix} a & b^{\mathrm{T}} \\ c & D \end{pmatrix}\right) = \tilde{\Omega}\begin{pmatrix} a^*a + (c^*)^{\mathrm{T}}c & a^*b^{\mathrm{T}} + (c^*)^{\mathrm{T}}D \\ b^*a + D^*c & b^*b^{\mathrm{T}} + D^*D \end{pmatrix} = \Omega(b^*b^{\mathrm{T}} + D^*D) \geq 0,$$

since by the first part the matrix b^*b^{T} is positive. For the last part we consider $\phi_0 = 0$ and $\phi_1, \ldots, \phi_n \in \mathcal{E}_{\mathscr{A}}$. Then by the second part

$$0 \leq \tilde{\Omega}\Big((\langle \phi_i, \phi_j\rangle_{\mathscr{A}})_{i,j=0,\ldots,n}\Big) = \Omega\Big((\langle \phi_i, \phi_j\rangle_{\mathscr{A}})_{i,j=1,\ldots,n}\Big)$$

shows the n-positivity. □

Remark 2.1.13 For certain classes of *-algebras we have the reverse implication: a positive inner product is automatically completely positive. Here C is an example by Corollary 2.1.11. Also C^*-algebras have this very nice property, see e.g. [79, Lem. 4.2] and Exercise 3.3.5. However, the proof requires a fair amount of rather specific properties of C^*-algebras and hence we can not transfer it to our completely algebraic situation. In general, it is not clear whether complete positivity is already implied by positivity. In practice, we will have to check this by hand as we will see in the following examples.

In the following we will need the *complete positivity* for a reasonable and useful definition of a pre-Hilbert module. This will become clear when we consider tensor products in Section 3.1.2.

Definition 2.1.14 (Pre-Hilbert module) *Let $\mathscr{A}$ be a *-algebra over C and $(\mathscr{E}_{\mathscr{A}}, \langle \cdot\,, \cdot \rangle_{\mathscr{A}})$ an inner-product module over $\mathscr{A}$. Then $(\mathscr{E}_{\mathscr{A}}, \langle \cdot\,, \cdot \rangle_{\mathscr{A}})$ is called pre-Hilbert module if $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is completely positive.*

Note that we do not require positive definiteness, non-degeneracy will be sufficient. Indeed, there are *-algebras where this makes a subtle difference:

Example 2.1.15 (Graßmann algebra) Let $\mathscr{A} = \Lambda^\bullet(\mathsf{C}^n)$ be the Graßmann algebra over C with n generators. Denote by $e_1, \ldots, e_n \in \mathsf{C}^n$ the canonical basis. Then by $e_i^* = e_i$ one determines a *-involution for $\mathscr{A}$ making the Graßmann algebra a *-algebra. We consider now $\mathscr{E}_{\mathscr{A}} = \mathscr{A}$ with the canonical inner product $\langle \alpha, \beta \rangle = \alpha^* \wedge \beta$. As we shall see in the next section, this is a completely positive inner product which is non-degenerate since $\langle \mathbb{1}, \alpha \rangle = \alpha$. On the other hand $\langle e_i, e_i \rangle = 0$, and thus it is *not* positive definite.

Positivity behaves well with respect to direct orthogonal sums and restrictions to submodules, see Exercise 2.4.6:

Lemma 2.1.16 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) A direct orthogonal sum of right $\mathscr{A}$-modules with positive (n-positive, completely positive) $\mathscr{A}$-valued inner products has again a positive (n-positive, completely positive) $\mathscr{A}$-valued inner product.

ii.) A restriction of a positive (n-positive, completely positive, positive definite) $\mathscr{A}$-valued inner product to a right $\mathscr{A}$-submodule is again positive (n-positive, completely positive, positive definite).

On the other hand, there are examples of (rather weird) *-algebras $\mathscr{A}$ where the direct orthogonal sum of positive definite $\mathscr{A}$-valued inner products is not necessarily positive definite anymore, see [29, Remark 3.3] and Exercise 2.4.7.

If a non-degenerate positive inner product is restricted to a submodule, then it might become degenerate. Here one can also find easy pathological examples, see Exercise 2.4.8. In order to avoid these phenomena we sometimes restrict ourselves to more well-behaved *-algebras:

Definition 2.1.17 (Admissible *-algebra) *A *-algebra $\mathscr{A}$ is called admissible if for any right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ with positive $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ one has*

$$\mathscr{E}_{\mathscr{A}}^{\perp} = \big\{\phi \in \mathscr{E}_{\mathscr{A}} \;\big|\; \langle \phi, \phi \rangle_{\mathscr{A}} = 0\big\}. \tag{2.1.18}$$

Clearly, the inclusion $\subseteq$ is fulfilled for any *-algebra. For an admissible *-algebra $\mathscr{A}$ the inner product on a pre-Hilbert module is automatically positive definite. Thus also the restriction to a submodule is positive definite and hence non-degenerate. This implies that a submodule of a pre-Hilbert module over $\mathscr{A}$ is again a pre-Hilbert module: it is this statement we want to be true.

From Example 2.1.15 we see that the Graßmann algebra is *not* admissible. In general, the above definition makes it of course difficult to decide whether a given *-algebra is admissible or not. However, the following proposition gives a sufficient criterion, see also [26, Lem. 5.21]:

Proposition 2.1.18 *Let $\mathscr{A}$ be a unital *-algebra with sufficiently many positive linear functionals. Then $\mathscr{A}$ is admissible.*

PROOF: Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with positive inner product $\langle \cdot, \cdot \rangle_{\mathscr{A}}$. Moreover, let $\phi_0 \in \mathscr{E}_{\mathscr{A}}$ satisfy $\langle \phi_0, \phi_0 \rangle_{\mathscr{A}} = 0$. We have to show $\phi_0 \in \mathscr{E}_{\mathscr{A}}^{\perp}$. To this end we consider a positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ and define

$$\langle \phi, \psi \rangle_{\omega} = \omega(\langle \phi, \psi \rangle_{\mathscr{A}})$$

for $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$. This gives a positive semi-definite inner product on $\mathscr{E}_{\mathscr{A}}$: indeed, the sesquilinearity is clear and we have

$$\langle \phi, \psi \rangle_{\omega} = \omega(\langle \phi, \psi \rangle_{\mathscr{A}}) = \omega((\langle \psi, \phi \rangle_{\mathscr{A}})^*) = \overline{\omega(\langle \psi, \phi \rangle_{\mathscr{A}})} = \overline{\langle \psi, \phi \rangle_{\omega}},$$

since for a *unital* *-algebra and a positive linear functional we have $\omega(a^*) = \overline{\omega(a)}$ for all $a \in \mathscr{A}$. Finally, $\langle \phi, \phi \rangle_{\omega} = \omega(\langle \phi, \phi \rangle_{\mathscr{A}}) \geq 0$ follows by the positivity of $\langle \phi, \phi \rangle_{\mathscr{A}} \in \mathscr{A}^+$. Thus we have the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_{\omega}$, i.e.

$$\langle \phi, \psi \rangle_{\omega} \overline{\langle \phi, \psi \rangle_{\omega}} \leq \langle \phi, \phi \rangle_{\omega} \langle \psi, \psi \rangle_{\omega}$$

for all $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$. Applied to ϕ_0 we see that $0 = \langle \psi, \phi_0 \rangle_{\omega} = \omega(\langle \psi, \phi_0 \rangle_{\mathscr{A}})$ for all ψ since $\langle \phi_0, \phi_0 \rangle_{\omega} = 0$. Since

$$2\langle \psi, \phi_0 \rangle_{\mathscr{A}} = \underbrace{\langle \psi, \phi_0 \rangle_{\mathscr{A}} + \langle \phi_0, \psi \rangle_{\mathscr{A}}}_{=a} + \underbrace{\langle \psi, \phi_0 \rangle_{\mathscr{A}} - \langle \phi_0, \psi \rangle_{\mathscr{A}}}_{=b},$$

and $a^* = a$ and $b^* = -b$ we see that $0 = 2\langle \psi, \phi_0 \rangle_{\omega} = \omega(a) + \omega(b)$. By $\overline{\omega(a)} = \omega(a)$ and $\overline{\omega(b)} = -\omega(b)$ we conclude that $\omega(a) = 0 = \omega(b)$. Thus by assumption, $a = 0 = b$ since ω was arbitrary. This shows $2\langle \psi, \phi_0 \rangle_{\mathscr{A}} = 0$ and by Corollary 1.2.10 we can conclude $\phi_0 \in \mathscr{E}_{\mathscr{A}}^{\perp}$. □

We conclude this section with an alternative formulation for the complete positivity requirement following [114, Sect. 4]. Thus let $\mathscr{E}_{\mathscr{A}}$ be again a right $\mathscr{A}$-module with $\mathscr{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathscr{A}}$ and let ${}_{\mathscr{A}}\overline{\mathscr{E}}$ be the corresponding complex conjugated left $\mathscr{A}$-module. Out of $\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\overline{\mathscr{E}}$ we can build the $(\mathscr{A}, \mathscr{A})$-bimodule

$${}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}} = {}_{\mathscr{A}}\overline{\mathscr{E}} \otimes \mathscr{E}_{\mathscr{A}}, \tag{2.1.19}$$

where the tensor product is taken over C. Clearly, ${}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$ is a $(\mathscr{A}, \mathscr{A})$-bimodule. For ${}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$ we can define a complex conjugation I as follows. For $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$ we set

$$I(\overline{\phi} \otimes \psi) = \overline{\psi} \otimes \phi, \tag{2.1.20}$$

and extend this to a C-antilinear map $I\colon {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$. One easily confirms that (2.1.20) is well-defined. Now we can define the *positive elements* in ${}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$ by setting

$${}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}^{+} = \left\{ \sum\nolimits_i \beta_i \overline{\phi}_i \otimes \phi_i \;\middle|\; \beta_i > 0 \text{ and } \phi_i \in \mathscr{E}_{\mathscr{A}} \right\}. \tag{2.1.21}$$

By the very definition this is a convex cone in ${}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$. We have now the following result:

Proposition 2.1.19 *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module.*

i.) The map $I\colon {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$ is an involutive antilinear $(\mathscr{A}, \mathscr{A})$-bimodule antiautomorphism, i.e.

$$I(a \cdot \Phi \cdot b) = b^* \cdot I(\Phi) \cdot a^* \tag{2.1.22}$$

for $a, b \in \mathscr{A}$ and $\Phi \in {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$. One has $I({}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}^{+}) \subseteq {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}^{+}$.

ii.) *An $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ on $\mathscr{E}_{\mathscr{A}}$ corresponds to a unique $(\mathscr{A}, \mathscr{A})$-bimodule morphism*

$$P \colon {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \tag{2.1.23}$$

with the property $P(I(\Phi)) = P(\Phi)^*$ via

$$\langle \phi, \psi \rangle_{\mathscr{A}} = P(\overline{\phi} \otimes \psi). \tag{2.1.24}$$

iii.) *An $\mathscr{A}$-valued inner product is positive if and only if the corresponding map P is positive in the sense that*

$$P\big({}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}^{+}\big) \subseteq \mathscr{A}^{+}. \tag{2.1.25}$$

Proof: For the first part we compute for factorizing tensors $\overline{\phi} \otimes \psi \in {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}}$

$$\begin{aligned}
I(a \cdot (\overline{\phi} \otimes \psi) \cdot b) &= I(\overline{\phi \cdot a^*} \otimes \psi \cdot b) \\
&= \overline{\psi \cdot b} \otimes \phi \cdot a^* \\
&= b^* \cdot (\overline{\psi} \otimes \phi) \cdot a^* \\
&= b^* \cdot I(\overline{\phi} \otimes \psi) \cdot a^*,
\end{aligned}$$

from which (2.1.22) follows in general. Clearly, I is involutive and C-antilinear, hence the first part is shown. Now let $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ be an $\mathscr{A}$-valued inner product. Then

$$P(\overline{\phi} \otimes \psi) = \langle \phi, \psi \rangle_{\mathscr{A}}$$

has the correct sesquilinearity properties to extend to a C-linear map $P \colon {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}} \longrightarrow \mathscr{A}$. From the properties of an $\mathscr{A}$-valued inner product it is immediate that P is an $(\mathscr{A}, \mathscr{A})$-bimodule morphism and satisfies $P(I(\Phi)) = P(\Phi)^*$. Conversely, if a map P with these properties is given, then one defines $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ by (2.1.24) and checks that this gives an $\mathscr{A}$-valued inner product. This completes the second part. The last part is obvious. □

The characterization of completely positive inner products is obtained by the following considerations: The direct sum $\mathscr{E}_{\mathscr{A}}^n$ is a right $\mathrm{M}_n(\mathscr{A})$-module in the canonical way, i.e. via

$$(\phi_i) \cdot (a_{ij}) = \left(\sum_{i=1}^{n} \phi_i \cdot a_{ij} \right). \tag{2.1.26}$$

Thus $\mathscr{F}^{(n)} = {}_{\mathscr{A}}\overline{\mathscr{E}}^n \otimes \mathscr{E}_{\mathscr{A}}^n$ becomes a $(\mathrm{M}_n(\mathscr{A}), \mathrm{M}_n(\mathscr{A}))$-bimodule and we can repeat the above construction for $\mathrm{M}_n(\mathscr{A})$ instead of $\mathscr{A}$. Elements in $\mathscr{F}^{(n)}$ are now C-linear combinations of factorizing elements of the form $(\overline{\phi_i}) \otimes (\psi_j)$ which we can represent as matrix $(\overline{\phi_i} \otimes \psi_j)_{i,j=1,\ldots,n}$. The positive elements in $\mathscr{F}^{(n)}$ are by (2.1.21) explicitly given by

$$\mathscr{F}^{(n),+} = \left\{ \sum\nolimits_{\alpha} \beta_\alpha \left(\overline{\phi_i^{(\alpha)}} \otimes \phi_j^{(\alpha)} \right) \;\middle|\; \beta_\alpha > 0 \text{ and } (\phi_1^{(\alpha)}, \ldots, \phi_n^{(\alpha)}) \in \mathscr{E}_{\mathscr{A}}^n \right\}. \tag{2.1.27}$$

The map I from (2.1.20) in this situation is explicitly given by

$$I\big((\overline{\phi_i} \otimes \psi_j)\big) = (\overline{\psi_j} \otimes \phi_i). \tag{2.1.28}$$

Finally, if P is an $(\mathscr{A}, \mathscr{A})$-bimodule morphism $P \colon {}_{\mathscr{A}}\mathscr{F}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ then applying P componentwise gives

$$P^{(n)}\big((\overline{\phi_i} \otimes \psi_j)\big) = \big(P(\overline{\phi_i} \otimes \psi_j)\big), \tag{2.1.29}$$

which is a $(\mathrm{M}_n(\mathscr{A}), \mathrm{M}_n(\mathscr{A}))$-bimodule morphism from $\mathscr{F}^{(n)}$ to $\mathrm{M}_n(\mathscr{A})$. This follows either from our general considerations or from an explicit observation using (2.1.29). Moreover, $P^{(n)}$ is compatible with the involution I if and only if P is compatible. The following proposition is now obvious:

Proposition 2.1.20 *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ and let $n \in \mathbb{N}$.*

i.) The inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is n-positive if and only if the corresponding map P is n-positive, i.e. $P^{(n)}$ is positive.

ii.) The inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is completely positive if and only if the corresponding map P is completely positive, i.e. $P^{(n)}$ is positive for all n.

PROOF: We only have to show the first part. If P is n-positive we have by definition $P^{(n)}\big(\mathscr{F}^{(n),+}\big) \subseteq \mathrm{M}_n(\mathscr{A})^+$ and

$$P^{(n)}\big((\overline{\phi_i} \otimes \phi_j)\big) = (\langle \phi_i, \phi_j \rangle_{\mathscr{A}}),$$

from which the first part follows immediately. □

The practical use of the Proposition 2.1.19 and Proposition 2.1.20 is rather limited. However, they show the relation to the theory of matrix-ordered spaces [104, Chap. 11], see also [114] for a more detailed discussion of this relation.

2.1.4 The Representation Theories *-Mod and *-Rep

Using the notion of a pre-Hilbert module we can enlarge our framework of *-representations of a *-algebra: in the following, $\mathscr{B}$ will be a *-algebra over C as before and $\mathscr{A}$ will be an additional *-algebra, typically even an admissible one in the sense of Definition 2.1.17. This auxiliary *-algebra will now play the role of the scalars:

Definition 2.1.21 (*-Representation) *Let $\mathscr{B}$ and $\mathscr{A}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) A *-representation of $\mathscr{B}$ on an inner-product right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ is a *-homomorphism

$$\pi\colon \mathscr{B} \longrightarrow \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}). \tag{2.1.30}$$

ii.) An intertwiner $T\colon (\mathscr{E}_{\mathscr{A}}, \pi) \longrightarrow (\mathscr{E}'_{\mathscr{A}}, \pi')$ between two *-representations of $\mathscr{B}$ on inner-product right $\mathscr{A}$-modules is an adjointable map $T \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}})$ such that for all $b \in \mathscr{B}$

$$T\pi(b) = \pi'(b)T. \tag{2.1.31}$$

iii.) The category of all *-representations of $\mathscr{B}$ on inner-product modules over $\mathscr{A}$ will be denoted by $^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$.

Remark 2.1.22 (Intertwiners) First we note that the composition of intertwiners is again an intertwiner and hence we indeed obtain a category. Since adjointable maps are automatically right $\mathscr{A}$-linear in the case of non-degenerate inner products, a *-representation $(\mathscr{E}_{\mathscr{A}}, \pi)$ of $\mathscr{B}$ on $\mathscr{E}_{\mathscr{A}}$ is equivalent to a $(\mathscr{B}, \mathscr{A})$-bimodule structure on $\mathscr{E}_{\mathscr{A}}$ with the additional feature that

$$\langle \pi(b)\phi, \psi \rangle_{\mathscr{A}} = \langle \phi, \pi(b^*)\psi \rangle_{\mathscr{A}} \tag{2.1.32}$$

for all $b \in \mathscr{B}$ and $\phi, \psi \in \mathscr{E}_{\mathscr{A}}$. In the following, we will frequently suppress the symbol π for the representation and simply write $b \cdot \phi$ instead. For later use we note that C-linear combinations of intertwiners are again intertwiners, too. This endows the space of morphisms from one *-representation to another one with the structure of a C-module structure. Clearly, the composition of intertwiners is bilinear with respect to this C-module structure.

Intertwiners are just adjointable $(\mathscr{B}, \mathscr{A})$-bimodule morphisms. This allows to forget about the inner product and the result is then just a $(\mathscr{B}, \mathscr{A})$-bimodule. It yields a forgetful functor

$$^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B}) \longrightarrow \mathsf{mod}_{\mathscr{A}}(\mathscr{B}) \tag{2.1.33}$$

into the category of representations of $\mathscr{B}$ on right $\mathscr{A}$-modules.

For any inner-product module $\mathscr{E}_{\mathscr{A}}$ we have the canonical left module structure of $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ on $\mathscr{E}_{\mathscr{A}}$. It is, by the very definition of the *-involution of $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$, a *-representation of $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ on $\mathscr{E}_{\mathscr{A}}$. This way, $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is the largest *-algebra which can be represented faithfully on $\mathscr{E}_{\mathscr{A}}$. The *-algebra $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is also represented faithfully on $\mathscr{E}_{\mathscr{A}}$ when we view it as a *-subalgebra of $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$.

In a next step we proceed as we did for the ring-theoretic module categories: we consider strongly non-degenerate *-representations on inner-product modules:

Definition 2.1.23 (Inner-product bimodule) *Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) A *-representation $(\mathscr{E}_{\mathscr{A}}, \pi)$ of $\mathscr{B}$ on an inner-product right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ is called strongly non-degenerate if

$$\pi(\mathscr{B})\mathscr{E}_{\mathscr{A}} = \mathscr{E}_{\mathscr{A}}. \tag{2.1.34}$$

ii.) A strongly non-degenerate *-representation of $\mathscr{B}$ on an inner-product right $\mathscr{A}$-module is also called an inner-product $(\mathscr{B}, \mathscr{A})$-bimodule.

iii.) The full sub-category of *-$\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ of inner-product $(\mathscr{B}, \mathscr{A})$-bimodules is denoted by *-$\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$.

Clearly, the forgetful functor from (2.1.33) restricts to a forgetful functor

$$^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B}) \longrightarrow \mathsf{Mod}_{\mathscr{A}}(\mathscr{B}). \tag{2.1.35}$$

In a last step we require the inner product to be completely positive. This will give us the representation theories on pre-Hilbert modules, again with or without strong non-degeneracy:

Definition 2.1.24 (Pre-Hilbert bimodule) *Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) The full sub-category of *-$\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ of *-representations of $\mathscr{B}$ on pre-Hilbert right $\mathscr{A}$-modules is denoted by *-$\mathsf{rep}_{\mathscr{A}}(\mathscr{B})$.

ii.) An inner-product $(\mathscr{B}, \mathscr{A})$-bimodule $(\mathscr{E}_{\mathscr{A}}, \pi)$ with completely positive inner product is called a pre-Hilbert $(\mathscr{B}, \mathscr{A})$-bimodule.

iii.) The full sub-category of *-$\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ of pre-Hilbert $(\mathscr{B}, \mathscr{A})$-bimodules is denoted by *-$\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$.

Again, we can forget that the inner product is completely positive. As the notion of an intertwiner only needs the notion of an adjoint but not of positivity, we get forgetful functors

$$^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B}) \quad \text{and} \quad {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B}), \tag{2.1.36}$$

which is now even *fully faithful*, contrary to the forgetful functors (2.1.33) and (2.1.35), since the additional adjointability of intertwiners is not affected by the complete positivity of the inner products.

Remark 2.1.25 Let $\mathscr{B}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then we get back our previous *-representation theories by taking the coefficient algebra to be C, i.e. we have

$$^*\text{-}\mathsf{rep}_{\mathsf{C}}(\mathscr{B}) = {}^*\text{-}\mathsf{rep}(\mathscr{B}) \quad \text{and} \quad {}^*\text{-}\mathsf{Rep}_{\mathsf{C}}(\mathscr{B}) = {}^*\text{-}\mathsf{Rep}(\mathscr{B}). \tag{2.1.37}$$

Thus *-$\mathsf{rep}_{\mathscr{A}}(\mathscr{B})$ provides indeed a generalization of *-$\mathsf{rep}(\mathscr{B})$ where the auxiliary *-algebra $\mathscr{A}$ plays now the role of the scalars C. The important point is that $\mathscr{A}$ might very well be noncommutative.

Note the asymmetry in the definition of a inner-product bimodule in Definition 2.1.23, *ii.)*, as well as in Definition 2.1.24, *ii.)*: we do *not* require a $\mathscr{B}$-valued inner product on an inner-product $(\mathscr{B}, \mathscr{A})$-bimodule. Moreover, we do *not* require $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$. This additional requirement will become important only in Section 4.3. Finally, note that as in the case of pre-Hilbert spaces the restriction to strongly non-degenerate *-representations is not severe: for unital *-algebras we can always decompose the representation space into a strongly non-degenerate one, where $\pi(\mathbb{1}) = \mathrm{id}$, and the zero-representation. Thus we are mainly interested in *-$\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ and *-$\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$, respectively.

Remark 2.1.26 Of course, an analogous framework for *-representations of $\mathscr{B}$ from the *right* can be established on inner-product *left* $\mathscr{A}$-modules. As usual, by complex conjugation we can easily pass from one to the other. Hence it will be sufficient to study $^*\text{-mod}_{\mathscr{A}}(\mathscr{B})$ and $^*\text{-rep}_{\mathscr{A}}(\mathscr{B})$.

For the following constructions it will be useful to allow for degenerate inner products as intermediate steps as already before. Thus let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Then we call $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ *compatible* with the left $\mathscr{B}$-module structure if

$$\langle b \cdot \phi, \psi \rangle_{\mathscr{A}} = \langle \phi, b^* \cdot \psi \rangle_{\mathscr{A}} \tag{2.1.38}$$

for all $b \in \mathscr{B}$ and $\phi, \psi \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. In this case the degeneracy space of $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is preserved by the left $\mathscr{B}$-multiplications:

Proposition 2.1.27 *Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule with a compatible $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$.*

i.) The left $\mathscr{B}$-multiplications preserve the degeneracy space, i.e. we have

$$\mathscr{B} \cdot \mathscr{E}_{\mathscr{A}}^{\perp} \subseteq \mathscr{E}_{\mathscr{A}}^{\perp}. \tag{2.1.39}$$

ii.) The quotient $\mathscr{E}_{\mathscr{A}} / \mathscr{E}_{\mathscr{A}}^{\perp}$ becomes an inner-product $(\mathscr{B}, \mathscr{A})$-bimodule.

iii.) If the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ was completely positive then the quotient is even a pre-Hilbert $(\mathscr{B}, \mathscr{A})$-bimodule.

PROOF: Let $b \in \mathscr{B}$, $\phi \in \mathscr{E}_{\mathscr{A}}^{\perp}$, and $\psi \in \mathscr{E}_{\mathscr{A}}$ be given. Then $\langle \psi, b \cdot \phi \rangle_{\mathscr{A}} = \langle b^* \cdot \psi, \phi \rangle_{\mathscr{A}} = 0$ shows that $b \cdot \phi \in \mathscr{E}_{\mathscr{A}}^{\perp}$, too. This proves the first part. But then it is clear that the quotient is a $(\mathscr{B}, \mathscr{A})$-bimodule. The induced left $\mathscr{B}$-module structure is still compatible with the induced inner product as this can be checked on representatives. Finally, the induced inner product stays completely positive since this can again be checked on representatives. □

This proposition will prove very useful in the construction of *-representations of *-algebras: the non-degeneracy of the inner products can always be achieved by a simple quotient procedure.

2.2 Examples of Pre-Hilbert Modules

In this section we collect some fundamental examples of inner-product modules and pre-Hilbert modules, starting with the canonical inner product on the free module $\mathscr{A}^n$. Beyond the case of unital *-algebras, one typically has to require some additional non-triviality conditions on the multiplication law in $\mathscr{A}$ as otherwise pathological behaviour occurs easily. It turns out that non-degenerate and idempotent *-algebras provide a good class, also for later applications in Morita theory. Passing from a non-degenerate to a strongly non-degenerate inner product will provide additional features simplifying the parametrization of all possible inner products drastically. However, for non-unital *-algebras such inner products will be typically rather rare.

2.2.1 First Examples and Constructions

We start with the most fundamental and simple example of a completely positive inner product: we consider $\mathscr{A}$ itself as a right $\mathscr{A}$-module via right multiplications and set for $a, b \in \mathscr{A}$

$$\langle a, b \rangle = a^* b. \tag{2.2.1}$$

We call this the *canonical inner product* on the *-algebra $\mathscr{A}$.

Lemma 2.2.1 *The canonical $\mathscr{A}$-valued inner product* (2.2.1) *is completely positive.*

Proof: First it is clear that $\langle \cdot\,, \cdot \rangle$ satisfies the remaining requirements of an $\mathscr{A}$-inner product. Thus consider $a_1, \ldots, a_n \in \mathscr{A}$. Then the matrix $(a_i^* a_j)$ is positive by Lemma 2.1.12, *i.)*. ☐

This simple example gives immediately further examples of completely positive inner products. The following is of equal major importance:

Example 2.2.2 (Canonical inner product) Let $n \in \mathbb{N}$. Then the *free* right $\mathscr{A}$-module $\mathscr{A}^n$ (with multiplication defined componentwise from the right) has an $\mathscr{A}$-valued inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i^* y_i, \tag{2.2.2}$$

still called the *canonical inner product.* It is completely positive by Lemma 2.1.16 since it is obtained as the finite direct orthogonal sum of (2.2.1). Sometimes we shall also make use of an infinite direct orthogonal sum $\mathscr{A}^{(\Lambda)}$ with arbitrary index set Λ. Of course, also $\mathscr{A}^{(\Lambda)}$ inherits a completely positive $\mathscr{A}$-valued inner product. Recall that a module of this form is called *free* and in case where $\#\Lambda < \infty$ it is called *finitely generated* in addition. In the free module $\mathscr{A}^{(\Lambda)}$ over a *unital* algebra we have a *canonical basis* $\{e_\lambda\}_{\lambda\in\Lambda}$ given by $e_\lambda = (e_{\lambda\lambda'})_{\lambda'\in\Lambda}$ with $e_{\lambda\lambda'} = \delta_{\lambda\lambda'}\mathbb{1}$. This basis is orthonormal with respect to the canonical inner product in the sense that

$$\langle e_\lambda, e_{\lambda'} \rangle = \delta_{\lambda\lambda'}\mathbb{1} \tag{2.2.3}$$

for all $\lambda, \lambda' \in \Lambda$.

The question whether (2.2.1) and thus (2.2.2) are non-degenerate depends strongly on the *-algebra $\mathscr{A}$. Here the following notions turn out to be useful:

Definition 2.2.3 (Non-degenerate and idempotent algebra) *A *-algebra $\mathscr{A}$ is called*

i.) *non-degenerate if $ab = 0$ for all $a \in \mathscr{A}$ implies $b = 0$,*

ii.) *idempotent if the elements of the form ab with $a, b \in \mathscr{A}$ span $\mathscr{A}$.*

Remark 2.2.4 Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) A unital *-algebra is non-degenerate and idempotent.

ii.) The non-degeneracy of $\mathscr{A}$ from the left or from the right coincide thanks to the presence of the *-involution. Thus we only need to state non-degeneracy from one side.

iii.) The canonical inner product (2.2.1) is non-degenerate if and only if $\mathscr{A}$ is non-degenerate. In this case $\mathscr{A}$ and also $\mathscr{A}^n$ becomes a pre-Hilbert module over $\mathscr{A}$ via the canonical inner products.

iv.) For an admissible non-degenerate *-algebra we have $a^*a = 0$ only for $a = 0$.

For a non-unital *-algebra there is a reasonable replacement of a unit element, given by local (Hermitian) unit elements:

Definition 2.2.5 (Local Hermitian unit elements) *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\{\mathsf{e}_i\}_{i\in I}$ be a collection of Hermitian elements in $\mathscr{A}$. They are called local Hermitian units if for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathscr{A}$ one finds an index $i \in I$ with*

$$\mathsf{e}_i a_k = a_k = a_k \mathsf{e}_i \tag{2.2.4}$$

for all $k = 1, \ldots, n$.

Proposition 2.2.6 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with local Hermitian units. Then $\mathscr{A}$ is non-degenerate and idempotent.*

Example 2.2.7 Consider a non-compact smooth manifold M. Then $\mathscr{C}^\infty_0(M)$ is non-unital but has local Hermitian units. Indeed, choose an exhausting sequence

$$K_0 \subseteq K_1^\circ \subseteq \cdots \subseteq K_{n-1} \subseteq K_n^\circ \subseteq \cdots M \tag{2.2.5}$$

of compact subsets K_n of M, i.e. we have $\bigcup_{n\in\mathbb{N}_0} K_n = M$. By the $\mathscr{C}^\infty$-Urysohn Lemma we find smooth functions $\chi_n = \overline{\chi}_n \in \mathscr{C}^\infty_0(M)$ with $\operatorname{supp}\chi_n \subseteq K_{n+1}$ but $\chi_n\big|_{K_n} = 1$. It is then easy to see that they form local Hermitian units for $\mathscr{C}^\infty_0(M)$. This has easy generalizations to other kinds of topological spaces and function algebras on them.

Still within differential geometry we have further important examples of completely positive inner products and pre-Hilbert modules. First we note the following result:

Proposition 2.2.8 *Let $n \in \mathbb{N}$. For a matrix-valued function $A \in \mathscr{C}^\infty(M, \mathrm{M}_n(\mathbb{C})) = \mathrm{M}_n(C^\infty(M))$ we have $A \in \mathrm{M}_n(C^\infty(M))^+$ if and only if $A(p) \in \mathrm{M}_n(\mathbb{C})^+$ for all $p \in M$.*

The proof of this proposition is contained in Exercise 1.4.17. The positive functionals of $\mathscr{C}^\infty(M)$ are determined by using Riesz' Representation Theorem.

Example 2.2.9 (Hermitian vector bundles) Consider $\mathscr{A} = \mathscr{C}^\infty(M)$ for a manifold M and let $E \longrightarrow M$ be a complex vector bundle. The sections $\Gamma^\infty(E)$ are a $\mathscr{C}^\infty(M)$-module in the usual way. Since $\mathscr{C}^\infty(M)$ is commutative we can choose whether we want to consider this as a left or as a right module. In order to fit to the current presentation, we choose a right module. Next, let h be a pseudo-Hermitian fiber metric. Then

$$\langle \phi, \psi \rangle(p) = h(p)(\phi(p), \psi(p)) \tag{2.2.6}$$

with $p \in M$ defines a smooth function $\langle \phi, \psi \rangle \in \mathscr{C}^\infty(M)$ for all sections $\Gamma^\infty(E)$. It is easy to see that this way we obtain a non-degenerate $\mathscr{C}^\infty(M)$-valued inner product. Hence $\Gamma^\infty(E)$ becomes an inner-product module over $\mathscr{C}^\infty(M)$. By Lemma 2.2.8 it is easy to see that for a *Hermitian* fiber metric, the inner product (2.2.6) is actually completely positive and positive definite. Thus $\Gamma^\infty(E)$ becomes a pre-Hilbert module over $\mathscr{C}^\infty(M)$ for a Hermitian vector bundle. We will come back to this example after introducing some more advanced technology and give an independent proof of complete positivity. Needless to say, this example is the starting point to investigate a deformation quantization of vector bundles with inner products.

The next example generalizes the canonical inner product on the free module by twisting it with a collection of module morphisms.

Example 2.2.10 Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{A}}$. Assume that it can be written as

$$\langle \phi, \psi \rangle_{\mathscr{A}} = \sum_{\alpha=1}^m P_\alpha(\phi)^* P_\alpha(\psi) \tag{2.2.7}$$

with $\mathscr{A}$-module morphisms $P_\alpha\colon \mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{A}$. We note that $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{A}}$ is an $\mathscr{A}$-valued inner product for any choice of such module morphisms. Moreover, $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{A}}$ is completely positive. Indeed, let $\phi_1, \ldots, \phi_n \in \mathscr{E}_{\mathscr{A}}$ be given then

$$(\langle \phi_i, \phi_j \rangle_{\mathscr{A}}) = \left(\sum_{\alpha=1}^m P_\alpha(\phi_i)^* P_\alpha(\phi_j) \right) = \sum_{\alpha=1}^m \big(P_\alpha(\phi_i)^* P_\alpha(\phi_j) \big) \in \mathrm{M}_n(\mathscr{A})^{++} \tag{2.2.8}$$

by Lemma 2.1.12, *i.)*, as already $(P_\alpha(\phi_i)^* P_\alpha(\phi_j)) \in \mathrm{M}_n(\mathscr{A})^{++}$ for each α. More generally, we can also use an infinite number $\{P_\alpha\}_{\alpha \in I}$ of such module morphisms and an infinite sum *provided* that for a fixed element $\phi \in \mathscr{E}_{\mathscr{A}}$ only finitely many $P_\alpha(\phi) \in \mathscr{A}$ are different from zero. Note that in general it is of course difficult to say whether such an inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is non-degenerate: this depends very much on the details of the maps P_α.

2.2.2 Strongly Non-Degenerate Inner Products

Also for an inner-product or pre-Hilbert module we can impose a stronger version of the non-degeneracy of the inner product. To formulate this, we first recall that for a right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ the *dual module* is defined by the set of right $\mathscr{A}$-module homomorphisms

$$\mathscr{E}^* = \mathrm{Hom}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{A}), \tag{2.2.9}$$

where we endow $\mathscr{A}$ with its canonical right $\mathscr{A}$-module structure as usual. Note that $\mathscr{E}^*$ is a *left* $\mathscr{A}$-module in a canonical way by setting

$$(a \cdot \chi)(\phi) = a\chi(\phi) \tag{2.2.10}$$

for $a \in \mathscr{A}$, $\phi \in \mathscr{E}_{\mathscr{A}}$, and $\chi \in \mathscr{E}^*$. In the following, we will emphasize this by writing ${}_{\mathscr{A}}\mathscr{E}^*$ instead of $\mathscr{E}^*$. Now if we have an inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ on $\mathscr{E}_{\mathscr{A}}$ then this gives a map

$$\flat \colon \mathscr{E}_{\mathscr{A}} \ni \phi \;\mapsto\; \phi^\flat = \langle \phi, \cdot \rangle_{\mathscr{A}} \in {}_{\mathscr{A}}\mathscr{E}^*, \tag{2.2.11}$$

which is an antilinear antihomomorphism of modules in the sense that

$$\phi \cdot a \;\mapsto\; a^* \cdot \langle \phi, \cdot \rangle_{\mathscr{A}} \tag{2.2.12}$$

for $a \in \mathscr{A}$ and $\phi \in \mathscr{E}_{\mathscr{A}}$. This antihomomorphism is called *musical* in analogy to the usual musical homomorphism of inner products. Clearly, the inner product is non-degenerate iff the map (2.2.11) is injective. This motivates the following definition:

Definition 2.2.11 (Strongly non-degenerate inner product) *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Then the inner product is called strongly non-degenerate if the induced map* (2.2.11) *is bijective.*

Thus it is the surjectivity which is the additional property of a strongly non-degenerate inner product compared to a non-degenerate one. The following is obvious:

Proposition 2.2.12 *A finite direct orthogonal sum of strongly non-degenerate inner products is again strongly non-degenerate.*

For an infinite direct orthogonal sum the statement is false in general. The simplest examples are already obtained for $\mathcal{A} = \mathsf{C} = \mathbb{C}$.

Example 2.2.13 Consider again a (finite-dimensional) vector bundle $E \longrightarrow M$ over a smooth manifold and endow the sections $\Gamma^\infty(E)$ with its usual right $\mathscr{C}^\infty(M)$-module structure. If h is a pseudo-Hermitian fiber metric then the inner product $h(p)$ on each fiber E_p is non-degenerate at every point. Since the fiber is finite-dimensional, the induced map from E_p to the dual fiber E_p^* is bijective for every $p \in M$. From this one concludes that the induced map for the sections $\Gamma^\infty(E) \longrightarrow \Gamma^\infty(E^*)$ is bijective as well. Finally, $\Gamma^\infty(E^*)$ is known to coincide with the dual module $\Gamma^\infty(E)^*$. Hence a pseudo-Hermitian fiber metric gives a strongly non-degenerate inner product. We see that if $h(p)$ is degenerate at some few points, the corresponding map $\Gamma^\infty(E) \longrightarrow \Gamma^\infty(E^*)$ can still be injective by the continuity of the sections. However, the surjectivity will be lost. Conversely, if we have a bijective map (2.2.11) for an inner product on $\Gamma^\infty(E)$ then the induced map on every fiber has to be bijective, too. Thus we arrive at the statement, that the pseudo-Hermitian fiber metrics on E correspond precisely to the *strongly non-degenerate* inner products on $\Gamma^\infty(E)$ with values in $\mathscr{C}^\infty(M)$. Again, there is an analogous statement in the continuous category.

If we have a strongly non-degenerate inner product on a right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ then we can parametrize all other inner products in terms of Hermitian elements of $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ as we know this from finite-dimensional complex vector spaces:

Proposition 2.2.14 *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with strongly non-degenerate inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$.*

i.) We have $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ with the unique *-involution determined by the condition

$$\langle x, Ay \rangle_{\mathscr{A}} = \langle A^* x, y \rangle_{\mathscr{A}} \tag{2.2.13}$$

for $A \in \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ and $x, y \in \mathscr{E}_{\mathscr{A}}$.

ii.) The $\mathscr{A}$-valued inner products $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ on $\mathscr{E}_{\mathscr{A}}$ are in bijection to the Hermitian elements $H \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ via

$$\langle \phi, \psi \rangle'_{\mathscr{A}} = \langle \phi, H\psi \rangle_{\mathscr{A}}. \tag{2.2.14}$$

iii.) The inner product $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ is non-degenerate if and only if H is injective.

iv.) The inner product $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ is strongly non-degenerate if and only if H is bijective.

v.) The inner product $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ is isometric to $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ if and only if there exists an invertible $U \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ with $H = U^*U$.

vi.) Let $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ be in addition completely positive. Then $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ is completely positive, too, if $H \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})^+$.

PROOF: For each $A \in \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ and each $\phi \in \mathscr{E}_{\mathscr{A}}$ the map $\psi \mapsto \langle \phi, A\psi \rangle_{\mathscr{A}}$ is right $\mathscr{A}$-linear and, by strong non-degeneracy, of the form $\psi \mapsto \langle A^*\phi, \psi \rangle^{\mathscr{E}}_{\mathscr{A}}$ with a unique $A^*\phi \in \mathscr{E}_{\mathscr{A}}$. This defines the map $A^*\colon \mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{E}_{\mathscr{A}}$. Thus A is adjointable. For the second part we argue analogously by noting that the map $\psi \mapsto \langle \phi, \psi \rangle'_{\mathscr{A}}$ is right $\mathscr{A}$-linear and thus of the form $\psi \mapsto \langle \tilde{\phi}, \psi \rangle_{\mathscr{A}}$ with a unique $\tilde{\phi} \in \mathscr{E}_{\mathscr{A}}$. Again, $\phi \mapsto \tilde{\phi}$ is right $\mathscr{A}$-linear. Hence it is of the form $\tilde{\phi} = H\phi$ with some unique $H \in \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. The symmetry of $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ then shows immediately that $H = H^*$. Conversely, it is clear that any Hermitian $H = H^* \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ induces a new inner product $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ via (2.2.14). This shows that second part. The third and fourth part are obvious. For the fifth part, we have to be a little bit more careful as the *-involution in $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ refers to the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. For an isometric isomorphism U *between* the two inner products $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ and $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ we have to find an adjointable and bijective map $U \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}})$ with

$$\langle \phi, \psi \rangle'_{\mathscr{A}} = \langle U\phi, U\psi \rangle_{\mathscr{A}}, \tag{$*$}$$

where $\mathscr{E}'_{\mathscr{A}}$ is the right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$, but now endowed with the inner product $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$. We denote the adjoint of U in $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}})$ by $U^\dagger \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}'_{\mathscr{A}}, \mathscr{E}_{\mathscr{A}})$, i.e.

$$\langle U\phi, \psi \rangle_{\mathscr{A}} = \langle \phi, U^\dagger \psi \rangle'_{\mathscr{A}}.$$

From the first part we know that $U \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}}) \subseteq \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}_{\mathscr{A}}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Thus U is adjointable with respect to $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$, too, and $(*)$ gives $H = U^*U$. Conversely, if $H = U^*U$ with an invertible U then we clearly have $\langle \phi, \psi \rangle'_{\mathscr{A}} = \langle U\phi, U\psi \rangle_{\mathscr{A}}$. Since by assumption U is invertible, we can write $\psi = U^{-1}U\psi$ leading to $\langle \phi, U^{-1}U\psi \rangle'_{\mathscr{A}} = \langle U\phi, U\psi \rangle_{\mathscr{A}}$ showing that U is also adjointable with respect to the involution † and satisfies $U^{-1} = U^\dagger$. Thus, in this sense, U is unitary as wanted, proving the fourth part. Now let $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ be completely positive and let $H \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})^+$. For given elements $\phi_1, \ldots, \phi_n \in \mathscr{E}_{\mathscr{A}}$ and a given positive linear functional $\Omega\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathsf{C}$ we consider the map

$$\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \ni A \;\mapsto\; \tilde{\Omega}(A) = \Omega(\langle \phi_i, A\phi_j \rangle_{\mathscr{A}}).$$

This is a positive functional since $\Omega(\langle \phi_i, A^* A\phi_j \rangle_{\mathscr{A}}) = \Omega(\langle A\phi_i, A\phi_j \rangle_{\mathscr{A}}) \geq 0$ by the complete positivity of $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ and the positivity of Ω. Thus for a positive element $H \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})^+$ we have

$$0 \leq \tilde{\Omega}(H) = \Omega(\langle \phi_i, H\phi_j \rangle_{\mathscr{A}}) = \Omega\big(\langle \phi_i, \phi_j \rangle'_{\mathscr{A}}\big),$$

which gives immediately the complete positivity of $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$. □

Remark 2.2.15 For the canonical pre-Hilbert module $\mathscr{A}^n$ and a unital *-algebra $\mathscr{A}$ one has the stronger result that $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ is completely positive if and only if $H \in \mathfrak{B}_{\mathscr{A}}(\mathscr{A}^n)^+ \cong \mathrm{M}_n(\mathscr{A})^+$. The only-if part follows easily since $(\langle e_i, e_j \rangle'_{\mathscr{A}}) = (\langle e_i, He_j \rangle_{\mathscr{A}}) = H$, hence by complete positivity of $\langle \cdot\,, \cdot \rangle'_{\mathscr{A}}$ the matrix H has to be positive.

2.3 Various K_0-Theories

In Example 2.2.10, the module morphisms P_α guaranteed the complete positivity of the inner product. However, in general it is not clear how one can possibly construct such maps. We shall now present a simple situation with a natural and explicit construction of such maps P_α, leading us into the realm of projective modules and K_0-theory. The following example will turn out to be of crucial importance:

Example 2.3.1 Let $P \in \mathrm{M}_n(\mathscr{A})$ be a projection, $P^2 = P = P^*$. Then we consider the image of P as a submodule of $\mathscr{A}^n$, i.e. let

$$\mathscr{E}_{\mathscr{A}} = P\mathscr{A}^n \subseteq \mathscr{A}^n \tag{2.3.1}$$

be endowed with the induced right $\mathscr{A}$-module structure. Elements in $\mathscr{E}_{\mathscr{A}}$ are of the form Px with $x \in \mathscr{A}^n$ arbitrary. The restriction of the canonical inner product $\langle \cdot\,, \cdot \rangle$ on $\mathscr{A}^n$ to $P\mathscr{A}^n$ is still completely positive by Lemma 2.1.16, *ii.)*. On the other hand, we have

$$\langle Px, Py \rangle = \sum_{i=1}^{n} (P(x))_i^* P(y)_i, \tag{2.3.2}$$

where the maps $P_i\colon Px \mapsto (P(x))_i = \sum_{j=1}^n P_{ij} x_j$ are right $\mathscr{A}$-module morphisms as in Example 2.2.10. Moreover, we have $\langle Px, Py \rangle = 0$ for all $Px \in P\mathscr{A}^n$ if and only if $\langle x, Py \rangle = 0$ for all $x \in \mathscr{A}^n$ since $P^*P = P$. Thus if $\mathscr{A}$ is non-degenerate as algebra then $Py = 0$ follows. In this case, the inner product on $P\mathscr{A}^n$ is non-degenerate and $P\mathscr{A}^n$ is a pre-Hilbert module for all projections P.

2.3.1 Projective Modules and Ring-Theoretic K_0-Theory

The construction in Example 2.3.1 will be of major importance in many places. Hence it is worth to take a closer look at the right $\mathscr{A}$-module structure obtained from a projection. We recall the following definition of a projective module:

Definition 2.3.2 (Projective module) *A right $\mathscr{A}$-module $\mathscr{E}$ is called projective if there exists another right $\mathscr{A}$-module $\mathscr{F}$ such that $\mathscr{E} \oplus \mathscr{F}$ is a free right $\mathscr{A}$-module, i.e. isomorphic to some $\mathscr{A}^{(\Lambda)}$ for suitable index set Λ.*

Note that the definition of a projective module does not refer to a *-involution, instead it applies for general rings. In order to keep things simple, we shall focus on *unital* rings and *-algebras in this section. The non-unital case will require some slightly more involved definition of the K_0-theories.

We collect some well-known properties of projective modules. For convenience of the reader, we sketch the proofs, see also e.g. the textbooks [78, Sect. 1.2] or [4] and [99, Chap. 1] for a detailed discussion of projective modules and K-theory:

Proposition 2.3.3 *Let $\mathscr{E}$ be a right $\mathscr{A}$-module over a unital ring $\mathscr{A}$. Then the following statements are equivalent:*

i.) The module $\mathscr{E}$ is projective.

ii.) There exists an index set Λ and an idempotent element $e \in \operatorname{End}_{\mathscr{A}}(\mathscr{A}^{(\Lambda)})$ such that $\mathscr{E} \cong e\mathscr{A}^{(\Lambda)}$ as right $\mathscr{A}$-modules.

iii.) There exist elements $\{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{E}$ and elements $\{e^\lambda\}_{\lambda\in\Lambda}$ in the dual (left) module $\mathscr{E}^*$ such that for a given $x \in \mathscr{E}$ only finitely many $e^\lambda(x) \in \mathscr{A}$ are different from zero and

$$x = \sum_{\lambda\in\Lambda} e_\lambda \cdot e^\lambda(x). \tag{2.3.3}$$

iv.) Let $\phi\colon \mathscr{M} \longrightarrow \mathscr{N}$ and $\psi\colon \mathscr{E} \longrightarrow \mathscr{N}$ be right $\mathscr{A}$-module morphisms with ϕ surjective. Then there exists a right $\mathscr{A}$-module morphism $\chi\colon \mathscr{E} \longrightarrow \mathscr{M}$ with $\phi \circ \chi = \psi$, i.e. the diagram

$$\begin{array}{ccccc} & & \mathscr{E} & & \\ & \overset{\chi}{\swarrow} & \big\downarrow \psi & & \\ \mathscr{M} & \underset{\phi}{\longrightarrow} & \mathscr{N} & \longrightarrow & 0 \end{array} \tag{2.3.4}$$

commutes.

Proof: We show *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *iv.)* $\implies$ *i.)*. Thus, let us assume *i.)* and let $\mathscr{E} \oplus \mathscr{F} = \mathscr{A}^{(\Lambda)}$ be a free module after the choice of an appropriate isomorphism. For $x \in \mathscr{A}^{(\Lambda)}$ we have a unique decomposition $x = x_\parallel + x_\perp$ where $x_\parallel \in \mathscr{E}$ and $x_\perp \in \mathscr{F}$. Since the decomposition is a direct sum as right $\mathscr{A}$-modules the map $e\colon x \mapsto x_\parallel$ is right $\mathscr{A}$-linear and clearly idempotent $e^2 = e$. It follows that $\mathscr{E} = \operatorname{im} e$ and thus *ii.)*. Now we assume *ii.)* and let $\{\mathsf{e}_\lambda\}_{\lambda\in\Lambda}$ be the canonical module basis of $\mathscr{A}^{(\Lambda)}$. Then $x = \sum_{\lambda\in\Lambda} \mathsf{e}_\lambda \cdot x^\lambda$ with unique coefficients x^λ where for each x only finitely many of them are non-zero. The map $\mathsf{e}^\lambda\colon x \mapsto x^\lambda$ is right $\mathscr{A}$-linear and thus e^λ is in the dual module $\mathscr{E}^*$. Now let $x = e(x) \in \mathscr{E} = \operatorname{im} e \subseteq \mathscr{A}^{(\Lambda)}$ with an idempotent $e \in \operatorname{End}_{\mathscr{A}}(\mathscr{A}^{(\Lambda)})$ be given then

$$x = e(x) = e\Big(\sum\nolimits_\lambda \mathsf{e}_\lambda \cdot \mathsf{e}^\lambda(x)\Big) = \sum\nolimits_\lambda e(\mathsf{e}_\lambda)\mathsf{e}^\lambda(x).$$

Thus we have found the elements $e_\lambda = e(\mathsf{e}_\lambda)$ and $e^\lambda = \mathsf{e}^\lambda\big|_{\operatorname{im} e} \in \mathscr{E}^*$ as wanted. Assuming *iii.)*, let $e_\lambda \in \mathscr{E}$ and $e^\lambda \in \mathscr{E}^*$ with (2.3.3) be given. Let $\phi\colon \mathscr{M} \longrightarrow \mathscr{N}$ be a surjective and let $\psi\colon \mathscr{E} \longrightarrow \mathscr{N}$ be an arbitrary right $\mathscr{A}$-module morphism. Since ϕ is surjective we find $m_\lambda \in \mathscr{M}$ with $\phi(m_\lambda) = \psi(e_\lambda)$. Then we define $\chi(x) = \sum_{\lambda\in\Lambda} m_\lambda \cdot e^\lambda(x)$. Clearly, for a given $x \in \mathscr{E}$ the sum is finite. Moreover, χ is right $\mathscr{A}$-linear since the e^λ are in $\mathscr{E}^*$. A simple computation using (2.3.3) shows $\phi(\chi(x)) = \psi(x)$ which is *iv.)*. Finally, assume *iv.)*. First we note that for every right $\mathscr{A}$-module $\mathscr{E}$ there exists a large enough free module $\mathscr{A}^{(\Lambda)}$ together with a surjective right $\mathscr{A}$-linear map $\phi\colon \mathscr{A}^{(\Lambda)} \longrightarrow \mathscr{E}$. This is clear as we can use $\Lambda = \mathscr{E}$ as index set and define

$$\phi\Big(\sum\nolimits_{x\in\mathscr{E}} \mathsf{e}_x \cdot a^x\Big) = \sum\nolimits_{x\in\mathscr{E}} x \cdot a^x.$$

Next, we set $\psi = \mathrm{id}\colon \mathscr{E} \longrightarrow \mathscr{E}$. Thus by assumption we find a right $\mathscr{A}$-linear map $\chi\colon \mathscr{E} \longrightarrow \mathscr{A}^{(\Lambda)}$ with $\phi \circ \chi = \mathrm{id}$. This implies that χ is injective with $\operatorname{im}\chi \cong \mathscr{E}$ as right $\mathscr{A}$-modules. Moreover, $\mathscr{F} = \ker\phi$ is a right $\mathscr{A}$-submodule of $\mathscr{A}^{(\Lambda)}$ which is complementary to $\operatorname{im}\chi$. Thus $\mathscr{E}$ is projective. □

Remark 2.3.4 (Projective modules) Let $\mathscr{A}$ be a unital ring.

i.) Mainly, we will be interested in *finitely generated* projective modules over $\mathscr{A}$. In this case, one can show that the index set Λ in Proposition 2.3.3 can be replaced by some suitable $n \in \mathbb{N}$, see Exercise 2.4.17.

ii.) The elements $e_\lambda \in \mathscr{E}$ and $e^\lambda \in \mathscr{E}^*$ from *iii.)* are called a *dual basis*. Note, however, that the e_λ are by far *not* $\mathscr{A}$-linearly independent: from $\sum_{\lambda\in\Lambda} e_\lambda \cdot a^\lambda = 0$ we can not conclude $a^\lambda = 0$ in general. If this would be true then the projective module is even a free module which in general needs not to be the case. Nevertheless, free modules are examples for projective ones. Moreover, in general the vectors e^λ do not even span the dual module $\mathscr{E}^*$, see also Exercise 2.4.15.

iii.) It is easy to see that the direct sum of projective modules is again projective. Moreover, it is again finitely generated whenever the direct sum was finite and each term was finitely generated.

The question whether two projective modules are isomorphic can be encoded in terms of the idempotents of Proposition 2.3.3.

Proposition 2.3.5 *Let $\mathscr{E}$ and $\mathscr{F}$ be finitely generated projective right modules over a unital ring $\mathscr{A}$ which we write without restrictions as $\mathscr{E} = e\mathscr{A}^n$ and $\mathscr{F} = f\mathscr{A}^n$ with the same $n \in \mathbb{N}$ and suitably chosen idempotents $e, f \in \mathrm{M}_n(\mathscr{A})$. Then the following statements are equivalent:*

i.) The right $\mathscr{A}$-modules $\mathscr{E}$ and $\mathscr{F}$ are isomorphic.

ii.) There are $u, v \in \mathrm{M}_n(\mathscr{A})$ with

$$e = uv \quad \text{and} \quad f = vu. \tag{2.3.5}$$

iii.) There exists an invertible matrix $V \in \mathrm{M}_{2n}(\mathscr{A})$ with

$$V\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}V^{-1} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3.6}$$

Proof: First we note that adding zero entries we can always bring two idempotents to the same size n without changing their images. Hence we can assume that n is the same without restriction from the beginning. We show *i.)* $\Longrightarrow$ *ii.)* $\Longrightarrow$ *iii.)* $\Longrightarrow$ *i.)*. First, we assume that $\phi\colon \mathscr{E} \longrightarrow \mathscr{F}$ is an isomorphism of right $\mathscr{A}$-modules. Now we set $v = f\phi e\colon \mathscr{A}^n \longrightarrow \mathscr{A}^n$ which is again right $\mathscr{A}$-linear. Hence we can identify v with a matrix in $\mathrm{M}_n(\mathscr{A})$. Analogously we define $u = e\phi^{-1}f \in \mathrm{M}_n(\mathscr{A})$. Since f is the identity on the image of ϕ (which is $\mathscr{F}$) we have

$$uv(x) = e\phi^{-1}ff\phi e(x) = e\phi^{-1}f\phi(e(x)) = ee(x) = e(x).$$

Similarly, one shows $vu = f$ and hence we obtain *ii.)*. Now assume *ii.)* and let $e, f \in \mathrm{M}_n(\mathscr{A})$ be idempotent elements with (2.3.5) for some $u, v \in \mathrm{M}_n(\mathscr{A})$. Then we define the block matrix

$$V = \begin{pmatrix} -v & \mathbb{1} - f \\ \mathbb{1} - e & u \end{pmatrix} \in \mathrm{M}_{2n}(\mathscr{A}).$$

An elementary computation shows that V is invertible with inverse given by

$$V^{-1} = \begin{pmatrix} -u & \mathbb{1} - e \\ \mathbb{1} - f & v \end{pmatrix}.$$

Using these explicit formulas, (2.3.6) is a straightforward computation thereby verifying *iii.)*. Finally, we assume (2.3.6). Then the projective modules $\mathscr{E}$ and $\mathscr{F}$ can also be considered as submodules of $\mathscr{A}^{2n}$ as images of the idempotents $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)$, respectively. Then (2.3.6) gives the desired isomorphism $V\big|_{\mathscr{E}} \colon \mathscr{E} \longrightarrow \mathscr{F}$. Note that this restriction gives indeed a map *into* $\mathscr{F}$. □

This property of idempotent elements is the starting point for the definition of the K_0-theory of an algebra $\mathscr{A}$ with unit. First we denote by $\underline{\mathsf{Proj}}(\mathscr{A})$ the category of all finitely generated and projective right $\mathscr{A}$-modules with module homomorphisms as morphisms. By $\mathsf{Proj}(\mathscr{A})$ we denote the class of isomorphism classes of finitely generated and projective modules over $\mathscr{A}$. Thanks to Proposition 2.3.5 we see that $\mathsf{Proj}(\mathscr{A})$ is in bijection to the *set* of all equivalence classes of idempotent elements in $\mathrm{M}_\infty(\mathscr{A})$: here two idempotents e, f are called *equivalent* if there exist $u, v \in M_\infty(\mathscr{A})$ such that

$$e = uv \quad \text{and} \quad f = vu. \tag{2.3.7}$$

Though it follows from Proposition 2.3.5 that this is indeed an equivalence relation, it is also a nice exercise to check this directly, see Exercise 2.4.16. Obviously, (2.3.7) means that there is a large enough $n \in \mathbb{N}$ such that $e, f, u, v \in \mathrm{M}_n(\mathscr{A}) \subseteq M_\infty(\mathscr{A})$. Alternatively, we can also use the third statement in Proposition 2.3.5 to define the equivalence of idempotents: e and f are equivalent if they are conjugate to each other after one has brought them to equal size in some sufficiently large $\mathrm{M}_n(\mathscr{A})$ by adding zeros.

The set $\mathsf{Proj}(\mathscr{A})$ has now an additional structure: we can take finite direct sums of projective modules which are again finitely generated and projective by Remark 2.3.4, *iii.)*. Then the direct sum $\oplus$ becomes an associative and commutative (only on the level of isomorphism classes) composition law. The commutativity and also the associativity is not fulfilled on the level of projective modules directly: we have to use the canonical isomorphism to obtain $\mathscr{E} \oplus \mathscr{F} \cong \mathscr{F} \oplus \mathscr{E}$ etc. Finally, the 0-module is the neutral element with respect to $\oplus$, again after using the isomorphisms $\mathscr{E} \oplus 0 \cong \mathscr{E} \cong 0 \oplus \mathscr{E}$. We summarize these considerations in the following proposition:

Proposition 2.3.6 *The set of isomorphism classes of finitely generated projective modules* $\mathsf{Proj}(\mathscr{A})$ *is an abelian semi-group with respect to* $\oplus$ *with neutral element* $[0]$. *Moreover,* $\mathsf{Proj}(\mathscr{A})$ *is isomorphic to the abelian semi-group of equivalence classes of idempotent elements in* $M_\infty(\mathscr{A})$ *where on the level of representatives* $e \in \mathrm{M}_n(\mathscr{A})$ *and* $f \in \mathrm{M}_m(\mathscr{A})$ *the direct sum is defined by* $e \oplus f = \left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) \in \mathrm{M}_{n+m}(\mathscr{A})$.

An abelian semi-group can always be turned into an abelian group by adding sufficiently many inverses. This process is well-known from the transition from the semi-group of natural numbers $\mathbb{N}_0$ to $\mathbb{Z}$. The resulting group is called the *Grothendieck group* of the semi-group, see Exercise 2.4.18.

Definition 2.3.7 (K_0-Theory) *The Grothendieck group of* $\mathsf{Proj}(\mathscr{A})$ *is denoted by* $\mathsf{K}_0(\mathscr{A})$ *and called the* K_0*-theory of* $\mathscr{A}$.

Remark 2.3.8 (K-Theory) The above algebraic definition of K-theory allows for many generalizations and specializations, we only presented the most simple version. On can extend the above construction to algebras without unit element, where a slightly different approach has to be taken. Moreover, there are higher K-groups $\mathsf{K}_n(\mathscr{A})$ which we will not need in the sequel. A detailed discussion can be found in monographs [4, 99].

2.3.2 The Serre-Swan Theorem

Before we define the Hermitian K_0-theory we will give a geometric interpretation of the construction of $\mathsf{K}_0(\mathscr{A})$ for the case where $\mathscr{A} = \mathscr{C}^\infty(M)$. The following classical theorem of Serre and Swan was originally formulated for commutative unital C^*-algebras, i.e. $\mathscr{A} = \mathscr{C}(X)$ with a compact Hausdorff space X, and in an algebraic-geometric situation, see [107, 110]. Ever since there have been various other formulations and contexts for this theorem, one of which we shall present here:

Theorem 2.3.9 (Serre-Swan Theorem) *Let M be a connected manifold.*

i.) The sections $\Gamma^\infty(E)$ of a complex vector bundle $\pi\colon E \longrightarrow M$ are a finitely generated and projective module over $\mathscr{C}^\infty(M)$.

ii.) If $\mathcal{E}_{\mathscr{C}^\infty(M)}$ is a finitely generated and projective module over $\mathscr{C}^\infty(M)$ then there exists a complex vector bundle $\pi\colon E \longrightarrow M$ such that $\mathcal{E}_{\mathscr{C}^\infty(M)} \cong \Gamma^\infty(E)_{\mathscr{C}^\infty(M)}$ with E being determined uniquely up to vector bundle isomorphisms over the identity of M.

iii.) Vector bundle homomorphisms over the identity of M correspond to module homomorphisms under the correspondence in part i.).

Proof: The first part is the most non-trivial one. We give here a proof which works for the case of compact M to simplify things. By definition of a vector bundle there is a vector bundle atlas $\{(U_i, \varphi_i)\}_{i\in I}$ where $U_i \subseteq M$ is open and $\varphi_i\colon \pi^{-1}(U_i) \subseteq E \longrightarrow U_i \times \mathbb{C}^k$ is a local trivialization. Here k is the fiber dimension. As we assume M to be compact, *finitely many* U_i already cover M, say $U_1, \ldots, U_N$. We choose a subordinate quadratic partition of unity $\{\chi_i\}_{i=1,\ldots,N}$, i.e. smooth functions $\chi_i \in \mathscr{C}^\infty(M)$ with $\operatorname{supp}\chi_i \subseteq U_i$ and $\chi_1^2 + \cdots + \chi_N^2 = 1$. Moreover, let $\mathrm{e}_{i,\alpha} \in \Gamma^\infty(E|_{U_i})$ be the local base sections given by the trivialization and denote the corresponding dual base sections by $\mathrm{e}_i^\alpha \in \Gamma^\infty(E^*|_{U_i})$. Here and in the following $\alpha = 1, \ldots, k$. We define now the *global* sections

$$e_{i,\alpha} = \chi_i \mathrm{e}_{i,\alpha} \in \Gamma^\infty(E) \quad \text{and} \quad e^{i,\alpha} = \chi_i \mathrm{e}_i^\alpha \in \Gamma^\infty(E^*).$$

Indeed, since $\operatorname{supp}\chi_i \subseteq U_i$ these sections are extended smoothly from U_i to M by setting them equal to zero outside of U_i. Now let $\phi \in \Gamma^\infty(E)$ be an arbitrary section. Then for all $p \in M$ we have

$$\sum_{i,\alpha} e_{i,\alpha} \cdot e^{i,\alpha}(\phi)\big|_p = \sum_i \chi_i^2(p) \sum_\alpha \mathrm{e}_{i,\alpha}(p)\mathrm{e}_i^\alpha(\phi)\big|_p = \sum_i \chi_i^2(p)\phi\big|_p = \phi\big|_p,$$

where we have used that either $p \in U_i$ so that we can use the locally defined base sections or p is not in U_i, in which case $\chi_i(p) = 0$. Together, this means

$$\phi = \sum_{i,\alpha} e_{i,\alpha} \cdot e^{i,\alpha}(\phi).$$

Since the natural pairing of $e^{i,\alpha} \in \Gamma^\infty(E^*)$ with ϕ is $\mathscr{C}^\infty(M)$-linear we have found a finite dual basis in the sense of Remark 2.3.4, *ii.)*. Thus by Proposition 2.3.3 the $\mathscr{C}^\infty(M)$-module $\Gamma^\infty(E)$ is finitely generated and projective. For the second part we consider $\mathcal{E}_{\mathscr{C}^\infty(M)} = e\mathscr{C}^\infty(M)^N$ with some idempotent $e = e^2 \in \mathrm{M}_N(\mathscr{C}^\infty(M)) = \mathscr{C}^\infty(M, \mathrm{M}_N(\mathbb{C}))$. Then the image $e\mathscr{C}^\infty(M)^N$ is a submodule of the free module $\mathscr{C}^\infty(M)^N = \Gamma^\infty(M \times \mathbb{C}^N)$. Since we can interpret e as a vector bundle endomorphism of the trivial vector bundle $M \times \mathbb{C}^N$, the projective module is the image of a vector bundle homomorphism. Since for $p \in M$ we have $\dim(\operatorname{im} e(p)) = \operatorname{tr}(e(p))$ and since $p \mapsto \operatorname{tr}(e(p))$ is smooth, we see that the dimension of the image is locally constant and hence constant. Thus the image has constant rank which shows that it defines a vector bundle. Clearly, E is unique up to isomorphism and $\Gamma^\infty(E) \cong \mathcal{E}$. The third part is well-known, see e.g. [116, Thm. 2.2.24] for a detailed proof. □

Remark 2.3.10 The above proof of the first part relies on the fact that we can find a finite vector bundle atlas. If the vector bundle atlas is not finite then the proof still gives a projective module, which might not be finitely generated. In particular, by the σ-compactness of second-countable manifolds we always find a countable vector bundle atlas. However, it can be shown that even in the non-compact case, one always can find a *finite* vector bundle atlas. Hence the above proof also applies in the non-compact case, see e.g. [121, Prop. 4.1]. An alternative proof can be found by using the Whitney embedding theorem for the (non-compact) tangent bundle TE of E. Since the vector bundle $E \longrightarrow M$ can be viewed as the vertical bundle of $TE \longrightarrow E$, restricted to the zero section, one can use the orthogonal complement inside the large $\mathbb{R}^N$. As the tangent bundle of $\mathbb{R}^N$ is trivial, the same holds for its restriction to M and hence we have found a complementary bundle to E inside a trivial bundle. From this, the Serre-Swan Theorem easily follows. We also note that an analogous statement holds in the continuous case of topological vector bundles over compact Hausdorff spaces, see Exercise 2.4.20. Finally, we note that there is an elaborate theory of the topological version of K-theory based on vector bundles on topological spaces in general, see e.g. the classical textbook [71].

Corollary 2.3.11 *Passing to sections $E \mapsto \Gamma^\infty(E)$ gives an equivalence of categories*

$$\Gamma^\infty \colon \underline{\mathsf{Vect}}(M) \xrightarrow{\cong} \underline{\mathsf{Proj}}(\mathscr{C}^\infty(M)), \tag{2.3.8}$$

where $\underline{\mathsf{Vect}}(M)$ denotes the category of smooth vector bundles over M with vector bundle morphisms over $\mathrm{id} \colon M \longrightarrow M$ as morphisms. Moreover, it is compatible with Whitney sums of vector bundles and direct sums of projective modules, respectively. Finally, it descends to an isomorphism

$$\mathsf{Vect}(M) \ni [E] \;\mapsto\; [\Gamma^\infty(E)] \in \mathsf{Proj}(C^\infty(M)) \tag{2.3.9}$$

of semi-groups, where $\mathsf{Vect}(M)$ denotes the semi-group of isomorphism classes of vector bundles over M.

In particular, the group $\mathsf{K}_0(\mathscr{C}^\infty(M))$ has the interpretation of classifying smooth vector bundles over M. The Grothendieck group of $\mathsf{Vect}(M)$ is called the (smooth) topological K-theory of M which is denoted by $\mathsf{K}^0(M)$. Thus the Serre-Swan Theorem states that these two groups are isomorphic. In fact, one can show that the restriction to smooth vector bundles is superfluous: the smooth vector bundles and the topological vector bundles over a manifold given the same K-theory.

2.3.3 Hermitian K_0-Theory

After this excursion to the ring-theoretic definition of K-theory, we will now take the *-involution as well as the positivity structures into account, leading to two notions of a "Hermitian" K_0-theory. It turns out that the strong non-degeneracy plays a crucial role.

Proposition 2.3.12 *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Then the following statements are equivalent:*

i.) The inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is strongly non-degenerate and $\mathscr{E}_{\mathscr{A}}$ is finitely generated and projective.

ii.) There exists a finite Hermitian dual basis $e_1, \ldots, e_n, f_1, \ldots, f_n \in \mathscr{E}_{\mathscr{A}}$, i.e. we have for all $x \in \mathscr{E}_{\mathscr{A}}$

$$x = \sum_{\alpha=1}^{n} e_\alpha \cdot \langle f_\alpha, x \rangle_{\mathscr{A}}. \tag{2.3.10}$$

iii.) The inner product is non-degenerate and one has $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$.

Proof: Assume that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is strongly non-degenerate and let $\{e_\alpha, e^\alpha\}_{\alpha=1,\ldots,n}$ be a finite dual basis thanks to Proposition 2.3.3, *iii.)*. Then there are uniquely determined $f_\alpha \in \mathscr{E}_{\mathscr{A}}$ with $e^\alpha = \langle f_\alpha, \cdot \rangle_{\mathscr{A}}$ and hence (2.3.10) follows. Conversely, assume (2.3.10). Since $e^\alpha \colon x \mapsto \langle f_\alpha, x \rangle$ is right $\mathscr{A}$-linear, (2.3.10) provides a finite dual basis. Thus $\mathscr{E}_{\mathscr{A}}$ is finitely generated and projective. Now let $\chi \in \mathscr{E}^*$ be given. Then for all $x \in \mathscr{E}_{\mathscr{A}}$

$$\chi(x) = \chi\Big(\sum\nolimits_\alpha e_\alpha \cdot \langle f_\alpha, x \rangle_{\mathscr{A}}\Big) = \sum\nolimits_\alpha \chi(e_\alpha) \langle f_\alpha, x \rangle_{\mathscr{A}} = \Big\langle \sum\nolimits_\alpha f_\alpha \cdot \chi(e_\alpha)^*, x \Big\rangle_{\mathscr{A}},$$

showing that the map $y \mapsto \langle y, \cdot \rangle_{\mathscr{A}}$ is surjective. The injectivity is obvious from (2.3.10). Finally, assume $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Since the finite-rank operators are a *-ideal in the unital *-algebra of all adjointable operators for an inner-product module, this is equivalent to $\mathrm{id}_{\mathscr{E}} \in \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Thus there exist finitely many $e_\alpha, f_\alpha \in \mathscr{E}_{\mathscr{A}}$ with $\mathrm{id}_{\mathscr{E}} = \sum_\alpha \Theta_{e_\alpha, f_\alpha}$. But this is precisely (2.3.10) and hence we have a dual Hermitian basis. Conversely, (2.3.10) shows that $\mathrm{id}_{\mathscr{E}} \in \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ and by the first part, the inner product is non-degenerate. □

For a vector bundle with a pseudo-Hermitian fiber metric we obtain a Hermitian dual basis from this proposition:

Example 2.3.13 (Pseudo-Hermitian vector bundles) Let $E \longrightarrow M$ be a complex vector bundle. According to Example 2.2.13 we know that the pseudo-Hermitian fiber metrics on E correspond exactly to the strongly non-degenerate inner products. For an alternative proof of this, one can construct a Hermitian dual basis: Locally this is certainly possible and given by a local frame $e_{i,\alpha}$ and $f_{i,\alpha}$ determined by the local dual frame via $e_i^\alpha(p) = h(p)(f_{i,\alpha}, \cdot)$ which indeed determines a smooth section $f_{i,\alpha} \in \Gamma^\infty(E|_{U_i})$. From here one can continue as in the proof of Theorem 2.3.9 to globalize the section without spoiling the property of a Hermitian dual basis. It turns out that such a Hermitian dual basis is often a very efficient tool for studying vector bundles.

This example also motivates the refined definition of Hermitian K-theory which takes into account the different isometry classes of inner products. We have two versions by either taking into account the additional complete positivity or not:

Definition 2.3.14 (Hermitian K_0-theory) *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) The category of finitely generated projective right $\mathscr{A}$-modules with strongly non-degenerate $\mathscr{A}$-valued inner products as objects and adjointable maps as morphisms is denoted by $\underline{\mathsf{Proj}}^*(\mathscr{A})$.

ii.) The category of finitely generated projective right $\mathscr{A}$-modules with strongly non-degenerate and completely positive $\mathscr{A}$-valued inner products as objects and adjointable maps as morphisms is denoted by $\underline{\mathsf{Proj}}^{\mathrm{str}}(\mathscr{A})$.

iii.) The corresponding semi-groups of isometric isomorphism classes are denoted by $\mathsf{Proj}^*(\mathscr{A})$ and $\mathsf{Proj}^{\mathrm{str}}(\mathscr{A})$, respectively.

iv.) The resulting Grothendieck groups are denoted by $\mathsf{K}_0^*(\mathscr{A})$ and $\mathsf{K}_0^{\mathrm{str}}(\mathscr{A})$. The group $\mathsf{K}_0^{\mathrm{str}}(\mathscr{A})$ is called the Hermitian K_0-theory of $\mathscr{A}$.

By forgetting the additional structures we get functors and on the level of isomorphism classes we get (semi-) group morphisms

$$
\begin{array}{ccc}
\mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) & \longrightarrow & \mathsf{Proj}^*(\mathscr{A}) \\
& \searrow \quad \swarrow & \\
& \mathsf{Proj}(\mathscr{A}) &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathsf{K}_0^{\mathrm{str}}(\mathscr{A}) & \longrightarrow & \mathsf{K}_0^*(\mathscr{A}) \\
& \searrow \quad \swarrow & \\
& \mathsf{K}_0(\mathscr{A}) &
\end{array}
\tag{2.3.11}
$$

which, geometrically speaking, encode how many non-isometric (pseudo-) Hermitian fiber metrics one has on the given vector bundles.

In general, the semi-group morphisms (2.3.11) are neither surjective nor injective. Clearly, if there is a completely positive inner product, then we can pass to a "completely negative" one by inserting a -1 into the definition. This shows that the semi-group morphism $\mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Proj}^*(\mathscr{A})$ is not surjective for somehow trivial reasons. Thus it is more interesting whether or not this is the only freedom we gain when passing from completely positive inner products to general ones. Instead, it could happen that we have possibilities for more complicated "signatures". For the same reason we expect the semi-group morphism $\mathsf{Proj}^*(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A})$ to be non-injective. Now the more interesting question is the injectivity and surjectivity of $\mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A})$.

2.3.4 The Properties (K) and (H)

In order to learn something about the bijectivity of the semi-group morphism $\mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A})$ we have to understand on which finitely generated projective modules there exists a strongly non-degenerate and completely positive $\mathscr{A}$-valued inner product. The complete positivity is rather easy to control since embedding a projective module in some $\mathscr{A}^n$ would give immediately a completely positive inner product by restricting the canonical one. The *strong* non-degeneracy is more difficult to guarantee. Here the following proposition gives a sufficient criterion:

Proposition 2.3.15 *Let $P = P^2 = P^* \in \mathrm{M}_n(\mathscr{A})$ be a projection. Then the restriction of the canonical inner product of $\mathscr{A}^n$ to $P\mathscr{A}^n$ is strongly non-degenerate.*

PROOF: Let $e_1, \ldots, e_n \in \mathscr{A}^n$ be the canonical basis and $x = Px \in P\mathscr{A}^n$. Using $P = P^*$ we compute

$$x = \sum\nolimits_i e_i \langle e_i, x\rangle = \sum\nolimits_i Pe_i \langle e_i, Px\rangle = \sum\nolimits_i Pe_i \langle Pe_i, x\rangle.$$

Since $Pe_i \in P\mathscr{A}^n$ for all i we have found a Hermitian dual basis $\{Pe_i, Pe_i\}_{i=1,\ldots,n}$ for $P\mathscr{A}^n$. Thus, by Proposition 2.3.12, the restriction of the canonical inner product is strongly non-degenerate. □

If e is only an idempotent element then the above proof does not apply since $e^*(e_i)$ may not be in $e\mathscr{A}^n$ any more. This raises the question whether or not for a given finitely generated projective module $\mathscr{E}_{\mathscr{A}}$ we can find a *projection* $P \in \mathrm{M}_n(\mathscr{A})$ such that $\mathscr{E}_{\mathscr{A}} \cong P\mathscr{A}^n$. As ultimately we are interested in an arbitrary such projective module we have to find for any idempotent element $e \in \mathrm{M}_n(\mathscr{A})$ an equivalent projection. In general, this needs not to be the case. However, there is a nice sufficient criterion due to Kaplansky [70, Thm. 26]:

Theorem 2.3.16 (Kaplansky) *Let $\mathscr{A}$ be a unital *-algebra over C. Assume that for all $n \in \mathbb{N}$ and for all $A \in \mathrm{M}_n(\mathscr{A})$ the elements $\mathbb{1} + A^*A \in \mathrm{M}_n(\mathscr{A})$ are invertible. Then every idempotent element in $\mathrm{M}_\infty(\mathscr{A})$ is equivalent to a projection.*

PROOF: Let $e = e^2 \in \mathrm{M}_n(\mathscr{A})$ be given and define

$$z = \mathbb{1} + (e - e^*)(e^* - e) = z^*.$$

By assumption, z is invertible with inverse $z^{-1} \in \mathrm{M}_n(\mathscr{A})$, which is again Hermitian. We have

$$\begin{aligned} ez &= e + e(e - e^*)(e^* - e) \\ &= e + (e - ee^*)(e^* - e) \\ &= e + ee^* - ee^*e^* - e + ee^*e \\ &= ee^*e, \end{aligned}$$

since $e^2 = e$ and $(e^*)^2 = e^*$. Analogously one computes $ze = ee^*e = ez$ and hence z and e commute. But this implies that also z and e^* commute and z^{-1} commutes with e and e^*, too. We define $p = ee^*z^{-1} = z^{-1}ee^* = ez^{-1}e^*$. Then

$$p^2 = ee^*z^{-1}ee^*z^{-1} = z^{-1}ee^*ee^*z^{-1} = z^{-1}zee^*z^{-1} = ee^*z^{-1} = p.$$

Clearly, $p^* = (ez^{-1}e^*)^* = ez^{-1}e^* = p$ showing that p is a projection. Finally, we have

$$pe = z^{-1}ee^*e = z^{-1}ez = e \quad \text{and} \quad ep = eee^*z^{-1} = ee^*z^{-1} = p,$$

which means that e and p are equivalent. □

The assumption in this theorem will be useful at many other places. Hence we state the following definition [29, Sect. 7A].

Definition 2.3.17 (Property (K)) *A unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ satisfies property* **(K)** *if for all $n \in \mathbb{N}$ and $A \in \mathrm{M}_n(\mathscr{A})$ the matrix $\mathbb{1} + A^*A \in \mathrm{M}_n(\mathscr{A})$ is invertible.*

We already know several examples of *-algebras with the property **(K)**:

Example 2.3.18 (The property (K))

i.) Every unital C^*-algebra $\mathscr{A}$ fulfills **(K)** since first $\mathrm{M}_n(\mathscr{A})$ is again a C^*-algebra and second, by the spectral calculus, the spectrum of $\mathbb{1} + A^*A$ is in $[1, \infty)$. Then such a matrix is invertible by spectral calculus.

ii.) For any manifold M the functions $\mathscr{C}^\infty(M)$ have the property **(K)**. Here one observes that the pointwise inverse of the matrix-valued function $\mathbb{1} + A^*A \in \mathscr{C}^\infty(M, \mathrm{M}_n(\mathbb{C}))$ is again a *smooth* function on M.

iii.) If $\mathscr{A}$ has the property **(K)** then also $\mathrm{M}_k(\mathscr{A})$ for all $k \in \mathbb{N}$ since $\mathrm{M}_n(\mathrm{M}_k(\mathscr{A})) \cong \mathrm{M}_{nk}(\mathscr{A})$.

iv.) The *-algebra $\mathsf{C} = \mathbb{Z}(\mathrm{i})$ does not satisfy **(K)** since $2 = 1 + 1^*1$ is not invertible in $\mathbb{Z}(\mathrm{i})$.

Corollary 2.3.19 *If $\mathscr{A}$ has the property* **(K)** *then the canonical semi-group morphism*

$$\mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A}) \tag{2.3.12}$$

is surjective.

Another simple application of the property **(K)** is given in the following proposition:

Proposition 2.3.20 *Let $\mathscr{A}$ be a unital *-algebra with the property* **(K)**. *Then for all $a_1, \ldots, a_N \in \mathscr{A}$ the algebraically positive element $\mathbb{1} + \sum_\alpha a_\alpha^* a_\alpha$ is invertible.*

Proof: We consider the $N \times N$-matrix $A \in \mathrm{M}_N(\mathscr{A})$ defined by

$$A = \begin{pmatrix} a_1 & & \\ \vdots & 0 & \\ a_N & & \end{pmatrix}$$

Then by **(K)** the matrix $\mathbb{1}_{N\times N} + A^*A$ is invertible. But this matrix is diagonal and the entry in the upper left corner is just $\mathbb{1}_{\mathscr{A}} + \sum_\alpha a_\alpha^* a_\alpha$. Thus this element is invertible, too. ☐

If there is a strongly non-degenerate inner product on $\mathscr{E}_{\mathscr{A}}$ we still have to answer the question how many non-isometric ones can be found. By Proposition 2.2.14 the strongly non-degenerate inner products are parametrized by Hermitian, invertible elements $H \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Isometric inner products are obtained for $H = U^*U$ with invertible U. In order to investigate this question, the following properties will turn out useful. First we define an *orthogonal partition of unity* in $\mathrm{M}_n(\mathscr{A})$ to be a finite collection of projections $P_\alpha = P_\alpha^2 = P_\alpha^* \in \mathrm{M}_n(\mathscr{A})$ with the property that

$$P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha \quad \text{and} \quad \sum\nolimits_\alpha P_\alpha = \mathbb{1}. \tag{2.3.13}$$

Using this we define the following properties of a unital *-algebra $\mathscr{A}$.

Definition 2.3.21 (Property (H)) *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then we define the following properties of $\mathscr{A}$:*

(H) *Let $n \in \mathbb{N}$ and let $H \in \mathrm{M}_n(\mathscr{A})^+$ be invertible. Furthermore, let $\{P_\alpha\}$ be an orthogonal partition of unity with $[P_\alpha, H] = 0$. Then there exists an invertible $U \in \mathrm{M}_n(\mathscr{A})$ with $H = U^*U$ and $[P_\alpha, U] = 0$.*

($\mathbf{H}^+$) *Let $n \in \mathbb{N}$ and let $H \in \mathrm{M}_n(\mathscr{A})^+$ be invertible. Then there exists an invertible $U \in \mathrm{M}_n(\mathscr{A})$ with $H = U^*U$ and $[P, U] = 0$ for all those projections $P \in \mathrm{M}_n(\mathscr{A})$ with $[P, H] = 0$.*

($\mathbf{H}^-$) *Let $n \in \mathbb{N}$, let $H \in \mathrm{M}_n(\mathscr{A})^+$ be invertible, and let $P \in \mathrm{M}_n(\mathscr{A})$ be a projection with $[P, H] = 0$. Then there exists an invertible $U \in \mathrm{M}_n(\mathscr{A})$ with $H = U^*U$ and $[P, U] = 0$.*

For a unital *-algebra $\mathscr{A}$ we obviously have the implications

$$(\mathbf{H}^+) \implies (\mathbf{H}) \implies (\mathbf{H}^-). \tag{2.3.14}$$

The property **($\mathbf{H}^-$)** can be seen as a more special case of **(H)** where we allow only for a partition of unity consisting of two projections P and $\mathbb{1} - P$ instead of an arbitrary (finite) number. On the other hand, in the case **($\mathbf{H}^+$)** the invertible element U is universal for all the projections P while in **(H)** it may depend on P.

Again, these properties are fulfilled by our primary two classes of examples:

Example 2.3.22 (The property (H))

i.) Every unital C^*-algebra fulfills property **($\mathbf{H}^+$)**. This is an immediate consequence of the spectral calculus. In this case, we can choose U to be the unique positive square root $\sqrt{H}$ of H which clearly commutes with all other elements in $\mathrm{M}_n(\mathscr{A})$ which commute with H.

ii.) Slightly more interesting is the algebra $\mathscr{C}^\infty(M)$ which also fulfills **($\mathbf{H}^+$)**. Indeed, if an *invertible* matrix-valued function H on M is given, such that H is positive, then we can define U to be the unique positive square root $\sqrt{H}$ point by point, which is again *smooth* as H is invertible. Note that in general, $\sqrt{H}$ would be continuous only. Also here $[\sqrt{H}, A] = 0$ for all matrix-valued functions A with $[H, A] = 0$.

With the properties **(H)**, **($\mathbf{H}^+$)**, and **($\mathbf{H}^-$)** one mimics certain aspects of the spectral calculus as available for C^*-algebras. However, only those aspects are required which are necessary for getting unique inner products up to isometries. Indeed, the first consequence of **($\mathbf{H}^-$)** is the following result:

Proposition 2.3.23 *Let $\mathscr{A}$ be a unital *-algebra with property* **($\mathbf{H}^-$)**. *If $P = P^2 = P^* \in \mathrm{M}_n(\mathscr{A})$ is a projection then every strongly non-degenerate, completely positive inner product on $P\mathscr{A}^n$ is isometric to the canonical inner product.*

Proof: Let $\langle \cdot\,, \cdot \rangle'$ be another such inner product on $P\mathscr{A}^n$. On $\mathscr{A}^n$ we consider the direct sum decomposition

$$\mathscr{A}^n = P\mathscr{A}^n \oplus (\mathbb{1} - P)\mathscr{A}^n, \tag{$*$}$$

on which we can define a new inner product $h(\cdot\,, \cdot)$ as follows. On the $P\mathscr{A}^n$-part we use $\langle \cdot\,, \cdot \rangle'$ and on the $(\mathbb{1} - P)\mathscr{A}^n$-part we use the canonical inner product $\langle \cdot\,, \cdot \rangle$. From Lemma 2.2.12 and Lemma 2.1.16 we conclude that h is strongly non-degenerate and completely positive, too. Thus there exists a unique invertible matrix $H \in \mathrm{M}_n(\mathscr{A})$ with

$$h(\phi, \psi) = \langle \phi, H\psi \rangle$$

for all $\phi, \psi \in \mathscr{A}^n$ by Proposition 2.2.14. By Remark 2.2.15, we conclude that H is a positive element $H \in \mathrm{M}_n(\mathscr{A})^+$. Since the decomposition $(*)$ is orthogonal with respect to h by construction, we conclude $[P, H] = 0$. Thus by **($\mathbf{H}^-$)** we obtain an invertible $U \in \mathrm{M}_n(\mathscr{A})$ with $H = U^*U$ and $[P, U] = 0$. It follows that $U\big|_{P\mathscr{A}^n} : P\mathscr{A}^n \longrightarrow P\mathscr{A}^n$ yields an isometry between $\langle \cdot\,, \cdot \rangle'$ and $\langle \cdot\,, \cdot \rangle$. Note that $[P, U] = 0$ is crucial for this argument. ☐

Corollary 2.3.24 *Let $\mathscr{A}$ be a unital* **-algebra with the properties* **(K)** *and* **(H⁻)***. Then*

$$\mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A}) \quad \textit{and} \quad \mathsf{K}_0^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{K}_0(\mathscr{A}) \tag{2.3.15}$$

are bijective.

Corollary 2.3.25 *Let $E \longrightarrow M$ be a complex vector bundle. Then there exists a Hermitian fiber metric on E and any two such fiber metrics are isometric.*

PROOF: There are of course more geometric proofs of this well-known fact but we take the opportunity to use the algebraic techniques developed so far. First we have $\Gamma^\infty(E) \cong e\mathscr{C}^\infty(M)^n$ by the Serre-Swan Theorem 2.3.9 with some idempotent $e \in \mathscr{C}^\infty(M, \mathrm{M}_n(\mathbb{C}))$. By Example 2.3.18, *ii.)*, we can assume that $e = P$ is a projection. Then Example 2.2.9 together with Proposition 2.3.15 shows the existence. With Example 2.3.22, *ii.)*, and Proposition 2.3.23 we conclude the uniqueness up to isometry. ☐

To conclude this section we give an example of a *-algebra which satisfies **(K)** but not **(H⁻)**:

Example 2.3.26 We consider the ordered ring $\mathbb{Q}$ and hence $\mathsf{C} = \mathbb{Q}(\mathrm{i})$ are the rational complex numbers. Clearly, C satisfies **(K)**. Indeed, for a matrix of the form $\mathbb{1} + A^*A$ with $A \in \mathrm{M}_n(\mathbb{Q}(\mathrm{i}))$ we know by spectral calculus that it is invertible in $\mathrm{M}_n(\mathbb{C})$. But the components of the inverse are obtained from rational combination of the components of A and hence the inverse is in $\mathrm{M}_n(\mathbb{Q}(\mathrm{i}))$. For **(H⁻)** we consider the canonical inner product $\langle \cdot\,, \cdot \rangle$ on $\mathbb{Q}(\mathrm{i})$ and

$$\langle z, w \rangle' = \langle z, 3w \rangle = 3\overline{z}w, \tag{2.3.16}$$

which is again completely positive and strongly non-degenerate as $3 = \overline{3} > 0$ is invertible and positive. However, $\langle \cdot\,, \cdot \rangle'$ is not isometric to $\langle \cdot\,, \cdot \rangle$. Indeed, assume that $3 = \overline{u}u$ for some $u = a + \mathrm{i}b \in \mathbb{Q}(\mathrm{i})$. Then we can write $a = \frac{r}{n}$ and $b = \frac{s}{n}$ with $r, s, n \in \mathbb{Z} \setminus \{0\}$ not all even. The equation we have to solve is then $3n^2 = r^2 + s^2$. Taking this equation modulo 4 gives a contradiction.

2.4 Exercises

Exercise 2.4.1 (Hilbert modules) Let $\mathfrak{A}$ be a C^*-algebra and $\mathscr{H}_{\mathfrak{A}}$ a pre-Hilbert right $\mathfrak{A}$-module.

i.) Show that
$$\|\phi\|_{\mathscr{H}_{\mathfrak{A}}} = \sqrt{\|\langle \phi, \phi \rangle_{\mathfrak{A}}\|_{\mathfrak{A}}} \tag{2.4.1}$$
defines a norm on $\mathscr{H}_{\mathfrak{A}}$, where $\|\cdot\|_{\mathfrak{A}}$ is the C^*-norm of $\mathfrak{A}$.

ii.) Show that the completion $\mathfrak{H}_{\mathfrak{A}} = \widehat{\mathscr{H}_{\mathfrak{A}}}$ of $\mathscr{H}_{\mathfrak{A}}$ is still a pre-Hilbert module: the module structure as well as the inner product extends canonically by continuity.
Hint: Show first that $\|\phi \cdot a\|_{\mathscr{H}_{\mathfrak{A}}} \le \|\phi\|_{\mathscr{H}_{\mathfrak{A}}} \|a\|_{\mathfrak{A}}$ for all $\phi \in \mathscr{H}_{\mathfrak{A}}$ and $a \in \mathfrak{A}$. Find a similar estimate for $\langle \phi, \psi \rangle_{\mathscr{H}_{\mathfrak{A}}}$. Why is the extension of the inner product still (completely) positive?

A complete pre-Hilbert module over a C^*-algebra is also called a *Hilbert C^*-module* or just Hilbert module.

Exercise 2.4.2 (Adjointable maps are continuous) Consider a C^*-algebra $\mathfrak{A}$ and Hilbert modules $\mathfrak{H}_{\mathfrak{A}}$ and $\mathfrak{H}'_{\mathfrak{A}}$ over $\mathfrak{A}$, see Exercise 2.4.1. Show that an adjointable map $B\colon \mathfrak{H}_{\mathfrak{A}} \longrightarrow \mathfrak{H}'_{\mathfrak{A}}$ is continuous with respect to the canonical norm topology of the Hilbert modules.
Hint: Use the closed graph theorem.

More on the rich and fascinating theory of C^*-Hilbert modules can be found e.g. in the textbooks [79, 87, 95].

Exercise 2.4.3 (Complex conjugate module) Provide the detailed proof of Proposition 2.1.2.

Exercise 2.4.4 (Degeneracy space) Prove Proposition 2.1.3.

Hint: Define the inner product on the quotient by means of representatives and the inner product on $\mathscr{E}_{\mathscr{A}}$. Show that this is well-defined and inherits all the necessary properties.

Exercise 2.4.5 (Non-degenerate and idempotent matrices) Let $\mathscr{A}$ be a non-degenerate and idempotent *-algebra. Show that for all $n \in \mathbb{N}$ the *-algebra $\mathrm{M}_n(\mathscr{A})$ is again non-degenerate and idempotent.

Hint: Show first $\Theta_{x,y}\Theta_{z,w} = \Theta_{x \cdot \langle y,z \rangle, w}$ for all $x, y, z, w \in \mathscr{A}^n$.

Exercise 2.4.6 (Direct orthogonal sum) Let $\mathscr{A}$ be a *-algebra.

i.) Show that the direct orthogonal sum of right $\mathscr{A}$-modules with positive (n-positive, completely positive) $\mathscr{A}$-valued inner products has again a positive (n-positive, completely positive) $\mathscr{A}$-valued inner product.

ii.) Show that the restriction of a positive (n-positive, completely positive) $\mathscr{A}$-valued inner product to a submodule stays positive (n-positive, completely positive).

iii.) Consider the direct orthogonal sum $\mathscr{E}_{\mathscr{A}} = \oplus_{i \in I} \mathscr{E}_{\mathscr{A}}^{(i)}$ of inner product right $\mathscr{A}$-modules. Show that the projections
$$P_i \colon \mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{E}_{\mathscr{A}}^{(i)} \tag{2.4.2}$$
are right $\mathscr{A}$-linear and adjointable with $P_i = P_i^*$.

iv.) Conversely, suppose that on an inner product module $\mathscr{E}_{\mathscr{A}}$ one has orthogonal projections $\{P_i\}_{i \in I}$ with the property that for $x \in \mathscr{E}_{\mathscr{A}}$ only finitely many $P_i x$ are different from zero and
$$x = \sum_{i \in I} P_i x \tag{2.4.3}$$
for all $x \in \mathscr{E}_{\mathscr{A}}$. Show that the images $\mathscr{E}_{\mathscr{A}}^{(i)} = \operatorname{im}(P_i) \subseteq \mathscr{E}_{\mathscr{A}}$ of the projections P_i are right $\mathscr{A}$-submodules such that the inner product of $\mathscr{E}_{\mathscr{A}}$ restricts to non-degenerate inner products on each $\mathscr{E}_{\mathscr{A}}^{(i)}$. Conclude that $\mathscr{E}_{\mathscr{A}}$ is the direct orthogonal sum of the $\mathscr{E}_{\mathscr{A}}^{(i)}$.

v.) Formulate and prove the universal property of the direct orthogonal sum of inner-product modules analogously to the universal property of the direct sum of vector spaces.

vi.) Show that all the above results stay valid after the obvious modifications if one considers inner-product $(\mathscr{B}, \mathscr{A})$-bimodules instead of right $\mathscr{A}$-modules alone.

Exercise 2.4.7 (Direct sum of positive definite inner products) Consider again the unital *-algebra $\mathscr{A} = \mathbb{Z}_2$ over $\mathbb{Z}(\mathrm{i})$ as in Exercise 1.4.16. Show that the canonical inner product on $\mathscr{A}^2$ is non-degenerate, completely positive, but not positive definite. Conclude that the direct sum of two positive definite inner products needs not to be positive definite anymore.

Exercise 2.4.8 (Degenerate submodules of pre-Hilbert modules) Consider the Grassmann algebra $\mathscr{A} = \Lambda^\bullet(\mathsf{C})$ in one dimension: it is the free C-module with basis 1 and x where the only non-trivial relation is $x^2 = 0$ and $x^* = x$. Endow $\mathscr{A}$ with the canonical positive inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$.

i.) Show that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is non-degenerate but not positive definite.

ii.) Show that $\operatorname{span}_{\mathsf{C}}\{x\} \subseteq \mathscr{A}$ is a submodule and the restriction of $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ to this submodule is degenerate.

Exercise 2.4.9 (Complex conjugate of completely positive inner products) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with a completely positive inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Show that on the complex conjugate left $\mathscr{A}$-module ${}_{\mathscr{A}}\overline{\mathscr{E}}$ the induced inner product as in (2.1.6) is again completely positive.

Exercise 2.4.10 (Complex conjugation is functorial) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\mathscr{E}_{\mathscr{A}}$, $\mathscr{E}'_{\mathscr{A}}$, and $\mathscr{E}''_{\mathscr{A}}$ be inner-product right $\mathscr{A}$-modules. Consider the corresponding inner-product left $\mathscr{A}$-modules ${}_{\mathscr{A}}\overline{\mathscr{E}}$, ${}_{\mathscr{A}}\overline{\mathscr{E}}'$, and ${}_{\mathscr{A}}\overline{\mathscr{E}}''$.

i.) Let $T\colon \mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{E}'_{\mathscr{A}}$ be an adjointable morphism. Show that

$$\overline{T}\colon {}_{\mathscr{A}}\overline{\mathscr{E}} \ni \overline{x} \mapsto \overline{T(x)} \in {}_{\mathscr{A}}\overline{\mathscr{E}}' \tag{2.4.4}$$

is an adjointable morphism again. Prove that $T \mapsto \overline{T}$ gives a C-antilinear map $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}}) \longrightarrow \mathfrak{B}_{\mathscr{A}}({}_{\mathscr{A}}\overline{\mathscr{E}}, {}_{\mathscr{A}}\overline{\mathscr{E}}')$.

ii.) Show that the analogous results hold for the complex conjugation of morphisms between inner-product left $\mathscr{A}$-modules.

iii.) Show that $\overline{\overline{T}} = T$ and $\overline{T^*} = (\overline{T})^*$.

iv.) Show that for a further adjointable morphism $S\colon \mathscr{E}'_{\mathscr{A}} \longrightarrow \mathscr{E}''_{\mathscr{A}}$ one has $\overline{S \circ T} = \overline{S} \circ \overline{T}$.

The conclusion is that the complex conjugation of inner-product modules is functorial. There are analogous statements and variants of this e.g. for inner-product bimodules, for pre-Hilbert modules etc.

Exercise 2.4.11 (Idempotents and matrices) Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $e \in \mathrm{M}_n(\mathscr{A})$ be an idempotent. Endow $\mathscr{A}^n$ with the canonical inner product.

i.) Show that for all $n \in \mathbb{N}$ one has $\mathrm{End}_{\mathscr{A}}(\mathscr{A}^n) \cong \mathrm{M}_n(\mathscr{A})$. Show also that a C-linear map $A\colon \mathscr{A}^n \longrightarrow \mathscr{A}^n$ is adjointable iff A is right $\mathscr{A}$-linear. Show that in this case the adjoint is the usual matrix adjoint when interpreting A and A^* as matrices in $\mathrm{M}_n(\mathscr{A})$.

ii.) Show that $\mathbb{1} - e$ is again an idempotent with $e(\mathbb{1} - e) = 0 = (\mathbb{1} - e)e$.

iii.) Show that also e^* is an idempotent.

iv.) Show that e induces a direct sum decomposition $\mathscr{A}^n = e\mathscr{A} \oplus (\mathbb{1} - e)\mathscr{A}^n$. Show that with respect to this decomposition one has the following induced decomposition

$$e\mathrm{M}_n(\mathscr{A})e \cong \mathrm{End}_{\mathscr{A}}(e\mathscr{A}^n), \tag{2.4.5}$$
$$(\mathbb{1} - e)\mathrm{M}_n(\mathscr{A})(\mathbb{1} - e) \cong \mathrm{End}_{\mathscr{A}}((\mathbb{1} - e)\mathscr{A}^n), \tag{2.4.6}$$
$$e\mathrm{M}_n(\mathscr{A})(\mathbb{1} - e) \cong \mathrm{Hom}_{\mathscr{A}}((\mathbb{1} - e)\mathscr{A}^n, e\mathscr{A}^n), \tag{2.4.7}$$

and

$$(\mathbb{1} - e)\mathrm{M}_n(\mathscr{A})e \cong \mathrm{Hom}_{\mathscr{A}}(e\mathscr{A}^n, (\mathbb{1} - e)\mathscr{A}^n). \tag{2.4.8}$$

Exercise 2.4.12 (Local Hermitian unit elements) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. A reasonable replacement for the existence of a unit element is sometimes the existence of *local Hermitian unit elements* $\{\mathsf{e}_\alpha\}_{\alpha \in I}$ as in Definition 2.2.5: This notion is borrowed from C^*-algebra theory where one only requires convergence $\mathsf{e}_\alpha a \longrightarrow a$ and $a\mathsf{e}_\alpha \longrightarrow a$ of a net $\{\mathsf{e}_\alpha\}_{\alpha \in I}$ instead of equality as in (2.2.4).

i.) Show that a unital *-algebra has local Hermitian units.

ii.) Find examples of non-unital *-algebras which have local Hermitian units.

Hint: Consider suitable non-compact topological spaces and continuous functions with compact support. Under which conditions on the space X do you find local Hermitian units for $\mathscr{C}_0(X)$?

iii.) Consider the infinite matrices $\mathrm{M}_\infty(\mathsf{C})$ with at most finitely many non-zero entries with its usual *-algebra structure. Show that $\mathrm{M}_\infty(\mathsf{C})$ has local Hermitian units.

iv.) More generally, suppose $\mathscr{A}$ has local Hermitian units. Show that for all $n \in \mathbb{N}$ the matrices $\mathrm{M}_n(\mathscr{A})$ have local Hermitian units. Do the infinite matrices $\mathrm{M}_\infty(\mathscr{A})$ with finitely many non-zero entries also have local Hermitian units?

v.) Discuss whether the additional requirement $\mathsf{e}_\alpha^2 = \mathsf{e}_\alpha$ for all $\alpha \in I$ would be an achievable modification in the above examples.

vi.) Suppose that $\mathscr{A}$ has local Hermitian units. Show that for every positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ one has $\omega(a^*) = \overline{\omega(a)}$.

Exercise 2.4.13 (Sufficiently many positive functionals) Let $\mathscr{A}$ be a *-algebra with sufficiently many positive functionals and local Hermitian unit elements $\{\mathsf{e}_\alpha\}_{\alpha\in I}$. Show that in this case the statement of Corollary 1.2.10 is still valid.

Hint: Use the vectors $\psi_{\mathsf{e}_\alpha} \in \mathcal{H}_\omega$ of a GNS representation π_ω for some positive functional ω.

Exercise 2.4.14 (Morphisms between projective modules) Let $\mathscr{E}_{\mathscr{A}}$ and $\mathscr{E}'_{\mathscr{A}}$ be finitely generated projective right modules over $\mathscr{A}$. Choose idempotents $e \in \mathrm{M}_n(\mathscr{A})$ and $f \in \mathrm{M}_m(\mathscr{A})$ such that $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n$ and $\mathscr{E}'_{\mathscr{A}} \cong f\mathscr{A}^m$.

i.) Show that the right $\mathscr{A}$-linear maps from $\mathscr{E}_{\mathscr{A}}$ to $\mathscr{E}'_{\mathscr{A}}$ can be identified with $f\mathrm{M}_{m\times n}(\mathscr{A})e \subseteq \mathrm{M}_{m\times n}(\mathscr{A})$.

ii.) If $\mathscr{E}''_{\mathscr{A}} \cong g\mathscr{A}^k$ with an idempotent $g \in \mathrm{M}_k(\mathscr{A})$ is another finitely generated projective right module over $\mathscr{A}$, how can one encode the composition of module morphisms $\mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{E}'_{\mathscr{A}}$ and $\mathscr{E}'_{\mathscr{A}} \longrightarrow \mathscr{E}''_{\mathscr{A}}$?

Exercise 2.4.15 (Projective modules and their duals) Let $\mathscr{A}$ be a unital ring.

i.) Assume that $\mathscr{E}_{\mathscr{A}}$ is a finitely generated and projective right $\mathscr{A}$-module. Show that in this case the dual left $\mathscr{A}$-module ${}_{\mathscr{A}}\mathscr{E}^*$ is a finitely generated and projective left $\mathscr{A}$-module.

Hint: Use that the notion of a dual basis is symmetric in $\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{E}^*$.

ii.) Let again $\mathscr{E}_{\mathscr{A}}$ be a finitely generated and projective right $\mathscr{A}$-module and choose an idempotent $e \in \mathrm{M}_n(\mathscr{A})$ with $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n$. Find a description of the dual module ${}_{\mathscr{A}}\mathscr{E}^*$ using e.

Hint: Exercise 2.4.14.

iii.) Give an example of a projective right $\mathscr{A}$-module and a dual basis $\{\mathsf{e}_\lambda, \mathsf{e}^\lambda\}_{\lambda\in\Lambda}$ for $\mathscr{E}_{\mathscr{A}}$ such that the elements $\mathsf{e}_\lambda \in {}_{\mathscr{A}}\mathscr{E}^*$ do *not* span ${}_{\mathscr{A}}\mathscr{E}^*$.

Exercise 2.4.16 (Equivalence of idempotents) Let $\mathscr{A}$ be a unital ring. Define two idempotents $e \in \mathrm{M}_n(\mathscr{A})$ and $f \in \mathrm{M}_m(\mathscr{A})$ to be *equivalent* if there exist (rectangular) matrices u and v with $e = uv$ and $f = vu$. Show that this defines indeed an equivalence relation. Find a reasonable adaption of this for a unital *-algebra and projections instead of general idempotents.

Exercise 2.4.17 (Finitely generated projective modules) Assume that $\mathscr{E}_{\mathscr{A}}$ is a projective right $\mathscr{A}$-module over a unital ring $\mathscr{A}$ which in addition is finitely generated with generators $e_1, \ldots, e_n \in \mathscr{E}_{\mathscr{A}}$. Show that there exists elements $e^1, \ldots, e^n$ in the dual module such that together with the $e_1, \ldots, e_n$ one has a finite dual basis.

Hint: Start with an arbitrary dual basis $\{f_\lambda, f^\lambda\}_{\lambda ni\Lambda}$ and express the elements f_λ by right $\mathscr{A}$-linear combinations of the generators $e_1, \ldots, e_n$.

Exercise 2.4.18 (The Grothendieck group) Consider the category AbSemiGroup of abelian semi-groups with the usual morphisms of semi-groups. For a semi-group S one considers on $S \times S$ the relation $\sim$ defined by

$$(s,t) \sim (s',t') \quad \text{if there exists a} \quad u \in S \quad \text{with} \quad s + t' + u = s' + t + u. \tag{2.4.9}$$

i.) Show that this defines an equivalence relation.

ii.) Suppose that S has the *cancellation property*, i.e. if $s + u = t + u$ holds then $s = t$. Show that in this case one can simplify the above construction and omit the usage of u in (2.4.9) to obtain the same relation.

iii.) Show that the semi-group addition $+$ passes to the quotient $\mathsf{G}(S) = (S \times S)\big/\sim$ and yields a group structure. The group $\mathsf{G}(S)$ is called the *Grothendieck group* of S.

iv.) Show that $S \ni s \mapsto [(s, 0)] \in \mathsf{G}(S)$ is a monoid morphism.

v.) Show that for a semi-group S with the cancellation property the canonical monoid morphism $S \longrightarrow \mathsf{G}(S)$ is injective.

vi.) Show that $S \mapsto \mathsf{G}(S)$ is functorial by specifying explicitly how semi-group morphisms pass to group morphisms between the corresponding Grothendieck groups. This yields a functor

$$\mathsf{G}\colon \mathsf{AbSemiGroup} \longrightarrow \mathsf{Ab}. \tag{2.4.10}$$

Exercise 2.4.19 (K_0 for a field) Let $\Bbbk$ be a field. Compute the semi-group $\mathsf{Proj}(\Bbbk)$ and the corresponding K_0-group $\mathsf{K}_0(\Bbbk)$.

Hint: First show that every finitely generated projective module over $\Bbbk$ is actually a finite-dimensional vector space.

Exercise 2.4.20 (Serre-Swan Theorem in the continuous case) Adapt the proof of the Serre-Swan Theorem 2.3.9 for topological vector bundles over a connected compact Hausdorff space X.

Exercise 2.4.21 (Property (K)) Let R be an ordered ring and $\mathsf{C} = \mathsf{R}(\mathrm{i})$ as usual. Show that $\mathrm{M}_n(\mathsf{C})$ satisfies property **(K)** whenever R is an ordered field. Does the same statement hold also for $\mathsf{R} = \mathbb{Z}$?

Exercise 2.4.22 (Property (H)) Let R be a real closed field, see e.g. [66, ????], with its canonical ordering. Then we know that $\mathsf{C} = \mathsf{R}(\mathrm{i})$ is algebraically closed. Show that $\mathrm{M}_n(\mathsf{C})$ satisfies property $\mathbf{(H^+)}$ for all $n \in \mathbb{N}$.

Exercise 2.4.23 (A Banach *-algebra without property (K)) Show that the Banach *-algebra $\mathscr{A}$ from Exercise 1.4.20 does not satisfy **(K)**.

Chapter 3

Tensor Products

In this chapter we describe various tensor product constructions for inner-product and pre-Hilbert modules. The first construction of internal tensor products builds on the tensor product of bimodules over the algebra in the middle. Here we follow Rieffel to introduce a new inner product on this tensor product once we have inner products on the two factors. A step of major importance will be to show that complete positivity is preserved under this internal tensor product. The second construction extends the external tensor product (over C) of *-algebras to bimodules and their inner products. This construction will provide us many interesting examples of inner-product bimodules.

3.1 Internal Tensor Products

Following Rieffel's original construction of a tensor product of Hilbert modules over C^*-algebras, see [97, 98], we can cast his approach now into our algebraic framework, following essentially [26, 29]. As before, we consider a ring $\mathsf{C} = \mathsf{R}(\mathrm{i})$ as scalars where R is an ordered ring and $\mathrm{i}^2 = -1$.

3.1.1 Construction of the Internal Tensor Product

We start with a $(\mathscr{B}, \mathscr{A})$-bimodule with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ compatible with the left $\mathscr{B}$-module structure. Moreover, let $\mathscr{F}_{\mathscr{B}}$ be a right $\mathscr{B}$-module with $\mathscr{B}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{F}}_{\mathscr{B}}$. Then the tensor product $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a right $\mathscr{A}$-module in the usual way. For elementary tensors we define

$$\big\langle y \otimes x, y' \otimes x' \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} = \big\langle x, \langle y, y' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x' \big\rangle^{\mathscr{E}}_{\mathscr{A}}. \tag{3.1.1}$$

Lemma 3.1.1 *The sesquilinear extension of* (3.1.1) *yields a well-defined $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}$ on $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$.*

Proof: We have to show that (3.1.1) is well-defined on the $\mathscr{B}$-tensor product. Let $x, x' \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and $y, y' \in \mathscr{F}_{\mathscr{B}}$ as well as $b \in \mathscr{B}$. Then we have

$$\begin{aligned}
\big\langle (y \cdot b) \otimes x, y' \otimes x' \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} &= \big\langle x, \langle y \cdot b, y' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle x, \big(b^* \langle y, y' \rangle^{\mathscr{F}}_{\mathscr{B}}\big) \cdot x' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle x, b^* \cdot \big(\langle y, y' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x'\big) \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle b \cdot x, \langle y, y' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle y \otimes (b \cdot x), y' \otimes x' \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}},
\end{aligned}$$

and analogously $\langle y \otimes x, (y' \cdot b) \otimes x' \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}} = \langle y \otimes x, y' \otimes (b \cdot x') \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$. Thus the inner product is well-defined over $\otimes_{\mathscr{B}}$. Let us now show the properties of an algebra-valued inner product. First, it is clear (on elementary tensors and by construction for general ones) that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$ is C-antilinear in the first and C-linear in the second argument. Moreover, we have

$$\begin{aligned}
\big\langle y \otimes x, (y' \otimes x') \cdot a \big\rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}} &= \big\langle x, \langle y, y' \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot (x' \cdot a) \big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \big\langle x, \big(\langle y, y' \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x'\big) \cdot a \big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \big\langle x, \langle y, y' \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x' \big\rangle_{\mathscr{A}}^{\mathscr{E}} a \\
&= \big\langle y \otimes x, y' \otimes x' \big\rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}} a
\end{aligned}$$

as well as

$$\begin{aligned}
\Big(\big\langle y \otimes x, y' \otimes x' \big\rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}\Big)^* &= \Big(\big\langle x, \langle y, y' \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x' \big\rangle_{\mathscr{A}}^{\mathscr{E}}\Big)^* \\
&= \big\langle \langle y, y' \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x', x \big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \big\langle x', \big(\langle y, y' \rangle_{\mathscr{B}}^{\mathscr{F}}\big)^* \cdot x \big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \big\langle x', \langle y', y \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x \big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \big\langle y' \otimes x', y \otimes x \big\rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}.
\end{aligned}$$

This implies that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$ is indeed an $\mathscr{A}$-valued inner product on $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. ☐

If one defines (3.1.1) just over the C-tensor product then the above Lemma can be interpreted in such a way that tensors of the form $y \otimes (b \cdot x) - (y \cdot b) \otimes x$ are in the *degeneracy space* of the inner product. Thus the passage to the tensor product over $\mathscr{B}$ can be seen as being part of the passage to the quotient by the degeneracy space as we did that before in Proposition 2.1.3. However, the next example shows that the degeneracy space can be strictly larger:

Example 3.1.2 Let $\mathsf{R} = (\mathbb{R}[x])[y] = \mathbb{R}[x, y]$ be the polynomials in two variables. Then we can endow R with the structure of an ordered ring by viewing it as a subring of $(\mathbb{R}[\![x]\!])[\![y]\!]$, where the latter is endowed with the ordering according to Example 1.1.3, *ii.)*, applied twice. As usual $\mathsf{C} = \mathbb{C}[x, y]$. Now we consider $\mathscr{H} \subseteq \mathsf{C}$ being the ideal generated by x and y, i.e. $\mathscr{H} = x\mathsf{C} + y\mathsf{C}$. Geometrically, this is the vanishing ideal of $(0, 0)$ in the (x, y)-plane. Being an ideal, $\mathscr{H}$ is a module over C and being a submodule it inherits the canonical positive definite inner product. Thus it is a pre-Hilbert space over C. We consider now the tensor product $\mathscr{H} \otimes \mathscr{H}$ with the induced inner product according to (3.1.1). It is well-known that in $\mathscr{H} \otimes \mathscr{H}$ the two elements $x \otimes y$ and $y \otimes x$ are different. However,

$$\langle x \otimes y - y \otimes x, f \otimes g \rangle = xy\overline{f}g - yx\overline{f}g = 0$$

shows that $x \otimes y - y \otimes x$ is in the degeneracy space of (3.1.1). Thus these torsion effects cause (3.1.1) to be *degenerate* even though the inner products on each of the two factors are non-degenerate.

From this example we see that in general it may happen that the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$ is degenerate. Thus we divide by the degeneracy space according to Proposition 2.1.3 to obtain a non-degenerate inner product, i.e. an inner-product module:

Definition 3.1.3 (Internal tensor product) *Let $\mathscr{F}_{\mathscr{B}}$ be a right $\mathscr{B}$-module with $\mathscr{B}$-valued inner product and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule with compatible $\mathscr{A}$-valued inner product. Then we define the internal tensor product of $\mathscr{F}$ and $\mathscr{E}$ by*

$$\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = \mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \big/ (\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})^{\perp}, \tag{3.1.2}$$

endowed with the non-degenerate induced $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$.

As usual, we shall drop the explicit reference to the algebra $\mathscr{B}$ and simply write $\mathscr{F} \mathbin{\widehat{\otimes}} \mathscr{E}$ if the participating algebras are clear from the context. Moreover, we will see more particular situations where automatically $(\mathscr{F} \otimes \mathscr{E})^\perp = \{0\}$ and thus the quotient (3.1.2) is unnecessary as opposed to Example 3.1.2.

Lemma 3.1.4 *Let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ be a $(\mathscr{C}, \mathscr{B})$-bimodule with compatible $\mathscr{B}$-valued inner product and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ as before. Then the canonical left $\mathscr{C}$-module structure on ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is compatible with $\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}$ and on the inner-product $\mathscr{A}$-module ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}}_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ we obtain a *-representation of $\mathscr{C}$.*

Proof: Clearly, it is sufficient to consider elementary tensors $y \otimes x$, $y' \otimes x' \in \mathscr{F} \otimes \mathscr{E}$. We have for $c \in \mathscr{C}$

$$\begin{aligned}
\big\langle c \cdot (y \otimes x), y' \otimes x' \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} &= \big\langle (c \cdot y) \otimes x, y' \otimes x' \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle x, \langle c \cdot y, y' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle x, \langle y, c^* \cdot y' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle y \otimes x, (c^* \cdot y') \otimes x' \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle y \otimes x, c^* \cdot (y' \otimes x') \big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}},
\end{aligned}$$

showing that the left $\mathscr{C}$-module structure is compatible with $\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}$. Using this, Proposition 2.1.27 shows that the $\mathscr{C}$-module structure passes to the quotient and yields a *-representation of $\mathscr{C}$ on $\mathscr{F} \mathbin{\widehat{\otimes}} \mathscr{E}$. □

The next nice property of $\widehat{\otimes}$ is the "associativity". As in the usual case of $\otimes$, the associativity only holds up to a canonical isomorphism. We will be slightly pedantic here as later we will need precisely this canonical isomorphism to formulate a bicategorical approach to representation theory. Moreover, due to the additional quotient procedure needed in $\widehat{\otimes}$, some more care is needed:

Proposition 3.1.5 *Let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be bimodules with algebra-valued inner products which are compatible with the corresponding left actions of $\mathscr{C}$ and $\mathscr{B}$, respectively. Moreover, let $\mathscr{G}_{\mathscr{C}}$ be a right $\mathscr{C}$-module with $\mathscr{C}$-valued inner product.*

i.) The C-linear map

$$\big(\mathscr{G}_{\mathscr{C}} \otimes_{\mathscr{C}} {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}\big) \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \big/ \big((\mathscr{G}_{\mathscr{C}} \otimes_{\mathscr{C}} {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}) \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big)^\perp \longrightarrow \big(\mathscr{G}_{\mathscr{C}} \mathbin{\widehat{\otimes}}_{\mathscr{C}} {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}\big) \mathbin{\widehat{\otimes}}_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, \tag{3.1.3}$$

determined by $[(z \otimes y) \otimes x] \mapsto [[z \otimes y] \otimes x]$, is a well-defined isometric isomorphism of inner product modules over $\mathscr{A}$.

ii.) Analogously, the map

$$\mathscr{G}_{\mathscr{C}} \otimes_{\mathscr{C}} \big({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big) \big/ \big(\mathscr{G}_{\mathscr{C}} \otimes_{\mathscr{C}} \big({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big)\big)^\perp \longrightarrow \mathscr{G}_{\mathscr{C}} \mathbin{\widehat{\otimes}}_{\mathscr{C}} \big({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}}_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big), \tag{3.1.4}$$

determined by $[z \otimes (y \otimes x)] \mapsto [z \otimes [y \otimes x]]$, gives a well-defined isometric isomorphism of inner product modules over $\mathscr{A}$.

iii.) The canonical isomorphism of right $\mathscr{A}$-modules

$$(\mathscr{G} \otimes_{\mathscr{C}} \mathscr{F}) \otimes_{\mathscr{B}} \mathscr{E} \ni (z \otimes y) \otimes x \ \mapsto\ z \otimes (y \otimes x) \in \mathscr{G} \otimes_{\mathscr{C}} (\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E}) \tag{3.1.5}$$

induces a well-defined isometric isomorphism

$$\mathsf{a}\colon \big(\mathscr{G} \mathbin{\widehat{\otimes}}_{\mathscr{C}} \mathscr{F}\big) \mathbin{\widehat{\otimes}}_{\mathscr{B}} \mathscr{E} \longrightarrow \mathscr{G} \mathbin{\widehat{\otimes}}_{\mathscr{C}} \big(\mathscr{F} \mathbin{\widehat{\otimes}}_{\mathscr{B}} \mathscr{E}\big) \tag{3.1.6}$$

of inner product modules over $\mathscr{A}$. In particular, we have for equivalence classes of elementary tensors

$$\mathsf{a}\big([[z \otimes y] \otimes x]\big) = [z \otimes [y \otimes x]]. \tag{3.1.7}$$

Proof: To show the well-definedness of the above maps we have to consider $\phi \in ((\mathscr{G} \otimes \mathscr{F}) \otimes \mathscr{E})^\perp$ and show that ϕ becomes an element in the degeneracy space of $(\mathscr{G} \mathbin{\widehat{\otimes}} \mathscr{F}) \otimes \mathscr{E}$. Denote the image of ϕ in $(\mathscr{G} \mathbin{\widehat{\otimes}} \mathscr{F}) \otimes \mathscr{E}$ by $\tilde{\phi}$. A general vector in $(\mathscr{G} \mathbin{\widehat{\otimes}} \mathscr{F}) \otimes \mathscr{E}$ is a linear combination of vectors of the form $[z \otimes y] \otimes x$. Thus for $\phi = \sum_i (z_i \otimes y_i) \otimes x_i$ we compute

$$
\begin{aligned}
\big\langle \tilde{\phi}, [z \otimes y] \otimes x \big\rangle_{\mathscr{A}}^{(\mathscr{G} \widehat{\otimes} \mathscr{F}) \otimes \mathscr{E}}
&= \sum\nolimits_i \big\langle [z_i \otimes y_i] \otimes x_i, [z \otimes y] \otimes x \big\rangle_{\mathscr{A}}^{(\mathscr{G} \widehat{\otimes} \mathscr{F}) \otimes \mathscr{E}} \\
&= \sum\nolimits_i \Big\langle x_i, \big\langle [z_i \otimes y_i], [z \otimes y] \big\rangle_{\mathscr{B}}^{\mathscr{G} \widehat{\otimes} \mathscr{F}} \cdot x \Big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \sum\nolimits_i \Big\langle x_i, \langle z_i \otimes y_i, z \otimes y \rangle_{\mathscr{B}}^{\mathscr{G} \otimes \mathscr{F}} \cdot x \Big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \sum\nolimits_i \big\langle (z_i \otimes y_i) \otimes x_i, (z \otimes y) \otimes x \big\rangle_{\mathscr{A}}^{(\mathscr{G} \otimes \mathscr{F}) \otimes \mathscr{E}}, \\
&= 0
\end{aligned}
$$

by the assumption that ϕ is orthogonal to all vectors. It follows that $\tilde{\phi} \in ((\mathscr{G} \mathbin{\widehat{\otimes}} \mathscr{F}) \otimes \mathscr{E})^\perp$ and thus (3.1.3) is well-defined. Moreover, this map is clearly right $\mathscr{A}$-linear. The analogous computation shows that (3.1.3) is isometric. Since an isometric map between inner-product modules with *non-degenerate* inner products is injective we conclude that (3.1.3) is injective, too, see Exercise 3.3.1. The surjectivity is clear and thus we have an isometric bijection. This implies that the inverse coincides with the adjoint, showing the first part. The second statement is proved analogously. For the third part we compute that on the level of representatives that (3.1.5) is isometric, since

$$
\begin{aligned}
\big\langle (z \otimes y) \otimes x, (z' \otimes y') \otimes x' \big\rangle_{\mathscr{A}}^{(\mathscr{G} \otimes \mathscr{F}) \otimes \mathscr{E}}
&= \Big\langle x, \big\langle z \otimes y, z' \otimes y' \big\rangle_{\mathscr{B}}^{\mathscr{G} \otimes \mathscr{F}} \cdot x' \Big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \Big\langle x, \big\langle y, \langle z, z' \rangle_{\mathscr{C}}^{\mathscr{G}} \cdot y' \big\rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x' \Big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \Big\langle y \otimes x, \Big(\langle z, z' \rangle_{\mathscr{C}}^{\mathscr{G}} \cdot y' \Big) \otimes x' \Big\rangle_{\mathscr{A}}^{\mathscr{F} \otimes \mathscr{E}} \\
&= \Big\langle y \otimes x, \langle z, z' \rangle_{\mathscr{C}}^{\mathscr{G}} \cdot (y' \otimes x') \Big\rangle_{\mathscr{A}}^{\mathscr{F} \otimes \mathscr{E}} \\
&= \big\langle z \otimes (y \otimes x), z' \otimes (y' \otimes x') \big\rangle_{\mathscr{A}}^{\mathscr{G} \otimes (\mathscr{F} \otimes \mathscr{E})}
\end{aligned}
$$

for all $x, x' \in \mathscr{E}$, $y, y' \in \mathscr{F}$, and $z, z' \in \mathscr{G}$. As (3.1.5) is clearly a right $\mathscr{A}$-linear isomorphism (with the obvious inverse) we obtain also in the quotient an isometric isomorphism

$$
\big((\mathscr{G} \otimes \mathscr{F}) \otimes \mathscr{E}\big) \big/ \big((\mathscr{G} \otimes \mathscr{F}) \otimes \mathscr{E}\big)^\perp \longrightarrow \big(\mathscr{G} \otimes (\mathscr{F} \otimes \mathscr{E})\big) \big/ \big(\mathscr{G} \otimes (\mathscr{F} \otimes \mathscr{E})\big)^\perp.
$$

Together with the first and second part we obtain the isometric isomorphism a which encodes the associativity of $\widehat{\otimes}$. The last equation is clear from the construction. □

We can now discuss the compatibility of the internal tensor product with adjointable (bi-) module morphisms. Here we formulate the following lemma directly for three *-algebras $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ with corresponding bimodules. Analogous statements are also true for the case of only two *-algebras with the first module being only a right module. This case is obtained by setting $\mathscr{C} = \mathsf{C}$.

Lemma 3.1.6 *Let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$, ${}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}}$, ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$, and ${}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ be bimodules with corresponding algebra-valued inner products compatible with the corresponding left-module structures. Moreover, let $S \in \mathfrak{B}_{\mathscr{B}}({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}, {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}})$ and $T \in \mathfrak{B}_{\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}})$ be adjointable bimodule morphisms with adjoints S^* and T^* being bimodule morphisms, too. Then the algebraic tensor product*

$$
S \otimes_{\mathscr{B}} T \colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \tag{3.1.8}
$$

induces an adjointable bimodule morphism

$$S \mathbin{\widehat{\otimes}_{\mathscr{B}}} T \colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}, \tag{3.1.9}$$

whose adjoint is given by $S^* \mathbin{\widehat{\otimes}_{\mathscr{B}}} T^*$. *If* S *and* T *are surjective isometric (not necessarily adjointable) then* $S \mathbin{\widehat{\otimes}_{\mathscr{B}}} T$ *is well-defined and unitary.*

Proof: Let S and T be adjointable bimodule morphisms with bimodule morphisms S^* and T^* as adjoints. Then

$$\begin{aligned}
\big\langle (S \otimes T)(y \otimes x), y' \otimes x' \big\rangle_{\mathscr{A}}^{\mathscr{F}' \otimes \mathscr{E}'} &= \big\langle S(y) \otimes T(x), y' \otimes x' \big\rangle_{\mathscr{A}}^{\mathscr{F}' \otimes \mathscr{E}'} \\
&= \Big\langle T(x), \big\langle S(y), y' \big\rangle_{\mathscr{B}}^{\mathscr{F}'} \cdot x' \Big\rangle_{\mathscr{A}}^{\mathscr{E}'} \\
&= \Big\langle x, T^*\Big(\big\langle y, S^*(y') \big\rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x' \Big) \Big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \Big\langle x, \big\langle y, S^*(y') \big\rangle_{\mathscr{B}}^{\mathscr{F}} \cdot T^*(x') \Big\rangle_{\mathscr{A}}^{\mathscr{E}} \\
&= \big\langle y \otimes x, S^*(y') \otimes T^*(x') \big\rangle_{\mathscr{A}}^{\mathscr{F} \otimes \mathscr{E}} \\
&= \big\langle y \otimes x, (S^* \otimes T^*)(y' \otimes x') \big\rangle_{\mathscr{A}}^{\mathscr{F} \otimes \mathscr{E}},
\end{aligned}$$

since T^* is left $\mathscr{B}$-linear. This shows that $S^* \otimes T^*$ is an adjoint of $S \otimes T$. By an analogous argument as in Proposition 2.1.27, the maps $S \otimes T$ and $S^* \otimes T^*$ preserve the degeneracy space and hence give well-defined adjointable bimodule morphisms in the quotient. This shows the first part. The case of surjective isometric bimodule morphisms (adjointable or not) follows from Exercise 3.3.1. ☐

The above results can now be summarized as follows if we insist on non-degenerate inner products, i.e. *-representations on inner-product modules, from the beginning. In this case the adjointable endomorphisms $\mathfrak{B}_{\mathscr{A}}(\mathscr{H}_{\mathscr{A}})$ form a *-algebra themselves and the adjoints are unique. This leads to the following theorem [29]:

Theorem 3.1.7 (Internal tensor product) *Let* $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{C}$ *be* **-algebras over* C. *Then the internal tensor product* $\widehat{\otimes}$ *yields a covariant functor*

$$\widehat{\otimes}_{\mathscr{B}} \colon {}^*\text{-}\mathsf{mod}_{\mathscr{B}}(\mathscr{C}) \times {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{C}), \tag{3.1.10}$$

where the **-representations are tensored by means of Lemma 3.1.4. The morphisms are tensored using Lemma 3.1.6.*

Proof: We have already seen that the internal tensor product of *-representations gives again a *-representation. Moreover, the internal tensor product of intertwiners yields an intertwiner. Thus it remains to show that the internal tensor product preserves the identity morphisms and the composition of morphisms: for $x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and $y \in {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ we have

$$(S \mathbin{\widehat{\otimes}_{\mathscr{B}}} T)([y \otimes x]) = [S(y) \otimes T(x)]$$

by construction. Since the equivalence classes of elementary tensors also span the quotient ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \widehat{\otimes}_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$, it suffices to check the compatibility with the composition on such equivalence classes of elementary tensors where it is trivial. ☐

Remark 3.1.8 The probably remarkable point is that the additional structure of the inner products allows to absorb torsion effects of the ring-theoretic tensor product as in Example 3.1.2 in a universal way by replacing this tensor product with the internal tensor product $\widehat{\otimes}$. The price is of course the additional quotient procedure needed in $\widehat{\otimes}$.

For strongly non-degenerate *-representations we obtain the following result. Note that the right factor can be arbitrary.

Corollary 3.1.9 *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be *-algebras over C. The internal tensor product restricts to a functor*

$$\widehat{\otimes}_{\mathscr{B}} \colon {}^*\text{-}\mathsf{Mod}_{\mathscr{B}}(\mathscr{C}) \times {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{C}). \tag{3.1.11}$$

Proof: Let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{B}}(\mathscr{C})$ be a strongly non-degenerate *-representation of $\mathscr{C}$ on a inner-product right $\mathscr{B}$-module and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ be arbitrary. Let $y \in \mathscr{F}$ and $x \in \mathscr{E}$. Then we find $c_i \in \mathscr{C}$ and $y_i \in \mathscr{F}$ with $y = c_1 \cdot y_1 + \cdots + c_n \cdot y_n$. Thus

$$\sum\nolimits_i c_i \cdot (y_i \otimes x) = \sum\nolimits_i (c_i \cdot y_i) \otimes x = y \otimes x$$

shows that $\mathscr{C} \cdot (\mathscr{F} \otimes \mathscr{E}) = \mathscr{F} \otimes \mathscr{E}$. From this, $\mathscr{C} \cdot (\mathscr{F} \mathbin{\widehat{\otimes}} \mathscr{E}) = \mathscr{F} \mathbin{\widehat{\otimes}} \mathscr{E}$ follows immediately. ☐

3.1.2 Complete Positivity of the Internal Tensor Product

We can now formulate the main question of this section, namely whether and how the internal tensor product is compatible with our positivity requirements. Let the inner products $\langle \cdot\,, \cdot \rangle^{\mathscr{F}}_{\mathscr{B}}$ and $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ be positive. Then the map

$$b \mapsto \langle x, b \cdot x \rangle^{\mathscr{E}}_{\mathscr{A}} \tag{3.1.12}$$

is clearly a positive map. Indeed, $\langle x, (b^*b) \cdot x \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle b \cdot x, b \cdot x \rangle^{\mathscr{E}}_{\mathscr{A}} \in \mathscr{A}^+$ by the positivity of the inner product. Hence it follows from $\langle y, y \rangle^{\mathscr{F}}_{\mathscr{B}} \in \mathscr{B}^+$ that

$$\langle y \otimes x, y \otimes x \rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}} = \big\langle x, \langle y, y \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x \big\rangle^{\mathscr{E}}_{\mathscr{A}} \in \mathscr{A}^+. \tag{3.1.13}$$

Thus the inner product takes positive values on the elementary tensors $y \otimes x \in \mathscr{F} \otimes \mathscr{E}$. However, not every element in $\mathscr{F} \otimes \mathscr{E}$ is of the form $y \otimes x$. In general, we need linear combinations $\phi = \sum_{i=1}^n y_i \otimes x_i$. Now the sesquilinear evaluation of $\langle \phi, \phi \rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}}$ gives off-diagonal terms $\langle x_i, \langle y_i, y_j \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x_j \rangle^{\mathscr{E}}_{\mathscr{A}}$ which in general are non-zero but not necessarily positive. Thus the mere positivity of the inner products $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ and $\langle \cdot\,, \cdot \rangle^{\mathscr{F}}_{\mathscr{B}}$ does not seem to guarantee a positive inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}}$ directly. This problem is the ultimate reason that we required *complete* positivity for the inner products instead of just positivity. In the case of completely positive inner products, we obtain the following result [29, Thm. 4.7]:

Theorem 3.1.10 (Complete positivity of the internal tensor product) *Let $\mathscr{F}_{\mathscr{B}}$ be a right $\mathscr{B}$-module with $\mathscr{B}$-valued inner product and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule with compatible $\mathscr{A}$-valued inner product. If both inner products $\langle \cdot\,, \cdot \rangle^{\mathscr{F}}_{\mathscr{B}}$ and $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ are completely positive then $\langle \cdot\,, \cdot \rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}}$ is completely positive, too.*

Proof: Let $\phi^{(1)}, \ldots, \phi^{(n)} \in \mathscr{F} \otimes \mathscr{E}$ be given. Then we can write these vectors as

$$\phi^{(\alpha)} = \sum_{i=1}^{N} y_i^{(\alpha)} \otimes x_i^{(\alpha)}$$

with $y_i^{(\alpha)} \in \mathscr{F}$ and $x_i^{(\alpha)} \in \mathscr{E}$. Without restriction we can assume that N is the same for all α. We have to show that the matrix $\big(\langle \phi^{(\alpha)}, \phi^{(\beta)} \rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}}\big) \in \mathrm{M}_n(\mathscr{A})$ is positive. To this end, we consider the map

$$f \colon \mathrm{M}_n(\mathrm{M}_N(\mathscr{B})) \longrightarrow \mathrm{M}_n(\mathrm{M}_N(\mathscr{A}))$$

defined by

$$f(B) = \left(\left\langle x_i^{(\alpha)}, B_{ij}^{\alpha\beta} \cdot x_j^{(\beta)} \right\rangle^{\mathscr{E}}_{\mathscr{A}} \right),$$

where $B = (B_{ij}^{\alpha\beta})$ with the Greek indices running from 1 to n and the Latin ones running from 1 to N. We compute $f(B^*B)$ explicitly yielding

$$\begin{aligned} f(B^*B) &= \left(\left\langle x_i^{(\alpha)}, (B^*B)_{ij}^{\alpha\beta} \cdot x_j^{(\beta)} \right\rangle_{\mathscr{A}}^{\mathscr{E}} \right) \\ &= \left(\left\langle x_i^{(\alpha)}, \left(\sum_{\gamma=1}^{n} \sum_{k=1}^{N} (B_{ki}^{\gamma\alpha})^* B_{kj}^{\gamma\beta} \right) \cdot x_j^{(\beta)} \right\rangle_{\mathscr{A}}^{\mathscr{E}} \right) \\ &= \sum_{\gamma=1}^{n} \sum_{k=1}^{N} \underbrace{\left(\left\langle B_{ki}^{\gamma\alpha} \cdot x_i^{(\alpha)}, B_{kj}^{\gamma\beta} \cdot x_j^{(\beta)} \right\rangle_{\mathscr{A}}^{\mathscr{E}} \right)}_{= C_k^\gamma} \end{aligned}$$

with $C_k^\gamma \in \mathrm{M}_n(\mathrm{M}_N(\mathscr{A}))^+$, since $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{E}}$ is completely positive. This implies $f(B^*B) \in \mathrm{M}_n(\mathrm{M}_N(\mathscr{A}))^+$ for all B and thus the map f is a positive map. Since the matrix $\left(\left\langle y_i^{(\alpha)}, y_j^{(\beta)} \right\rangle_{\mathscr{B}}^{\mathscr{F}} \right) \in \mathrm{M}_n(\mathrm{M}_N(\mathscr{B}))^+$ is positive by the complete positivity of $\langle \cdot\,, \cdot \rangle_{\mathscr{B}}^{\mathscr{F}}$ we conclude that also the matrix

$$f\left(\left(\left\langle y_i^{(\alpha)}, y_j^{(\beta)} \right\rangle_{\mathscr{B}}^{\mathscr{F}} \right) \right) = \left(\left\langle x_i^{(\alpha)}, \left\langle y_i^{(\alpha)}, y_j^{(\beta)} \right\rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x_j^{(\beta)} \right\rangle_{\mathscr{A}}^{\mathscr{E}} \right) \in \mathrm{M}_n(\mathrm{M}_N(\mathscr{A}))^+$$

is positive. Now we use the canonical isomorphism $\mathrm{M}_n(\mathrm{M}_N(\mathscr{A})) \cong \mathrm{M}_{nN}(\mathscr{A}) \cong \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ as well as the *positive* map $\tau\colon \mathrm{M}_N(\mathrm{M}_n(\mathscr{A})) \longrightarrow \mathrm{M}_n(\mathscr{A})$ from Example 1.1.14. Thus also

$$\tau\Big(f\Big(\Big(\langle y_i^{(\alpha)}, y_j^{(\beta)} \rangle_{\mathscr{B}}^{\mathscr{F}} \Big) \Big) \Big) = \sum_{i,j=1}^{N} \left(\left\langle x_i^{(\alpha)}, \langle y_i^{(\alpha)}, y_j^{(\beta)} \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x_j^{(\beta)} \right\rangle_{\mathscr{A}}^{\mathscr{E}} \right) = \left(\langle \phi^{(\alpha)}, \phi^{(\beta)} \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}} \right) \in \mathrm{M}_n(\mathscr{A})^+$$

is positive which finishes the proof. ☐

From this theorem we immediately obtain the following corollary as complete positivity is preserved when passing to quotients:

Corollary 3.1.11 *If the algebra-valued inner products on $\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ are completely positive then $\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a pre-Hilbert module over $\mathscr{A}$.*

Corollary 3.1.12 *The internal tensor product yields functors*

$$\widehat{\otimes}_{\mathscr{B}}\colon {}^*\text{-}\mathsf{rep}_{\mathscr{B}}(\mathscr{C}) \times {}^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{C}) \tag{3.1.14}$$

and

$$\widehat{\otimes}_{\mathscr{B}}\colon {}^*\text{-}\mathsf{Rep}_{\mathscr{B}}(\mathscr{C}) \times {}^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{C}). \tag{3.1.15}$$

By fixing one of the two arguments of the functor $\widehat{\otimes}$ we obtain further functors which allow to move between *-representation theories of *-algebras. The two following examples will partially answer the questions asked in Section 1.3.

Example 3.1.13 (Rieffel induction) Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{B})$ be fixed. Then tensoring with ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ from the left gives a functor

$$\mathsf{R}_{\mathscr{E}} = {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \widehat{\otimes}_{\mathscr{A}}\colon {}^*\text{-}\mathsf{rep}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathscr{D}}(\mathscr{B}), \tag{3.1.16}$$

the so-called *Rieffel induction*. For the particular case $\mathscr{D} = \mathsf{C}$ we simply obtain the Rieffel induction

$$\mathsf{R}_{\mathscr{E}} = {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \widehat{\otimes}_{\mathscr{A}}\colon {}^*\text{-}\mathsf{rep}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{rep}(\mathscr{B}), \tag{3.1.17}$$

which will allow to compare (or at least relate) the representation theories of $\mathscr{A}$ and $\mathscr{B}$. Let us unwind the definition of $\mathsf{R}_{\mathscr{E}}$ more precisely: on objects it is simply the internal tensor product with ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and on morphisms it is the internal tensor product with the identity morphism on ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$, i.e. for $T\colon {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \longrightarrow {}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}}$ we set

$$\mathsf{R}_{\mathscr{E}}(T) = \mathrm{id}_{\mathscr{E}} \mathbin{\widehat{\otimes}} T. \tag{3.1.18}$$

By Theorem 3.1.7 this is indeed a morphism $\mathsf{R}_{\mathscr{E}}(T)\colon \mathsf{R}_{\mathscr{E}}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) \longrightarrow \mathsf{R}_{\mathscr{E}}({}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}})$ and $\mathsf{R}_{\mathscr{E}}$ is compatible with composition of morphisms. Thus $\mathsf{R}_{\mathscr{E}}$ is a functor. If we only require ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ then we still get a functor

$$\mathsf{R}_{\mathscr{E}}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{B}) \tag{3.1.19}$$

for each auxiliary *-algebra $\mathscr{D}$. If in addition ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$ or ${}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$, respectively, we obtain a functor preserving the strong non-degeneracy of the *-representations.

The Rieffel induction was originally formulated by Rieffel for the case of C^*-algebras, see [96, 97], with the auxiliary *-algebra $\mathscr{D}$ being just the complex numbers $\mathscr{D} = \mathbb{C}$. In addition to our algebraic framework he required analytic features like completeness with respect to norm topologies induced by the inner products. However, for C^*-algebras these kind of requirements are automatic as e.g. any *-representation of a C^*-algebra on a pre-Hilbert space is continuous and can be extended to a *-representation on the Hilbert space completion. More details on the C^*-algebraic version can be found e.g. in the textbooks [79, 81, 95] as well as in the Exercises 2.4.1 and 2.4.2.

The second example is interesting when we want to study the representation theory of a fixed *-algebra $\mathscr{A}$ but on pre-Hilbert modules over different auxiliary *-algebras.

Example 3.1.14 (Change of base ring) Let $\mathscr{D}$ and $\mathscr{D}'$ be two *-algebras replacing the scalars C and let ${}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'} \in {}^*\text{-}\mathsf{rep}_{\mathscr{D}'}(\mathscr{D})$. Then we obtain a functor

$$\mathsf{S}_{\mathscr{G}} = \widehat{\otimes}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'}\colon {}^*\text{-}\mathsf{rep}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathscr{D}'}(\mathscr{A}) \tag{3.1.20}$$

for any *-algebra $\mathscr{A}$: on objects we set $\mathsf{S}_{\mathscr{G}}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \mathbin{\widehat{\otimes}_{\mathscr{D}}} {}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'}$ and on intertwiners $T\colon {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \longrightarrow {}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}}$ we tensor with the identity, i.e. $\mathsf{S}_{\mathscr{G}}(T) = T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{G}}$. Analogously, we have variants of $\mathsf{S}_{\mathscr{G}}$ for *-mod and *-Rep as well as *-Mod, too, see also Exercise 3.3.2.

The associativity of the internal tensor product (up to the isomorphism a) according to Proposition 3.1.5 can now be used to interchange the functors $\mathsf{R}_{\mathscr{E}}$ and $\mathsf{S}_{\mathscr{G}}$. Here we have to be slightly more careful as the functors do not just commute but they only commute up to a natural transformation:

Proposition 3.1.15 *Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ and ${}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}'}(\mathscr{D})$ be given. Then the functors $\mathsf{R}_{\mathscr{E}}$ and $\mathsf{S}_{\mathscr{G}}$ commute up to the natural isomorphism*

$$\mathsf{a}\colon \mathsf{S}_{\mathscr{G}} \circ \mathsf{R}_{\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{E}} \circ \mathsf{S}_{\mathscr{G}} \tag{3.1.21}$$

induced by the associativity map a *from Proposition 3.1.5.*

PROOF: We have to show that changing the brackets in Proposition 3.1.5 gives a natural isomorphism between $\mathsf{F} = \mathsf{S}_{\mathscr{G}} \circ \mathsf{R}_{\mathscr{E}}$ and $\mathsf{G} = \mathsf{R}_{\mathscr{E}} \circ \mathsf{S}_{\mathscr{G}}$. Let ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A})$ be given. Then

$$(\mathsf{S}_{\mathscr{G}} \circ \mathsf{R}_{\mathscr{E}})({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = \big({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}\big) \mathbin{\widehat{\otimes}_{\mathscr{D}}} {}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'}$$

and

$$(\mathsf{R}_{\mathscr{E}} \circ \mathsf{S}_{\mathscr{G}})({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} \big({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \mathbin{\widehat{\otimes}_{\mathscr{D}}} {}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'}\big).$$

For a morphism $T\colon {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \longrightarrow {}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}}$ we have

$$(\mathsf{S}_{\mathscr{G}} \circ \mathsf{R}_{\mathscr{E}})(T) = \big(\mathrm{id}_{\mathscr{E}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} T\big) \mathbin{\widehat{\otimes}_{\mathscr{D}}} \mathrm{id}_{\mathscr{G}}$$

and

$$(\mathsf{R}_{\mathscr{E}} \circ \mathsf{S}_{\mathscr{G}})(T) = \mathrm{id}_{\mathscr{E}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} \big(T \mathbin{\widehat{\otimes}_{\mathscr{D}}} \mathrm{id}_{\mathscr{G}}\big).$$

Now we define $\mathsf{a}_{\mathscr{H}}\colon (\mathsf{S}_{\mathscr{G}} \circ \mathsf{R}_{\mathscr{E}})({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) \longrightarrow (\mathsf{R}_{\mathscr{E}} \circ \mathsf{S}_{\mathscr{G}})({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}})$ on elementary tensors before taking the quotient needed for $\widehat{\otimes}$ by

$$\mathsf{a}_{\mathscr{H}}((x \otimes \phi) \otimes y) = x \otimes (\phi \otimes y),$$

and use the induced map on $\widehat{\otimes}$-tensor products according to Proposition 3.1.5. By putting things together, it is clear that

$$(\mathsf{R}_{\mathscr{E}} \circ \mathsf{S}_{\mathscr{G}})(T) \circ \mathsf{a}_{\mathscr{H}} = \mathsf{a}_{\mathscr{H}'} \circ (\mathsf{S}_{\mathscr{G}} \circ \mathsf{R}_{\mathscr{E}})(T),$$

which shows that a is a natural transformation. Finally, since all the maps $\mathsf{a}_{\mathscr{H}}$ are even isometric isomorphisms (with the obvious inverses) and hence unitary intertwiners by Proposition 3.1.5, we have a natural isomorphism as claimed. □

We can rephrase the statement of the proposition in the following way: the diagram of functors

$$\begin{array}{ccc} {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A}) & \xrightarrow{\ \mathsf{R}_{\mathscr{E}}\ } & {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{B}) \\ {\scriptstyle \mathsf{S}_{\mathscr{G}}}\big\downarrow & \overset{\mathsf{a}}{\Searrow} & \big\downarrow{\scriptstyle \mathsf{S}_{\mathscr{E}}} \\ {}^*\text{-}\mathsf{mod}_{\mathscr{D}'}(\mathscr{A}) & \xrightarrow{\ \mathsf{R}_{\mathscr{G}}\ } & {}^*\text{-}\mathsf{mod}_{\mathscr{D}'}(\mathscr{B}) \end{array} \tag{3.1.22}$$

commutes up to the natural isomorphism given by the associativity a of the internal tensor product.

We conclude this section with a remark on the necessity to use completely positive inner products: this is unavoidable if one insists to obtain pre-Hilbert spaces out of Rieffel induction. To this end we need the following lemma which is also of independent interest, see [31, Lem. 3.2]:

Lemma 3.1.16 *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $n \in \mathbb{N}$.*

i.) If $\Omega\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathsf{C}$ is a positive linear functional then there exists a strongly non-degenerate *-representation π on a pre-Hilbert space $\mathcal{H}$ of $\mathscr{A}$ and vectors $\phi_1, \ldots, \phi_n \in \mathcal{H}$ with

$$n\Omega(A) = \sum_{i,j=1}^{n} \langle \phi_i, \pi(a_{ij})\phi_j \rangle \tag{3.1.23}$$

for all $A = (a_{ij}) \in \mathrm{M}_n(\mathscr{A})$.

ii.) Conversely, if $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathcal{H})$ is a *-representation on a pre-Hilbert space and $\phi_1, \ldots, \phi_n \in \mathcal{H}$ then

$$A \;\mapsto\; \sum_{i,j=1}^{n} \langle \phi_i, \pi(a_{ij})\phi_j \rangle \tag{3.1.24}$$

is a positive linear functional on $\mathrm{M}_n(\mathscr{A})$.

Proof: For the first part we consider the elementary matrices $E_{ij} \in \mathrm{M}_n(\mathscr{A})$ with $\mathbb{1}$ at the (i,j)-th position and zero elsewhere. Since $E_{ij}^* = E_{ji}$ and $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$ we have

$$nA = \sum\nolimits_{i,j,k,\ell} E_{ji}^* a_{i\ell} E_{k\ell}.$$

Now let $(\mathcal{H}_\Omega, \Pi_\Omega)$ be the GNS representation of $\mathrm{M}_n(\mathscr{A})$ with respect to Ω as in Proposition 1.2.6. The map $a \mapsto \Pi_\Omega(a\mathbb{1}_{n\times n})$ defines a strongly non-degenerate *-representation of $\mathscr{A}$ on $\mathcal{H}_\Omega$. Now we consider the vectors $\phi_i = \sum_j \psi_{E_{ji}} \in \mathcal{H}_\Omega$ where as usual ψ_A denotes the equivalence class of $A \in \mathrm{M}_n(\mathscr{A})$ in the

GNS pre-Hilbert space. With these vectors, (3.1.23) is a simple computation. The second part is a straightforward computation as well. For $A = (a_{ij}) \in \mathrm{M}_n(\mathscr{A})$ we have

$$\begin{aligned} \sum\nolimits_{i,j} \langle \phi_i, \pi((A^*A)_{ij})\phi_j \rangle &= \sum\nolimits_{i,j,k} \langle \phi_i, \pi(a_{ki}^* a_{kj})\phi_j \rangle \\ &= \sum\nolimits_{i,j,k} \langle \phi(a_{ki})\phi_i, \pi(a_{kj})\phi_j \rangle \\ &= \sum\nolimits_{k} \langle \psi_k, \psi_k \rangle \geq 0 \end{aligned}$$

with $\psi_k = \sum_i \pi(a_{ki})\phi_i$. This shows that (3.1.24) is a positive linear functional. □

Using this lemma we can now formulate the necessity of completely positive inner products for Rieffel induction, at least for the case of unital *-algebras:

Proposition 3.1.17 *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{E}}$. Then the following statements are equivalent:*

i.) For all strongly non-degenerate *-representations $(\mathcal{H}, \pi) \in {}^*\text{-}\mathsf{Rep}(\mathscr{A})$ of $\mathscr{A}$, the inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}\otimes\mathcal{H}}$ on $\mathscr{E}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathcal{H}$ is positive.

ii.) The inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{E}}$ is completely positive.

Proof: We have already shown *ii.)* $\implies$ *i.)* in Theorem 3.1.10 in even larger generality for arbitrary $(\mathscr{H}, \pi) \in {}^*\text{-}\mathsf{rep}_{\mathscr{D}}(\mathscr{A})$ with arbitrary auxiliary *-algebra $\mathscr{D}$. Thus assume *i.)* and let $x_1, \ldots, x_n \in \mathscr{E}_{\mathscr{A}}$ be given. Let $\Omega\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathsf{C}$ be positive and let $(\mathcal{H}, \pi) \in {}^*\text{-}\mathsf{Rep}(\mathscr{A})$ be a *-representation such that (3.1.23) holds, according to Lemma 3.1.16. Then for $A = \big(\langle x_i, x_j \rangle_{\mathscr{A}}^{\mathscr{E}}\big) \in \mathrm{M}_n(\mathscr{A})$ we have

$$n\Omega(A) = \sum\nolimits_{i,j} \langle \phi_i, \pi\big(\langle x_i, x_j \rangle_{\mathscr{A}}^{\mathscr{E}}\big)\phi_j \rangle = \left\langle \sum\nolimits_i x_i \otimes \phi_i, \sum\nolimits_j x_j \otimes \phi_j \right\rangle^{\mathscr{E}\otimes\mathcal{H}} \geq 0$$

by assumption. But this implies $\Omega(A) \geq 0$ for all positive Ω. Hence $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{E}}$ is completely positive. □

This algebraic observation can now be used to show that positive inner products for a C^*-algebra are automatically completely positive since *-representations are known to be orthogonal sums of cyclic representations, see Exercise 3.3.5.

3.2 External Tensor Products

In this short section we present yet another possibility to construct interesting bimodules, the external tensor product.

As it is well-known, the tensor product $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ over C of two *-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ is again a *-algebra via the multiplication

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2) \tag{3.2.1}$$

and the *-involution

$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*. \tag{3.2.2}$$

In this sense we have $\mathscr{A} \otimes \mathrm{M}_n(\mathsf{C}) \cong \mathrm{M}_n(\mathscr{A})$ as a first example. The construction is functorial in the obvious sense: If $\Phi_i\colon \mathscr{A}_i \longrightarrow \mathscr{B}_i$ are *-homomorphisms for $i = 1, 2$ then the tensor product

$$\Phi = \Phi_1 \otimes \Phi_2\colon \mathscr{A}_1 \otimes \mathscr{A}_2 \longrightarrow \mathscr{B}_1 \otimes \mathscr{B}_2 \tag{3.2.3}$$

is again a *-homomorphism. For $\Psi_i\colon \mathscr{B}_i \longrightarrow \mathscr{C}_i$ we have

$$\Psi \circ \Phi = (\Psi_1 \otimes \Psi_2) \circ (\Phi_1 \otimes \Phi_2) = (\Psi_1 \circ \Phi_1) \otimes (\Psi_2 \circ \Phi_2) \tag{3.2.4}$$

as well as

$$\mathrm{id}_{\mathscr{A}_1} \otimes \mathrm{id}_{\mathscr{A}_2} = \mathrm{id}_{\mathscr{A}_1 \otimes \mathscr{A}_2} . \tag{3.2.5}$$

Thus the tensor product $\otimes$ yields a functor

$$\otimes\colon {}^*\text{-}\mathsf{alg} \times {}^*\text{-}\mathsf{alg} \longrightarrow {}^*\text{-}\mathsf{alg}, \tag{3.2.6}$$

which yields a functor

$$\otimes\colon {}^*\text{-}\mathsf{Alg} \times {}^*\text{-}\mathsf{Alg} \longrightarrow {}^*\text{-}\mathsf{Alg} \tag{3.2.7}$$

in the case of unital *-algebras since $\mathbb{1}_{\mathscr{A}_1} \otimes \mathbb{1}_{\mathscr{A}_2}$ is clearly a unit element for the external tensor product $\mathscr{A}_1 \otimes \mathscr{A}_2$ if $\mathscr{A}_1$ and $\mathscr{A}_2$ are unital. Clearly, the external tensor product of two unital *-homomorphisms is again unital.

Remark 3.2.1 (Composite systems) From a physical point of view, the external tensor product corresponds to composite systems: If $\mathscr{A}_1$ is the observable algebra of the first subsystem and $\mathscr{A}_2$ is the observable algebra of the second subsystem then the combined physical system has $\mathscr{A}_1 \otimes \mathscr{A}_2$ as observable algebra. As usual, some idealizations are made: for classical physics with phase spaces M_1 and M_2 the external tensor product $\mathscr{C}^\infty(M_1) \otimes \mathscr{C}^\infty(M_2)$ is not quite $\mathscr{C}^\infty(M_1 \times M_2)$ but only a dense subalgebra. Similarly, in the quantum mechanical situation with two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ the combined system has the *Hilbert space tensor product* $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$ as Hilbert space, being the completion of the algebraic tensor product with respect to the canonical inner product on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. For the bounded operators as observable algebras we have $\mathfrak{B}(\mathfrak{H}_1) \otimes \mathfrak{B}(\mathfrak{H}_2) \subseteq \mathfrak{B}(\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2)$ where in infinite dimension the inclusion is proper, but the algebraic tensor product is dense in various topologies. In any case, the algebraic tensor product gives a reasonable observable algebra for the combined system, though maybe not containing all of them. In our completely algebraic approach, the algebraic external tensor product is all we can discuss here.

3.2.1 External Tensor Product of Inner-Product Bimodules

We shall now extend this construction to modules and bimodules. To this end we consider *-algebras $\mathscr{A}_i$ and $\mathscr{B}_i$ for $i = 1, 2$ and set $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ and $\mathscr{B} = \mathscr{B}_1 \otimes \mathscr{B}_2$, respectively. Moreover, let ${}_{\mathscr{B}_1}\mathscr{E}^{(1)}_{\mathscr{A}_1}$ and ${}_{\mathscr{B}_2}\mathscr{E}^{(2)}_{\mathscr{A}_2}$ be bimodules over $(\mathscr{B}_1, \mathscr{A}_1)$ and $(\mathscr{B}_2, \mathscr{A}_2)$, respectively.

Lemma 3.2.2 *The tensor product $\mathscr{E} = \mathscr{E}^{(1)} \otimes \mathscr{E}^{(2)}$ becomes a $(\mathscr{B}, \mathscr{A})$-bimodule via*

$$(b_1 \otimes b_2) \cdot (x_1 \otimes x_2) = (b_1 \cdot x_1) \otimes (b_2 \cdot x_2) \tag{3.2.8}$$

and

$$(x_1 \otimes x_2) \cdot (a_1 \otimes a_2) = (x_1 \cdot a_1) \otimes (x_2 \otimes a_2), \tag{3.2.9}$$

where $b_i \in \mathscr{B}_i$, $x_i \in \mathscr{E}^{(i)}$ and $a_i \in \mathscr{A}_i$ for $i = 1, 2$.

Proof: This is a trivial verification. □

Lemma 3.2.3 *Let ${}_{\mathscr{B}_i}\mathscr{E}^{(i)}_{\mathscr{A}_i}$ be $(\mathscr{B}_i, \mathscr{A}_i)$-bimodules with compatible $\mathscr{A}_i$-valued inner products $\langle \cdot\,, \cdot \rangle^{\mathscr{E}^{(i)}}_{\mathscr{A}_i}$ for $i = 1, 2$. Then the C-sesquilinear extension of*

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x_1, y_1 \rangle^{\mathscr{E}^{(1)}}_{\mathscr{A}_1} \otimes \langle x_2, y_2 \rangle^{\mathscr{E}^{(2)}}_{\mathscr{A}_2} \tag{3.2.10}$$

endows $\mathscr{E} = \mathscr{E}^{(1)} \otimes \mathscr{E}^{(2)}$ with an $\mathscr{A}$-valued inner product which is compatible with the left $\mathscr{B}$-module structure.

Proof: Again, this is a simple computation. □

As expected, the construction above is functorial in the best sense. Here we can show the following lemma:

Lemma 3.2.4 *Let $\mathscr{A}_i$, $\mathscr{B}_i$ be *-algebras and let ${}_{\mathscr{B}_i}\mathscr{E}^{(i)}_{\mathscr{A}_i}$ and ${}_{\mathscr{B}_i}\mathscr{F}^{(i)}_{\mathscr{A}_i}$ be $(\mathscr{B}_i, \mathscr{A}_i)$-bimodules with $\mathscr{A}_i$-valued inner products $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{E}^{(i)}}_{\mathscr{A}_i}$ and $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{F}^{(i)}}_{\mathscr{A}_i}$ for $i = 1, 2$. Moreover, let $\Phi_i\colon \mathscr{E}^{(i)} \longrightarrow \mathscr{F}^{(i)}$ be adjointable bimodule morphisms with adjoints Φ_i^* being bimodule morphisms, too. Then*

$$\Phi = \Phi_1 \otimes \Phi_2 \colon \mathscr{E}^{(1)} \otimes \mathscr{E}^{(2)} \longrightarrow \mathscr{F}^{(1)} \otimes \mathscr{F}^{(2)} \tag{3.2.11}$$

is an adjointable $(\mathscr{B}, \mathscr{A})$-bimodule morphism with adjoint given by

$$\Phi^* = \Phi_1^* \otimes \Phi_2^*. \tag{3.2.12}$$

Moreover, $\mathrm{id}_{\mathscr{E}^{(1)}} \otimes \mathrm{id}_{\mathscr{E}^{(2)}} = \mathrm{id}_{\mathscr{E}}$ and

$$(\Psi_1 \otimes \Psi_2) \circ (\Phi_1 \otimes \Phi_2) = (\Psi_1 \circ \Phi_1) \otimes (\Psi_2 \circ \Phi_2) \tag{3.2.13}$$

for the composition of adjointable bimodule morphisms.

Proof: We only compute the adjoint of Φ, the remaining statements are trivial. We have

$$\begin{aligned}
\langle y_1 \otimes y_2, \Phi(x_1 \otimes x_2)\rangle^{\mathscr{F}}_{\mathscr{A}} &= \langle y_1 \otimes y_2, \Phi_1(x_1) \otimes \Phi_2(x_2)\rangle^{\mathscr{F}}_{\mathscr{A}} \\
&= \langle y_1, \Phi_1(x_1)\rangle^{\mathscr{F}^{(1)}}_{\mathscr{A}_1} \otimes \langle y_2, \Phi_2(x_2)\rangle^{\mathscr{F}^{(2)}}_{\mathscr{A}_2} \\
&= \langle \Phi_1^*(y_1), x_1\rangle^{\mathscr{F}^{(1)}}_{\mathscr{A}_1} \otimes \langle \Phi_2^*(y_2), x_2\rangle^{\mathscr{F}^{(2)}}_{\mathscr{A}_2} \\
&= \langle \Phi_1(y_1) \otimes \Phi_2(y_2), x_1 \otimes x_2\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \langle (\Phi_1^* \otimes \Phi_2^*)(y_1 \otimes y_2), x_1 \otimes x_2\rangle^{\mathscr{E}}_{\mathscr{A}}
\end{aligned}$$

for all elementary tensors $x_1 \otimes x_2 \in \mathscr{E}$ and $y_1 \otimes y_2 \in \mathscr{F}$. This gives (3.2.12). ☐

As already for the internal tensor product, it may happen that the inner product (3.2.10) is degenerate. In fact, if the *-algebras are both C then there is no difference between the internal and external tensor product and $\mathsf{C} \otimes \mathsf{C} = \mathsf{C}$. Thus the Example 3.1.2 also applies for the external tensor product.

In any case, we know how to handle a possible degeneracy of the inner product: we have to divide by the degeneracy space $\mathscr{E}^{\perp}$. This way, we end up with a *-representation of $\mathscr{B}$ on an inner-product module over $\mathscr{A}$. We call this bimodule the *external tensor product*

$$\mathscr{E}^{(1)} \otimes_{\mathsf{ext}} \mathscr{E}^{(2)} = (\mathscr{E}^{(1)} \otimes \mathscr{E}^{(2)}) \big/ (\mathscr{E}^{(1)} \otimes \mathscr{E}^{(2)})^{\perp}. \tag{3.2.14}$$

Obviously, this is still compatible with morphisms and we obtain the following functor:

Proposition 3.2.5 *For all *-algebras $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{B}_1$, and $\mathscr{B}_2$, the external tensor product $\otimes_{\mathsf{ext}}$ is a functor*

$$\otimes_{\mathsf{ext}}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{A}_1}(\mathscr{B}_1) \times {}^*\text{-}\mathsf{mod}_{\mathscr{A}_2}(\mathscr{B}_2) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{A}_1 \otimes \mathscr{A}_2}(\mathscr{B}_1 \otimes \mathscr{B}_2). \tag{3.2.15}$$

3.2.2 External Tensor Products and Complete Positivity

We consider now the positivity requirements for the inner products. In order to show the complete positivity of $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{E}}_{\mathscr{A}}$ once the inner products $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{E}^{(i)}}_{\mathscr{A}_i}$ are completely positive, we need the following proposition which is also of independent interest. In fact, this statement is one of the main motivations to consider completely positive maps instead of just positive maps:

Proposition 3.2.6 *Let $\mathscr{A}_i$ and $\mathscr{B}_i$ be *-algebras over C with $i = 1, 2$ and let $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ and $\mathscr{B} = \mathscr{B}_1 \otimes \mathscr{B}_2$.*

i.) *For $a_i \in \mathscr{A}_i^{++}$ we have $a_1 \otimes a_2 \in \mathscr{A}^{++}$.*

ii.) *For $a_i \in \mathscr{A}_i^{+}$ we have $a_1 \otimes a_2 \in \mathscr{A}^{+}$.*

iii.) *For completely positive maps $\Phi_i \colon \mathscr{A}_i \longrightarrow \mathscr{B}_i$, the tensor product $\Phi = \Phi_1 \otimes \Phi_2 \colon \mathscr{A} \longrightarrow \mathscr{B}$ is completely positive, too.*

Proof: Let $a_i = \sum_{k=1}^{N_i} \alpha_{ik} b_{ik}^* b_{ik}$ with $\alpha_{ik} > 0$ and $b_{ik} \in \mathscr{A}_i$. Then

$$a_1 \otimes a_2 = \sum_{k=1}^{N_1} \sum_{\ell=1}^{N_2} \alpha_{1k} \alpha_{2\ell} (b_{1k} \otimes b_{2\ell})^* (b_{1k} \otimes b_{2\ell}) \in \mathscr{A}^{++}$$

shows the first part. For the second part we observe that the map $a_1 \mapsto a_1 \otimes a_2^* a_2$ is a positive map $\mathscr{A}_1 \longrightarrow \mathscr{A}$ for all $a_2 \in \mathscr{A}_2$: indeed $a_1^* a_1 \mapsto (a_1^* a_1) \otimes (a_2^* a_2) = (a_1 \otimes a_2)^* (a_1 \otimes a_2) \in \mathscr{A}^+$. By Exercise 1.4.6, *ii.)*, this is enough to conclude that the map $a_1 \mapsto a_1 \otimes a_2^* a_2$ is positive. Hence for all $a_1 \in \mathscr{A}_1^+$ also $a_1 \otimes a_2^* a_2 \in \mathscr{A}^+$. But this means that the map $a_2 \mapsto a_1 \otimes a_2$ is a positive map for all $a_1 \in \mathscr{A}_1^+$. Hence, by the same argument, $a_1 \otimes a_2 \in \mathscr{A}^+$ for all $a_i \in \mathscr{A}_i^+$ follows. The third part requires slightly more work and illustrates again the importance of the notion of a completely positive map. Let $n \in \mathbb{N}$ and $A \in \mathrm{M}_n(\mathscr{A}_1 \otimes \mathscr{A}_2)$ be given. We write A as matrix $A = (A^{\alpha\beta})$ with matrix entries $A^{\alpha\beta} = \sum_{k=1}^{N} a_{1k}^{\alpha\beta} \otimes a_{2k}^{\alpha\beta}$ and $a_{ik}^{\alpha\beta} \in \mathscr{A}_i$. Without restriction we can assume that N is the same for all α and β. Then we have

$$\begin{aligned}
\Phi^{(n)}(A^* A) &= \Phi^{(n)}\left(\sum_{\gamma=1}^{n} (A^{\gamma\alpha})^* A^{\gamma\beta} \right) \\
&= \sum_{\gamma=1}^{n} \Phi\left(\sum_{k,\ell=1}^{N} \left(\left(a_{1k}^{\gamma\alpha}\right)^* \otimes \left(a_{2k}^{\gamma\alpha}\right)^* \right) \left(a_{1\ell}^{\gamma\beta} \otimes a_{2\ell}^{\gamma\beta} \right) \right) \\
&= \sum_{\gamma=1}^{n} \sum_{k,\ell=1}^{N} \Phi\Big(\left(\left(a_{1k}^{\gamma\alpha}\right)^* a_{1\ell}^{\gamma\beta} \right) \otimes \left(\left(a_{2k}^{\gamma\alpha}\right)^* a_{2\ell}^{\gamma\beta} \right) \Big) \\
&= \sum_{\gamma=1}^{n} \sum_{k,\ell=1}^{N} \left(\Phi_1\left(\left(a_{1k}^{\gamma\alpha}\right)^* a_{1\ell}^{\gamma\beta} \right) \otimes \Phi_2\left(\left(a_{2k}^{\gamma\alpha}\right)^* a_{2\ell}^{\gamma\beta} \right) \right). \qquad (*)
\end{aligned}$$

We need now some auxiliary maps. First note that the matrix

$$B_i^{\gamma} = \left(\Phi_i\left(\left(a_{ik}^{\gamma\alpha}\right)^* a_{i\ell}^{\gamma\beta} \right) \right) \in \mathrm{M}_{nN}(\mathscr{A}_i)^+$$

is positive for all γ and $i = 1, 2$: for all γ the matrix $\left(\left(a_{ik}^{\gamma\alpha}\right)^* a_{i\ell}^{\gamma\beta} \right)$ is positive by Lemma 2.1.12, *i.)*, and Φ_i is completely positive by assumption. Now we use the following map

$$\Delta \colon \mathrm{M}_m(\mathscr{B}_1) \otimes \mathrm{M}_m(\mathscr{B}_2) \longrightarrow \mathrm{M}_m(\mathscr{B}_1 \otimes \mathscr{B}_2), \qquad (3.2.16)$$

defined by "evaluating on the diagonal", i.e.

$$\left(b_1^{\alpha\beta} \right) \otimes \left(b_2^{\alpha'\beta'} \right) \mapsto \left(b_1^{\alpha\beta} \otimes b_2^{\alpha\beta} \right),$$

where $\alpha, \beta, \alpha', \beta' = 1, \ldots, m$ denote the matrix indices. We claim that Δ is positive. Indeed, let

$$B = \sum_{r=1}^{N} \left(B_{1r}^{\alpha\beta} \otimes B_{2r}^{\alpha'\beta'} \right) \in \mathrm{M}_m(\mathscr{B}_1) \otimes \mathrm{M}_m(\mathscr{B}_2)$$

be given. Then we have

$$
\begin{aligned}
\Delta(B^*B) &= \sum_{r,s=1}^{N} \Delta((B_{1r} \otimes B_{2r})^*(B_{1s} \otimes B_{2s})) \\
&= \sum_{r,s=1}^{N} \sum_{\gamma,\gamma'=1}^{m} \Delta\Big(\Big((B_{1r}^{\gamma\alpha})^* B_{1s}^{\gamma\beta}\Big) \otimes \Big(\Big(B_{2r}^{\gamma'\alpha'}\Big)^* B_{2s}^{\gamma'\beta'}\Big)\Big) \\
&= \sum_{r,s=1}^{N} \sum_{\gamma,\gamma'=1}^{m} \Big(\Big((B_{1r}^{\gamma\alpha})^* B_{1s}^{\gamma\beta}\Big) \otimes \Big(\Big(B_{2r}^{\gamma'\alpha}\Big)^* B_{2s}^{\gamma'\beta}\Big)\Big) \\
&= \sum_{\gamma,\gamma'=1}^{m} \left(\left(\sum_{r=1}^{N} B_{1r}^{\gamma\alpha} \otimes B_{2r}^{\gamma'\beta}\right)^* \left(\sum_{s=1}^{N} B_{1s}^{\gamma\beta} \otimes B_{2s}^{\gamma'\beta}\right)\right) \\
&= \sum_{\gamma,\gamma'=1}^{m} \Big((b_{\gamma\gamma'}^{\alpha})^* \Big(b_{\gamma\gamma'}^{\beta}\Big)\Big),
\end{aligned}
$$

where $b_{\gamma\gamma'}^{\alpha} = \sum_{r=1}^{N} B_{1r}^{\gamma\alpha} \otimes B_{2r}^{\gamma'\alpha} \in \mathscr{B}_1 \otimes \mathscr{B}_2$. But then by Lemma 2.1.12, *i.)*, we finally see that the matrix $\big((b_{\gamma\gamma'}^{\alpha})^* (b_{\gamma\gamma'}^{\beta})\big)$ is (even algebraically) positive for all γ and γ'. Thus $\Delta(B^*B) \in \mathrm{M}_m(\mathscr{B}_1 \otimes \mathscr{B}_2)^{++}$. This shows that Δ is a positive map. Note that Δ is *not* a *-homomorphism. Now we can evaluate $(*)$ further and get

$$
\begin{aligned}
\Phi^{(n)}(A^*A) &= \sum_{\gamma=1}^{n} \sum_{k,\ell=1}^{N} \Big((B_1^{\gamma})_{k\ell}^{\alpha\beta} \otimes (B_2^{\gamma})_{k\ell}^{\alpha\beta}\Big) \\
&= \sum_{\gamma=1}^{n} \sum_{k,\ell=1}^{N} \Big(\Delta(B_1^{\gamma} \otimes B_2^{\gamma})_{k\ell}^{\alpha\beta}\Big) \\
&= \sum_{\gamma=1}^{n} \Big(\tau(\Delta(B_1^{\gamma} \otimes B_2^{\gamma}))^{\alpha\beta}\Big),
\end{aligned}
$$

where we have used on one hand the positive map $\Delta\colon \mathrm{M}_{nN}(\mathscr{B}_1) \otimes \mathrm{M}_{nN}(\mathscr{B}_2) \longrightarrow \mathrm{M}_{nN}(\mathscr{B}_1 \otimes \mathscr{B}_2)$ and on the other hand the positive map $\tau\colon \mathrm{M}_{nN}(\mathscr{B}_1 \otimes \mathscr{B}_2) = \mathrm{M}_N(\mathrm{M}_n(\mathscr{B}_1 \otimes \mathscr{B}_2)) \longrightarrow \mathrm{M}_n(\mathscr{B}_1 \otimes \mathscr{B}_2)$ from Example 1.1.14. As for each γ the matrices B_i^{γ} are positive, their tensor product $B_1^{\gamma} \otimes B_2^{\gamma}$ is positive by the second part. Thus applying the positive maps Δ and τ results in a positive matrix $\Phi^{(n)}(A^*A)$ which proves the complete positivity of Φ. □

Corollary 3.2.7 *Let $\omega_i\colon \mathscr{A}_i \longrightarrow \mathsf{C}$ be positive linear functionals. Then the linear functional $\omega_1 \otimes \omega_2\colon \mathscr{A}_1 \otimes \mathscr{A}_2 \longrightarrow \mathsf{C}$ is positive as well.*

Proof: This follows easily from the Proposition 3.2.6, *iii.)*, as positive linear functionals are completely positive maps and $\mathsf{C} \otimes \mathsf{C} = \mathsf{C}$. See also Exercise 3.3.3 for a more direct approach. □

Remark 3.2.8 (Entanglement) In general, the algebra $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ has more positive linear functionals ω as those which are convex combinations of the form $\omega = \omega_1 \otimes \omega_2$ with positive linear functionals $\omega_i\colon \mathscr{A}_i \longrightarrow \mathsf{C}$ with $i = 1, 2$. The quantum mechanical interpretation is now the following: recall that the combined system is described by $\mathscr{A}$, while the two subsystems are described by $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. Then the corresponding states (viewed as positive linear functionals) of the total system of the form $\omega = \omega_1 \otimes \omega_2$ show *no* correlations between the two sub-systems. Convex combinations have correlations but these can be considered to be entirely classical. They are *not* *entangled.* The remaining positive functionals correspond to those states having correlations beyond the classical correlations, i.e. *entanglement.* The characterization of such states is one of the primary goals of quantum information theory and a highly non-trivial task, even for finite-dimensional matrix algebras $\mathscr{A}_i = \mathrm{M}_{n_i}(\mathbb{C})$, see again e.g. [21, 92].

After this short excursion we come to the main theorem of this section, see [29, Remark 4.2]:

Theorem 3.2.9 (Complete positivity of external tensor product) *Let $\mathscr{A}_i$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\mathscr{E}^{(i)}_{\mathscr{A}_i}$ be right $\mathscr{A}_i$-modules with $\mathscr{A}_i$-valued, completely positive inner products $\langle \cdot\,, \cdot \rangle^{\mathscr{E}^{(i)}}_{\mathscr{A}_i}$. Then the $\mathscr{A}$-valued external tensor product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ on $\mathscr{E}_{\mathscr{A}} = \mathscr{E}^{(1)}_{\mathscr{A}_1} \otimes \mathscr{E}^{(2)}_{\mathscr{A}_2}$ with $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ is completely positive, too.*

Proof: Let $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ for $\mathscr{E}_{\mathscr{A}} = \mathscr{E}^{(1)}_{\mathscr{A}_1} \otimes \mathscr{E}^{(2)}_{\mathscr{A}_2}$ be defined as in (3.2.10). Moreover, let $\phi^{(1)}, \ldots, \phi^{(n)} \in \mathscr{E}$ be given which we can write as

$$\phi^{(\alpha)} = \sum_{k=1}^{N} x^{\alpha}_{1k} \otimes x^{\alpha}_{2k}$$

with $x^{\alpha}_{ik} \in \mathscr{E}^{(i)}$. Then we have

$$\left\langle \phi^{(\alpha)}, \phi^{(\beta)} \right\rangle^{\mathscr{E}}_{\mathscr{A}} = \sum_{k,\ell=1}^{N} \left\langle x^{\alpha}_{1k} \otimes x^{\alpha}_{2\ell}, x^{\beta}_{1k} \otimes x^{\beta}_{2\ell} \right\rangle^{\mathscr{E}}_{\mathscr{A}} = \sum_{k,\ell=1}^{N} \left\langle x^{\alpha}_{1k}, x^{\beta}_{1\ell} \right\rangle^{\mathscr{E}^{(1)}}_{\mathscr{A}_1} \otimes \left\langle x^{\alpha}_{2k}, x^{\beta}_{2\ell} \right\rangle^{\mathscr{E}^{(2)}}_{\mathscr{A}_2}. \tag{$*$}$$

By assumption, the matrices

$$X^{(i)} = \left(\left\langle x^{\alpha}_{ik}, x^{\beta}_{i\ell} \right\rangle^{\mathscr{E}^{(i)}}_{\mathscr{A}_i} \right) \in \mathrm{M}_{nN}(\mathscr{A}_i)^{+}$$

are positive. By Proposition 3.2.6, *ii.)*, also their tensor product $X^{(1)} \otimes X^{(2)} \in \mathrm{M}_{nN}(\mathscr{A}_1) \otimes \mathrm{M}_{nN}(\mathscr{A}_2)$ is positive. After applying the positive diagonal map Δ from (3.2.16) we obtain a positive matrix

$$\Delta\Big(X^{(1)} \otimes X^{(2)}\Big) = \left(\left\langle x^{\alpha}_{1k}, x^{\beta}_{1\ell} \right\rangle^{\mathscr{E}^{(1)}}_{\mathscr{A}_1} \otimes \left\langle x^{\alpha}_{2k}, x^{\beta}_{2\ell} \right\rangle^{\mathscr{E}^{(2)}}_{\mathscr{A}_2} \right) \in \mathrm{M}_{nN}(\mathscr{A}_1 \otimes \mathscr{A}_2)^{+}. \tag{$**$}$$

Finally, applying the positive map $\tau \colon \mathrm{M}_{nN}(\mathscr{A}_1 \otimes \mathscr{A}_2) \cong \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}_1 \otimes \mathscr{A}_2)) \longrightarrow \mathrm{M}_n(\mathscr{A}_1 \otimes \mathscr{A}_2)$ from Example 1.1.14, which is just the summation over k and ℓ in this case, we get from $(*)$ and $(**)$

$$\left(\left\langle \phi^{(\alpha)}, \phi^{(\beta)} \right\rangle^{\mathscr{E}}_{\mathscr{A}} \right) = \tau\Delta\Big(X^{(1)} \otimes X^{(2)}\Big) \in \mathrm{M}_n(\mathscr{A}_1 \otimes \mathscr{A}_2)^{+},$$

which proves the complete positivity of $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$. □

Corollary 3.2.10 *The external tensor product $\otimes_{\mathsf{ext}}$ yields a functor*

$$\otimes_{\mathsf{ext}} \colon {}^*\text{-}\mathsf{rep}_{\mathscr{A}_1}(\mathscr{B}_1) \times {}^*\text{-}\mathsf{rep}_{\mathscr{A}_2}(\mathscr{B}_2) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathscr{A}_1 \otimes \mathscr{A}_2}(\mathscr{B}_1 \otimes \mathscr{B}_2). \tag{3.2.17}$$

Example 3.2.11 We can now view our canonical example of the $(\mathrm{M}_n(\mathscr{A}), \mathscr{A})$-bimodule ${}_{\mathrm{M}_n(\mathscr{A})}\mathscr{A}^n_{\mathscr{A}}$ with its canonical inner product from Example 2.2.2 as an external tensor product. Indeed, for a *-algebra $\mathscr{A}$ we have the bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ with completely positive $\mathscr{A}$-valued inner product $\langle a, b \rangle = a^* b$ which is clearly compatible with the canonical left $\mathscr{A}$-module structure. Moreover, C^n is a $(\mathrm{M}_n(\mathsf{C}), \mathsf{C})$-bimodule with the canonical completely positive inner product, also compatible with the left $\mathrm{M}_n(\mathsf{C})$-module structure. With the canonical identifications $\mathscr{A} \otimes \mathsf{C} \cong \mathscr{A}$ and $\mathscr{A} \otimes \mathrm{M}_n(\mathsf{C}) \cong \mathrm{M}_n(\mathscr{A})$ we see that

$${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \otimes_{\mathsf{ext}} {}_{\mathrm{M}_n(\mathsf{C})}\mathsf{C}^n_{\mathsf{C}} \;\cong\; {}_{\mathrm{M}_n(\mathscr{A})}\mathscr{A}^n_{\mathscr{A}}, \tag{3.2.18}$$

including all bimodule structures and inner products, see also Exercise 3.3.4.

3.3 Exercises

Exercise 3.3.1 (Isometries are injective) A situation we encounter quite often is to have an isometric C-linear map $T\colon \mathscr{E}_{\mathscr{A}} \longrightarrow \mathscr{E}'_{\mathscr{A}}$ between two right $\mathscr{A}$-modules with inner products which may be degenerate.

i.) Show that in this case
$$\ker T \subseteq \mathscr{E}_{\mathscr{A}}^{\perp}. \tag{3.3.1}$$
Conclude that for a non-degenerate inner product on the domain $\mathscr{E}_{\mathscr{A}}$ the map T is injective.

ii.) Assume in addition that T is surjective. Show that T maps $\mathscr{E}_{\mathscr{A}}^{\perp}$ into $\mathscr{E}'^{\perp}_{\mathscr{A}}$.

iii.) Assume again that T is surjective and assume that $\mathscr{E}'_{\mathscr{A}}$ is an inner-product module. Show that in this case $\ker T = \mathscr{E}_{\mathscr{A}}^{\perp}$.

iv.) Show that an isometric surjective T induces a unitary and hence adjointable right $\mathscr{A}$-linear map
$$T\colon \mathscr{E}_{\mathscr{A}} / \mathscr{E}_{\mathscr{A}}^{\perp} \longrightarrow \mathscr{E}'_{\mathscr{A}} / \mathscr{E}'^{\perp}_{\mathscr{A}}. \tag{3.3.2}$$

Exercise 3.3.2 (Change of base ring) Show that the change of base ring functor $\mathsf{S}_{\mathscr{G}}$ as in Example 3.1.14 can also be defined as a functor
$$\mathsf{S}_{\mathscr{G}}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{D}'}(\mathscr{A}), \tag{3.3.3}$$
whenever ${}_{\mathscr{D}}\mathscr{G}_{\mathscr{D}'} \in\in {}^*\text{-}\mathsf{mod}_{\mathscr{D}'}(\mathscr{D})$. Discuss the compatibility with strongly non-degenerate *-representations of $\mathscr{A}$, i.e. the compatibility with the subcategories ${}^*\text{-}\mathsf{Rep}(\mathscr{A})$ and ${}^*\text{-}\mathsf{Mod}(\mathscr{A})$, respectively.

Exercise 3.3.3 (Direct proof of Corollary 3.2.7) Give a direct proof for the fact that the tensor product of positive linear functionals is again a positive linear functional without using the results of Proposition 3.2.6.

Exercise 3.3.4 ($\mathscr{A}^n$ as external tensor product) Provide the isomorphism realizing (3.2.18) explicitly and check all its properties directly.

Exercise 3.3.5 (Positive inner products for C^*-algebras) Consider a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and a right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ with positive (but not necessarily completely positive) inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$.

i.) Suppose that $(\mathcal{H}, \pi) \in {}^*\text{-}\mathsf{Rep}(\mathscr{A})$ is a *cyclic* *-representation. Show that the induced inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}\otimes\mathcal{H}}$ on $\mathscr{E}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathcal{H}$ is positive again.

ii.) Suppose next that $(\mathcal{H}, \pi) \in {}^*\text{-}\mathsf{Rep}(\mathscr{A})$ is an orthogonal direct sum of *cyclic* *-representations. Show that also in this case $\langle \cdot\,, \cdot \rangle^{\mathscr{E}\otimes\mathcal{H}}$ on $\mathscr{E}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathcal{H}$ is positive.

iii.) Conclude that for a C^*-algebra $\mathfrak{A}$ over $\mathbb{C}$ every positive inner product is actually completely positive.

Hint: Use first that every (strongly non-degenerate) *-representation of a C^*-algebra is (a completion of) a direct sum of cyclic *-representations. Then use Proposition 3.1.17.

Exercise 3.3.6 (The internal tensor product and direct orthogonal sums) Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Moreover, let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ and $\{{}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}}\}_{i\in I}$ be inner-product bimodules.

i.) Show that one has an isometric isomorphism
$${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} \left(\bigoplus\nolimits_{i\in I} {}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}}\right) \cong \bigoplus\nolimits_{i\in I} {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}}. \tag{3.3.4}$$
Show analogously, that such an isomorphism also exists for a direct sum in the first argument of the internal tensor product.

Hint: It might be advantageous to use Exercise 2.4.6, *iv.)*, to construct the orthogonal projections needed for the right hand side directly.

ii.) Let now ${}_{\mathscr{C}}\tilde{\mathscr{F}}'_{\mathscr{B}}$ and $\{{}_{\mathscr{B}}\tilde{\mathscr{E}}^{(i)}_{\mathscr{A}}\}_{i\in I}$ be further inner-product modules with adjointable bimodule morphisms, i.e. intertwiners, $\psi\colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \longrightarrow {}_{\mathscr{C}}\tilde{\mathscr{F}}'_{\mathscr{B}}$ and $\phi_i\colon {}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\tilde{\mathscr{E}}^{(i)}_{\mathscr{A}}$ for all $i \in I$. Let $\phi = \bigoplus_{i\in I}\phi_i\colon \bigoplus_{i\in I}{}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}} \longrightarrow \bigoplus_{i\in I}{}_{\mathscr{B}}\tilde{\mathscr{E}}^{(i)}_{\mathscr{A}}$ be the direct sum of the intertwiners ϕ_i. Show that the diagram

$$
\begin{array}{ccc}
{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} \left(\bigoplus_{i\in I}{}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}}\right) & \longrightarrow & \bigoplus_{i\in I}{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}^{(i)}_{\mathscr{A}} \\
\Big\downarrow {\scriptstyle \psi\,\widehat{\otimes}\,\phi} & & \Big\downarrow {\scriptstyle \bigoplus_{i\in I}\psi\,\widehat{\otimes}\,\phi_i} \\
{}_{\mathscr{C}}\tilde{\mathscr{F}}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} \left(\bigoplus_{i\in I}{}_{\mathscr{B}}\tilde{\mathscr{E}}^{(i)}_{\mathscr{A}}\right) & \longrightarrow & \bigoplus_{i\in I}{}_{\mathscr{C}}\tilde{\mathscr{F}}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\tilde{\mathscr{E}}^{(i)}_{\mathscr{A}}
\end{array}
\tag{3.3.5}
$$

commutes. This is the naturality of the isomorphism in (3.3.4). Formulate and prove the analogous statement for the direct sum in the first argument of the tensor product.

iii.) Conclude that the Rieffel induction functor is compatible with direct orthogonal sums of *-representations.

Exercise 3.3.7 (Complex conjugation and $\otimes_{\mathsf{ext}}$) Let $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{B}_1$, and $\mathscr{B}_2$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. We set $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ as well as $\mathscr{B} = \mathscr{B}_1 \otimes \mathscr{B}_2$. Moreover, let ${}_{\mathscr{B}_1}\mathscr{E}^{(1)}_{\mathscr{A}_1}$ and ${}_{\mathscr{B}_2}\mathscr{E}^{(2)}_{\mathscr{A}_2}$ be inner-product bimodules with external tensor product ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = {}_{\mathscr{B}_1}\mathscr{E}^{(1)}_{\mathscr{A}_1} \otimes_{\mathsf{ext}} {}_{\mathscr{B}_2}\mathscr{E}^{(2)}_{\mathscr{A}_2}$.

i.) Show that there is a isometric isomorphism

$$
i\colon \overline{{}_{\mathscr{B}_1}\mathscr{E}^{(1)}_{\mathscr{A}_1}} \otimes_{\mathsf{ext}} \overline{{}_{\mathscr{B}_2}\mathscr{E}^{(2)}_{\mathscr{A}_2}} \longrightarrow \overline{{}_{\mathscr{B}_1}\mathscr{E}^{(1)}_{\mathscr{A}_1} \otimes_{\mathsf{ext}} {}_{\mathscr{B}_2}\mathscr{E}^{(2)}_{\mathscr{A}_2}},
\tag{3.3.6}
$$

mapping the equivalence class of $\overline{y} \otimes \overline{x}$ to the equivalence class of $\overline{y \otimes x}$.

ii.) Consider now additional inner-product bimodules ${}_{\mathscr{B}_1}\mathscr{F}^{(1)}_{\mathscr{A}_1}$ and ${}_{\mathscr{B}_2}\mathscr{F}^{(2)}_{\mathscr{A}_2}$ with external tensor product ${}_{\mathscr{B}}\mathscr{F}_{\mathscr{A}} = {}_{\mathscr{B}_1}\mathscr{F}^{(1)}_{\mathscr{A}_1} \otimes_{\mathsf{ext}} {}_{\mathscr{B}_2}\mathscr{F}^{(2)}_{\mathscr{A}_2}$. Suppose $T_1\colon {}_{\mathscr{B}_1}\mathscr{E}^{(1)}_{\mathscr{A}_1} \longrightarrow {}_{\mathscr{B}_1}\mathscr{F}^{(1)}_{\mathscr{A}_1}$ and $T_2\colon {}_{\mathscr{B}_2}\mathscr{E}^{(2)}_{\mathscr{A}_2} \longrightarrow {}_{\mathscr{B}_2}\mathscr{F}^{(2)}_{\mathscr{A}_2}$ are intertwiners. Show that the isomorphism i is natural in the sense that

$$
i \circ \overline{(T_1 \otimes_{\mathsf{ext}} T_2)} = (\overline{T_1} \otimes_{\mathsf{ext}} \overline{T_2}) \circ i
\tag{3.3.7}
$$

Chapter 4

Morita Equivalence

In this chapter, we first present a rather naive approach to Morita theory: we require the existence of certain bimodules which implements an equivalence relation between *-algebras. Beside the ring-theoretic version which we discuss later, we have two flavours of this equivalence relation: *-Morita equivalence and strong Morita equivalence. After this direct approach we discuss a more conceptual definition of Morita equivalence as isomorphism in an enlarged category of *-algebras: we enlarge the notion of *-homomorphism to certain inner-product or pre-Hilbert bimodules giving new categories in which we have the same objects but more morphisms. In particular, more *-algebras become isomorphic in these new categories, and isomorphism turns out to be precisely Morita equivalence. The actual construction of these enlarged categories can be done in essentially two ways. Either one has to use equivalence classes of bimodules to obtain an honest category, or one can stay with bimodules directly, paying a price in form of getting only a bicategory. We will discuss both versions in detail.

4.1 Strong and *-Morita Equivalence

Dealing directly with the bimodules, the approach is now rather simple: on one hand we have to ensure a symmetric situation for the two *-algebras $\mathscr{A}$ and $\mathscr{B}$. For ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ the *-algebras $\mathscr{A}$ and $\mathscr{B}$ enter quite differently: there is only an $\mathscr{A}$-valued inner product but not a $\mathscr{B}$-valued one. Thus we have to equip ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ with a $\mathscr{B}$-valued inner product as well, now of course *left* $\mathscr{B}$-linear and C-linear in the *first* argument since ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a *left* $\mathscr{B}$-module. On the other hand, the bimodule together with the inner products should be as non-trivial as possible in order to get a meaningful equivalence relation.

4.1.1 *-Equivalence and Strong Equivalence Bimodules

Concerning the non-triviality of an inner product one first observes the following lemma:

Lemma 4.1.1 *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Then*

$$\langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{A}} = \operatorname{span}_{\mathsf{C}} \big\{ \langle x, y \rangle_{\mathscr{A}} \;\big|\; x, y \in \mathscr{E} \big\} \subseteq \mathscr{A} \tag{4.1.1}$$

is a *-ideal in $\mathscr{A}$.

PROOF: With $\langle x, y\rangle_{\mathscr{A}} a = \langle x, y \cdot a\rangle_{\mathscr{A}} \in \langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{A}}$ it follows immediately that $\langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{A}}$ is a right ideal. Moreover, $(\langle x, y\rangle_{\mathscr{A}})^* = \langle y, x\rangle_{\mathscr{A}}$ shows that $\langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{A}}$ is stable under the *-involution. Hence it has to be a *-ideal. □

One possibility to encode the non-triviality of an inner product is to demand that the *-ideal $\langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{A}}$ coincides with the whole *-algebra $\mathscr{A}$. In fact, this will be the crucial condition, thus deserving its own name:

Definition 4.1.2 (Full inner product) *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$. Then the inner product is called full if*

$$\langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{A}} = \mathscr{A}. \tag{4.1.2}$$

Together with the non-degeneracy conditions on the inner product and with the symmetric situation in $\mathscr{A}$ and $\mathscr{B}$ this constitutes the definition of an equivalence bimodule:

Definition 4.1.3 (Strong and *-equivalence bimodule) *Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule with two algebra-valued inner products $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ and ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}$. Then the triple $({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\langle \cdot, \cdot \rangle^{\mathscr{E}}, \langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}})$ is called *-equivalence bimodule if the following conditions are fulfilled:*

i.) The inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ is full, non-degenerate, and compatible with the left $\mathscr{B}$-module structure.

ii.) The inner product ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}$ is full, non-degenerate, and compatible with the right $\mathscr{A}$-module structure.

iii.) The bimodule $\mathscr{E}$ is strongly non-degenerate both as right $\mathscr{A}$-module and as left $\mathscr{B}$-module, i.e.

$$\mathscr{B} \cdot \mathscr{E} = \mathscr{E} = \mathscr{E} \cdot \mathscr{A}. \tag{4.1.3}$$

iv.) For all $x, y, z \in \mathscr{E}$ one has

$$x \cdot \langle y, z \rangle^{\mathscr{E}}_{\mathscr{A}} = {}_{\mathscr{B}}\langle x, y \rangle^{\mathscr{E}} \cdot z. \tag{4.1.4}$$

If in addition both inner products are completely positive then $({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}, \langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}})$ is called a strong equivalence bimodule.

We remark that the notion of strong equivalence bimodules for C^*-algebras is due to Rieffel [96–98], see also e.g. [81,95] for textbooks on the C^*-algebraic version of strong Morita equivalence. Ara discussed *-equivalences bimodules for general rings with involution [1,2]. Our *-algebras are a particular case of this slightly more general situation. Nevertheless, as we will aim for strong Morita equivalence of *-algebras anyway, we have presented Ara's definition only in this context. Finally, the extension of the notion of strong equivalence bimodules to *-algebras over rings of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$ is due to Bursztyn and Waldmann [26,29]. In the C^*-algebraic case, Rieffel considered additional completeness conditions which in the end turn out to be easy to get. For a strong equivalence bimodule as above one has certain automatic continuity properties which allow for an immediate completion: two C^*-algebras are strongly Morita equivalent in Rieffel's original sense iff their minimal dense ideals, i.e. their *Pedersen ideals*, are *-Morita equivalent in Ara's sense or strongly Morita equivalent in the above sense, see [2] and [25, Lem. 3.1]. Taking the Pedersen ideals and building the completion can be extended to bimodules and yields good functorial behaviour discussed in [29, Sect. 6B].

If the context is clear we shall drop the explicit reference to the inner products and simply call ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ a *-equivalence or strong equivalence bimodule. Note that the $\mathscr{B}$-valued inner product ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}$ is C-linear and left $\mathscr{B}$-linear in the *first* argument, according to our considerations in Section 2.1.2.

The existence of an equivalence bimodule of either one of the above flavours gives now the (preliminary) definition of Morita equivalence:

Definition 4.1.4 (Strong and *-Morita equivalence) *Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) The *-algebras $\mathscr{A}$ and $\mathscr{B}$ are called *-Morita equivalent if there exists a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$.

*ii.) The *-algebras $\mathscr{A}$ and $\mathscr{B}$ are called strongly Morita equivalent if there exists a strong equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$.*

The first task is to justify the notion "equivalence". Thus we have to show that *-Morita equivalence as well as strong Morita equivalence yield relations which are reflexive, symmetric and transitive. We start with the following lemma:

Lemma 4.1.5 *The $(\mathscr{A},\mathscr{A})$-bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ with the canonical inner products*

$$ {}_{\mathscr{A}}\langle a,b\rangle = ab^* \quad \textit{and} \quad \langle a,b\rangle_{\mathscr{A}} = a^*b \tag{4.1.5}$$

is a strong equivalence bimodule if and only if the algebra $\mathscr{A}$ is idempotent and non-degenerate.

PROOF: It is straightforward to see that not only $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{A}}$ but also ${}_{\mathscr{A}}\langle\,\cdot\,,\,\cdot\,\rangle$ is an $\mathscr{A}$-valued inner product compatible with the left respectively right $\mathscr{A}$-module structure. Moreover, ${}_{\mathscr{A}}\langle a,b\rangle c = ab^*c = a\langle b,c\rangle_{\mathscr{A}}$ shows the compatibility (4.1.4) of the two inner products. Now the algebra is idempotent if and only if $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{A}}$ is full in which case also ${}_{\mathscr{A}}\langle\,\cdot\,,\,\cdot\,\rangle$ is full. The condition $\mathscr{A}\cdot{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} = {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} = {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}\cdot\mathscr{A}$ is also equivalent to $\mathscr{A}$ being idempotent. Finally, the inner products are non-degenerate if and only if $\mathscr{A}$ is non-degenerate. The complete positivity of the inner product has already been shown for $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{A}}$. For ${}_{\mathscr{A}}\langle\,\cdot\,,\,\cdot\,\rangle$ it follows analogously. □

In the following we do not just require reflexivity but we want ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ to be an equivalence bimodule directly. Thus we shall restrict to *idempotent and non-degenerate *-algebras* in the sequel. Taking a closer look at representation theory and Morita theory shows that *-algebras which are non-idempotent or degenerate behave rather pathological. Excluding them from our considerations is thus welcomed. In any case, the restriction is not very severe as e.g. unital *-algebras are idempotent and non-degenerate anyway, see also Proposition 2.2.6 and Exercise 2.4.12 for further non-unital examples.

The next lemma shows that complex conjugation turns equivalence bimodules into equivalence bimodules:

Lemma 4.1.6 *Let $({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{E}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{E}}_{\mathscr{A}})$ be a *-equivalence (strong equivalence) bimodule. Then $({}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}, {}_{\mathscr{A}}\langle\,\cdot\,,\,\cdot\,\rangle^{\overline{\mathscr{E}}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}})$ is a *-equivalence (strong equivalence) bimodule as well where the bimodule structure and the inner products are defined according to Proposition 2.1.2.*

PROOF: The proof consists in a simple verification of the required properties. Let $\mathscr{E}\ni x\mapsto\overline{x}\in\overline{\mathscr{E}}$ be the canonical map. Then clearly $a\cdot\overline{x} = \overline{x\cdot a^*}$ and $\overline{x}\cdot b = \overline{b^*\cdot x}$ define the $(\mathscr{A},\mathscr{B})$-bimodule structure. We have $\mathscr{A}\cdot\overline{\mathscr{E}} = \overline{\mathscr{E}} = \overline{\mathscr{E}}\cdot\mathscr{B}$. Moreover, the definitions

$$ {}_{\mathscr{A}}\langle\overline{x},\overline{y}\rangle^{\overline{\mathscr{E}}} = \langle x,y\rangle^{\mathscr{E}}_{\mathscr{A}} \quad\text{and}\quad \langle\overline{x},\overline{y}\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} = {}_{\mathscr{B}}\langle x,y\rangle^{\mathscr{E}} $$

yield algebra-valued inner products with the correct linearity properties. Clearly, they are still full and non-degenerate. If the original inner products are completely positive then also the inner products ${}_{\mathscr{A}}\langle\,\cdot\,,\,\cdot\,\rangle^{\overline{\mathscr{E}}}$ and $\langle\,\cdot\,,\,\cdot\,\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}}$ are completely positive, see Exercise 2.4.9. We have

$$ {}_{\mathscr{A}}\langle\overline{x}\cdot b,\overline{y}\rangle^{\overline{\mathscr{E}}} = {}_{\mathscr{A}}\langle\overline{b^*\cdot x},\overline{y}\rangle^{\overline{\mathscr{E}}} = \langle b^*\cdot x,y\rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x,b\cdot y\rangle^{\mathscr{E}}_{\mathscr{A}} = {}_{\mathscr{A}}\langle\overline{x},\overline{b\cdot y}\rangle^{\overline{\mathscr{E}}} = {}_{\mathscr{A}}\langle\overline{x},\overline{y}\cdot b^*\rangle^{\overline{\mathscr{E}}}, $$

showing that ${}_{\mathscr{A}}\langle\,\cdot\,,\,\cdot\,\rangle^{\overline{\mathscr{E}}}$ is compatible with the right $\mathscr{B}$-module structure. Analogously, one shows the compatibility of $\langle\,\cdot\,,\,\cdot\,\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}}$ with the left $\mathscr{A}$-module structure. For the compatibility of the inner products we compute

$$ \begin{aligned} \overline{x}\cdot\langle\overline{y},\overline{z}\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} &= \overline{\left(\langle\overline{y},\overline{z}\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}}\right)^*\cdot x} \\ &= \overline{\left({}_{\mathscr{B}}\langle y,z\rangle^{\mathscr{E}}\right)^*\cdot x} \end{aligned} $$

$$
\begin{aligned}
&= \overline{{}_{\mathscr{B}}\langle z, y\rangle^{\mathscr{E}} \cdot x} \\
&= \overline{z \cdot \langle y, x\rangle^{\mathscr{E}}_{\mathscr{A}}} \\
&= \overline{z \cdot \left(\langle x, y\rangle^{\mathscr{E}}_{\mathscr{A}}\right)^*} \\
&= {}_{\mathscr{A}}\langle \overline{x}, \overline{y}\rangle^{\overline{\mathscr{E}}} \cdot \overline{z}
\end{aligned}
$$

for all $\overline{x}, \overline{y}, \overline{z} \in \overline{\mathscr{E}}$. This completes the proof. ☐

The next lemma uses the non-degeneracy of the *-algebra as well as the fullness of the inner product:

Lemma 4.1.7 *Let $\mathscr{E}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with full $\mathscr{A}$-valued inner product. If $\mathscr{A}$ is non-degenerate then the map*

$$
\mathscr{A} \ni a \;\mapsto\; (x \mapsto x \cdot a) \in \operatorname{End}_{\mathsf{C}}(\mathscr{E}_{\mathscr{A}}) \tag{4.1.6}
$$

is injective. The analogous statement holds for left modules.

PROOF: Let $a \in \mathscr{A}$ with $x \cdot a = 0$ for all $x \in \mathscr{E}$ be given. Then $0 = \langle y, x \cdot a\rangle_{\mathscr{A}} = \langle y, x\rangle_{\mathscr{A}} a$ and hence $\langle \mathscr{E}, \mathscr{E}\rangle_{\mathscr{A}} a = 0$. Since by assumption $\langle \mathscr{E}, \mathscr{E}\rangle_{\mathscr{A}} = \mathscr{A}$ we conclude $\mathscr{A} a = 0$ which is only possible for $a = 0$ as $\mathscr{A}$ is non-degenerate. ☐

The associativity property of the inner products according to Definition 4.1.3, *iv.)*, is responsible for the following result which allows to get rid of possible degeneracy spaces:

Lemma 4.1.8 *Let $({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\langle \cdot\,, \cdot\rangle^{\mathscr{E}}, \langle \cdot\,, \cdot\rangle^{\mathscr{E}}_{\mathscr{A}})$ be a $(\mathscr{B}, \mathscr{A})$-bimodule with algebra-valued inner products satisfying all requirements of a *-equivalence bimodule except that the inner products may be degenerate. Moreover, assume $\mathscr{A}$ and $\mathscr{B}$ are non-degenerate *-algebras.*

i.) We have ${}_{\mathscr{B}}\mathscr{E}^{\perp}_{\mathscr{A}} = {}^{\perp}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ for the two degeneracy spaces

$$
{}_{\mathscr{B}}\mathscr{E}^{\perp}_{\mathscr{A}} = \left\{x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \;\middle|\; \langle x, \cdot\,\rangle^{\mathscr{E}}_{\mathscr{A}} = 0\right\} \quad \textit{and} \quad {}^{\perp}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = \left\{x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \;\middle|\; {}_{\mathscr{B}}\langle \cdot\,, x\rangle^{\mathscr{E}} = 0\right\}. \tag{4.1.7}
$$

ii.) The quotient bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big/{}_{\mathscr{B}}\mathscr{E}^{\perp}_{\mathscr{A}}$ with the induced inner products is a *-equivalence bimodule.

iii.) If both inner products ${}_{\mathscr{B}}\langle \cdot\,, \cdot\rangle^{\mathscr{E}}$ and $\langle \cdot\,, \cdot\rangle^{\mathscr{E}}_{\mathscr{A}}$ are in addition completely positive then the quotient bimodule is even a strong equivalence bimodule.

PROOF: For the first part we consider $x \in {}_{\mathscr{B}}\mathscr{E}^{\perp}_{\mathscr{A}}$ and y, z arbitrary. Then we have

$$
{}_{\mathscr{B}}\langle y, x\rangle^{\mathscr{E}} \cdot z = y \cdot \langle x, z\rangle^{\mathscr{E}}_{\mathscr{A}} = 0
$$

for all z implying ${}_{\mathscr{B}}\langle y, x\rangle^{\mathscr{E}} = 0$ by Lemma 4.1.7. Since this holds for all y we conclude $x \in {}^{\perp}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. The opposite inclusion follows analogously. Then the second part is clear as the two quotients ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big/{}_{\mathscr{B}}\mathscr{E}^{\perp}_{\mathscr{A}} = {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big/{}^{\perp}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ simply coincide. The first variant inherits the $\mathscr{A}$-valued inner product, the second the $\mathscr{B}$-valued inner product by Proposition 2.1.3. Thus overall we have both inner products well-defined on the quotient, now being both non-degenerate. The other properties stay valid and hence we end up with a *-equivalence bimodule. The third part follows as well since the complete positivity is preserved when passing to the quotient. ☐

For two *-equivalence bimodules ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ we can form the $\mathscr{B}$-tensor product ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. As we already know from Section 3.1, it is endowed with a canonically given $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}$. With the same construction we also obtain a $\mathscr{C}$-valued inner product, now C-linear and left $\mathscr{C}$-linear in the first argument, by setting

$$
{}_{\mathscr{C}}\big\langle y \otimes x, y' \otimes x'\big\rangle^{\mathscr{F}\otimes\mathscr{E}} = {}_{\mathscr{C}}\big\langle y \cdot {}_{\mathscr{B}}\langle x, x'\rangle^{\mathscr{E}}, y'\big\rangle^{\mathscr{F}} \tag{4.1.8}
$$

on elementary tensors $y \otimes x, y' \otimes x' \in \mathscr{F} \otimes_{\mathscr{B}} \mathscr{E}$ and extending this C-sesquilinearly to the whole tensor product. Analogous arguments as in Section 3.1.1 show that this is indeed well-defined and gives a $\mathscr{C}$-valued inner product compatible with the right $\mathscr{A}$-module structure.

Lemma 4.1.9 *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be *-algebras and let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be *-equivalence bimodules. Then ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ together with the inner products ${}_{\mathscr{C}}\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}$ and $\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}$ fulfills all requirements of a *-equivalence bimodule except that the inner products might be degenerate.*

Proof: The compatibility of the inner products with the module structures has already been discussed. Next, it is clear that $\mathscr{C} \cdot (\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E}) = \mathscr{F} \otimes_{\mathscr{B}} \mathscr{E} = (\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E}) \cdot \mathscr{A}$. This follows directly from $\mathscr{C} \cdot \mathscr{F} = \mathscr{F}$ and $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$. We show that $\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}$ is full, the proof for the fullness of ${}_{\mathscr{C}}\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}$ is analogous. Let $a \in \mathscr{A}$ then we find $x_i, x_i' \in \mathscr{E}$ with $a = \sum_i \langle x_i, x_i' \rangle^{\mathscr{E}}_{\mathscr{A}}$ since $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ is full. Since $\mathscr{B} \cdot \mathscr{E} = \mathscr{E}$ we find $b_{ij} \in \mathscr{B}$ and $x_{ij}'' \in \mathscr{E}$ with $x_i' = \sum_j b_{ij} \cdot x_{ij}''$. Since also $\langle \cdot\,, \cdot \rangle^{\mathscr{F}}_{\mathscr{B}}$ is full we find $y_{ijk}, y_{ijk}' \in \mathscr{F}$ with $b_{ij} = \sum_k \langle y_{ijk}, y_{ijk}' \rangle^{\mathscr{F}}_{\mathscr{B}}$. Now we compute

$$
\begin{aligned}
\sum\nolimits_{i,j,k} \langle y_{ijk} \otimes x_i, y_{ijk}' \otimes x_{ij}'' \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} &= \sum\nolimits_i \Big\langle x_i, \sum\nolimits_{j,k} \langle y_{ijk}, y_{ijk}' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x_{ij}'' \Big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \sum\nolimits_i \Big\langle x_i, \sum\nolimits_j b_{ij} \cdot x_{ij}'' \Big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \sum\nolimits_i \langle x_i, x_i' \rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= a,
\end{aligned}
$$

which proves fullness. The analogous argument works for ${}_{\mathscr{C}}\langle \cdot\,, \cdot \rangle^{\mathscr{F}\otimes\mathscr{E}}$ as well. It remains to show the compatibility of the two inner products. As usual it suffices to consider factorizing tensors. Thus let $y \otimes x$, $y' \otimes x'$ and $y'' \otimes x'' \in \mathscr{F} \otimes_{\mathscr{B}} \mathscr{E}$ be given. Then

$$
\begin{aligned}
{}_{\mathscr{C}}\langle y \otimes x, y' \otimes x' \rangle^{\mathscr{F}\otimes\mathscr{E}} \cdot (y'' \otimes x'') &= \Big({}_{\mathscr{C}}\big\langle y \cdot {}_{\mathscr{B}}\langle x, x' \rangle^{\mathscr{E}}, y' \big\rangle^{\mathscr{F}} \cdot y'' \Big) \otimes x'' \\
&= \Big({}_{\mathscr{C}}\big\langle y, y' \cdot \big({}_{\mathscr{B}}\langle x, x' \rangle^{\mathscr{E}}\big)^* \big\rangle^{\mathscr{F}} \cdot y'' \Big) \otimes x'' \\
&= \Big(y \cdot \big\langle y' \cdot \big({}_{\mathscr{B}}\langle x, x' \rangle^{\mathscr{E}}\big)^*, y'' \big\rangle^{\mathscr{F}}_{\mathscr{B}} \Big) \otimes x'' \\
&= y \otimes \Big(\big\langle y' \cdot \big({}_{\mathscr{B}}\langle x, x' \rangle^{\mathscr{E}}\big)^*, y'' \big\rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x'' \Big) \\
&= y \otimes \big(\big({}_{\mathscr{B}}\langle x, x' \rangle^{\mathscr{E}} \langle y', y'' \rangle^{\mathscr{F}}_{\mathscr{B}}\big) \cdot x'' \big) \\
&= y \otimes \Big({}_{\mathscr{B}}\big\langle x, \big(\langle y', y'' \rangle^{\mathscr{F}}_{\mathscr{B}}\big)^* \cdot x' \big\rangle^{\mathscr{E}} \cdot x'' \Big) \\
&= y \otimes \Big(x \cdot \big\langle \big(\langle y', y'' \rangle^{\mathscr{F}}_{\mathscr{B}}\big)^* \cdot x', x'' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \Big) \\
&= (y \otimes x) \cdot \langle x', \langle y', y'' \rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x'' \rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= (y \otimes x) \cdot \langle y' \otimes x', y'' \otimes x'' \rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}}
\end{aligned}
$$

shows that the inner products are compatible and the proof is finished. □

A priori it does not seem to be possible to guarantee that the inner products on $\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E}$ are non-degenerate. Thus we have to pass to the quotient by the degeneracy space which, by Lemma 4.1.8, is the same for both inner products. This way, we obtain a *-equivalence bimodule including both inner products. In order to emphasize the fact that we have two inner products to take care of, we use a different symbol

$$
{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \big/ \big({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big)^{\perp} \tag{4.1.9}
$$

for this tensor product of *-equivalence bimodules as for the internal tensor product $\widehat{\otimes}_{\mathscr{B}}$. We shall refer to $\widetilde{\otimes}$ as the *internal tensor product of *-equivalence bimodules.* Later, we will see that under certain assumptions the inner products on ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ for *-equivalence bimodules are automatically non-degenerate, and hence the quotient is not necessary in these cases.

Using the previous lemmas we can now show the main result of this section [1, 29]:

Theorem 4.1.10 (*-Morita and strong Morita equivalence) *For the class of idempotent and non-degenerate* **-algebras over* $\mathbb{C}$ *the notions of* **-Morita equivalence as well as strong Morita equivalence are equivalence relations.*

PROOF: Indeed, reflexivity is obtained from the strong equivalence bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ as in Lemma 4.1.5, symmetry follows via the complex conjugate bimodule according to Lemma 4.1.6. Finally, transitivity is obtained by taking the $\widetilde{\otimes}$-tensor product of equivalence bimodules. Clearly, all constructions are not only possible for *-equivalence bimodules but also for strong equivalence bimodules since complete positivity of inner products is preserved by complex conjugation and by $\widetilde{\otimes}$. □

4.1.2 First Examples of Strong Morita Equivalences

We now face the fundamental question which *-algebras are *-Morita or strongly Morita equivalent. Here we consider the following situation. Denote by $\operatorname{Iso}^*(\mathscr{B}, \mathscr{A})$ the (possibly empty) set of all *-isomorphisms from $\mathscr{A}$ to $\mathscr{B}$. For later use we set $\operatorname{Aut}^*(\mathscr{A}) = \operatorname{Iso}^*(\mathscr{A}, \mathscr{A})$. Assume that $\mathscr{A}$ and $\mathscr{B}$ are *-isomorphic and let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a *-isomorphism. Then we can consider the following $(\mathscr{B}, \mathscr{A})$-bimodule ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$: we endow $\mathscr{B}$ with the usual left $\mathscr{B}$-module structure and let $\mathscr{A}$ act on $\mathscr{B}$ via

$$b \cdot_{\Phi} a = b\Phi(a) \tag{4.1.10}$$

for $b \in \mathscr{B}$ and $a \in \mathscr{A}$. Clearly, this way $\mathscr{B}$ becomes a right $\mathscr{A}$-module. For the inner products we use the canonical $\mathscr{B}$-valued one

$${}_{\mathscr{B}}\langle b, b' \rangle^{\mathscr{B}^{\Phi}} = b(b')^*, \tag{4.1.11}$$

and for the $\mathscr{A}$-valued inner product we use

$$\langle b, b' \rangle^{\mathscr{B}^{\Phi}}_{\mathscr{A}} = \Phi^{-1}(b^* b'), \tag{4.1.12}$$

where $b, b' \in \mathscr{B}$. Using this bimodule we obtain the following theorem:

Theorem 4.1.11 (*-Isomorphism and strong Morita equivalence) *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are* **-isomorphic non-degenerate and idempotent* **-algebras then* $\mathscr{A}$ *and* $\mathscr{B}$ *are strongly Morita equivalent. In particular, for any* $\Phi \in \operatorname{Iso}^*(\mathscr{B}, \mathscr{A})$ *the bimodule* ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$ *with the inner products* ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{B}^{\Phi}}$ *and* $\langle \cdot\,, \cdot \rangle^{\mathscr{B}^{\Phi}}_{\mathscr{A}}$ *is a strong equivalence bimodule.*

PROOF: Since $\mathscr{A}$ (and hence necessarily also $\mathscr{B}$) is idempotent and non-degenerate and since Φ is a *-isomorphism, the $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{B}^{\Phi}}_{\mathscr{A}}$ is full, non-degenerate, and compatible with the left $\mathscr{B}$-action. Clearly, ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{B}^{\Phi}}$ is full and non-degenerate as well and compatible with the right $\mathscr{A}$-action. Here we have to use again that Φ is a *-isomorphism. The remaining properties are checked easily. □

Thus *-isomorphic *-algebras (always within our class of idempotent and non-degenerate ones) are strongly Morita equivalent. The reverse is not true in general as the following simple example shows:

Theorem 4.1.12 (Morita equivalence of $\mathscr{A}$ and $\mathrm{M}_n(\mathscr{A})$) *Let* $\mathscr{A}$ *be non-degenerate and idempotent. Then* ${}_{\mathrm{M}_n(\mathscr{A})}\mathscr{A}^n_{\mathscr{A}}$*, endowed with the usual* $\mathscr{A}$*-valued inner product and the canonical bimodule structure together with the* $\mathrm{M}_n(\mathscr{A})$*-valued inner product*

$${}_{\mathrm{M}_n(\mathscr{A})}\langle x, y \rangle^{\mathscr{A}^n} = (x_i y_j^*) = \Theta_{x,y} \tag{4.1.13}$$

for $x, y \in \mathscr{A}^n$*, is a strong Morita equivalence bimodule.*

PROOF: Again, this is a simple verification. First we note that $\mathrm{M}_n(\mathscr{A})$ is again non-degenerate and idempotent if $\mathscr{A}$ has these properties, see Exercise 2.4.5. Hence we stay in the correct framework. Next it is clear that ${}_{\mathrm{M}_n(\mathscr{A})}\mathscr{A}^n_{\mathscr{A}}$ is a $(\mathrm{M}_n(\mathscr{A}), \mathscr{A})$-bimodule with $\mathrm{M}_n(\mathscr{A}) \cdot \mathscr{A}^n = \mathscr{A}^n = \mathscr{A}^n \cdot \mathscr{A}$. Here we use that $\mathscr{A}$ is idempotent. Moreover, the canonical $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is clearly non-degenerate, full, and completely positive. The compatibility with the left $\mathrm{M}_n(\mathscr{A})$-module structure is obvious as the *-involution of $\mathrm{M}_n(\mathscr{A})$ is defined precisely this way. Now let $A = (a_{ij}) \in \mathrm{M}_n(\mathscr{A})$ be given. Then we find $b_{ijk}, c_{ijk} \in \mathscr{A}$ with $a_{ij} = \sum_k b_{ijk} c^*_{ijk}$ since $\mathscr{A}$ is idempotent. Now let $x_{ijk}, y_{ijk} \in \mathscr{A}^n$ be the vectors with b_{ijk} and c_{ijk}, respectively, at the i-th position and zeros elsewhere. Then $\Theta_{x_{ijk}, y_{ijk}}$ is the matrix with $b_{ijk} c^*_{ijk}$ at the (i, j)-th position and zeros elsewhere. Thus $A = \sum_{i,j,k} \Theta_{x_{ijk}, y_{ijk}}$ shows fullness. The compatibility of the two inner products is tautological as this is precisely the definition of the rank one operator $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_{\mathscr{A}}$. Let $x \in \mathscr{A}^n$ and assume that $\Theta_{x,y} = 0$ for all y. Then $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_{\mathscr{A}} = 0$ for all y, z. Since $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is full and $\mathscr{A}^n \cdot \mathscr{A} = \mathscr{A}^n$ it follows immediately that $x = 0$. Thus the $\mathrm{M}_n(\mathscr{A})$-valued inner product is non-degenerate. Finally, let $x^1, \ldots, x^N \in \mathscr{A}^n$ then the matrix

$$\left(\Theta_{x^\alpha, x^\beta}\right) = \left(x^\alpha_i \left(x^\beta_j\right)^*\right) \in \mathrm{M}_{nN}(\mathscr{A})^+ = \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))^+$$

is positive by Lemma 2.1.12, *i.)*. This shows the complete positivity of the $\mathrm{M}_n(\mathscr{A})$-valued inner product. □

Remark 4.1.13 (Commutativity) In particular, C and $\mathrm{M}_n(\mathsf{C})$ are strongly Morita equivalent via the strong equivalence bimodule C^n. Thus it follows that *commutativity* is not preserved by strong Morita equivalence. Moreover, the above theorem gives many examples of *-algebras which are strongly Morita equivalent but not *-isomorphic.

4.1.3 First Functorial Aspects

Later on, it will not only be of interest to decided whether two *-algebras are strongly Morita equivalent but also in how many ways: thus we consider the classes of equivalence bimodules between two given *-algebras and turn them into categories with appropriate morphisms: A morphism of *-equivalence bimodules $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ is, as usual, a structure preserving map. In this case we require T to be adjointable with respect to both inner products, i.e. there is *one* map $T^*\colon {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ with

$$\langle T(x), y' \rangle^{\mathscr{E}'}_{\mathscr{A}} = \langle x, T^*(y') \rangle^{\mathscr{E}}_{\mathscr{A}} \quad \text{and} \quad {}_{\mathscr{B}}\langle T(x), y' \rangle^{\mathscr{E}'} = {}_{\mathscr{B}}\langle x, T^*(y') \rangle^{\mathscr{E}} \tag{4.1.14}$$

for all $x \in \mathscr{E}$ and $y' \in \mathscr{E}'$. From the non-degeneracy of the inner products it follows at once that T and T^* are necessarily bimodule morphisms. Clearly, this way we obtain a good notion of morphisms allowing to state the following definition:

Definition 4.1.14 (Category of equivalence bimodules) *The category of *-equivalence bimodules from $\mathscr{A}$ to $\mathscr{B}$ is denoted by $\underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$. The category of strong equivalence bimodules is denoted by $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$.*

Of course it may well be that the category $\underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ or $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$ is empty, i.e. there is no equivalence bimodule between $\mathscr{A}$ and $\mathscr{B}$. Indeed, we shall see that the existence of a *-equivalence or even a strong equivalence bimodule is a highly non-trivial condition. The notation $\underline{\mathsf{Pic}}^*$ and $\underline{\mathsf{Pic}}^{\mathrm{str}}$ will become clear when we introduce the Picard groupoids in Chapter 5.

We conclude this section with some functorial aspects of the tensor product $\widetilde{\otimes}$ and the complex conjugation of equivalence bimodules.

Proposition 4.1.15 *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be non-degenerate and idempotent *-algebras.*

i.) The complex conjugation gives an equivalence (in fact: an isomorphism) of categories

$$\overline{}\colon \underline{\mathsf{Pic}}^*(\mathscr{B},\mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^*(\mathscr{A},\mathscr{B}) \tag{4.1.15}$$

and

$$\overline{}\colon \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B},\mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{A},\mathscr{B}), \tag{4.1.16}$$

where for a morphism $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ *one sets*

$$\overline{T}\colon {}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}} \ni \overline{x} \mapsto \overline{T}(\overline{x}) = \overline{T(x)} \in {}_{\mathscr{A}}\overline{\mathscr{E}}'_{\mathscr{B}}. \tag{4.1.17}$$

The inverse of complex conjugation is again the complex conjugation and we have

$$\overline{T^*} = \overline{T}^*. \tag{4.1.18}$$

ii.) The internal tensor product $\widetilde{\otimes}$ *yields functors*

$$\widetilde{\otimes}_{\mathscr{B}}\colon \underline{\mathsf{Pic}}^*(\mathscr{C},\mathscr{B}) \times \underline{\mathsf{Pic}}^*(\mathscr{B},\mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^*(\mathscr{C},\mathscr{A}) \tag{4.1.19}$$

and

$$\widetilde{\otimes}_{\mathscr{B}}\colon \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{C},\mathscr{B}) \times \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B},\mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{C},\mathscr{A}), \tag{4.1.20}$$

where $\widetilde{\otimes}_{\mathscr{B}}$ *is defined in the usual way on morphisms. We have*

$$(S \mathbin{\widetilde{\otimes}_{\mathscr{B}}} T)^* = S^* \mathbin{\widetilde{\otimes}_{\mathscr{B}}} T^*. \tag{4.1.21}$$

PROOF: For the first part we have to show that the complex conjugation is functorial. Here we can rely on Exercise 2.4.10 and modify those arguments slightly to incorporate both inner products. Thus let $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ be an adjointable bimodule morphism with adjoint T^* with respect to both inner products. Then we have

$$a \cdot \overline{T}(\overline{x}) = a \cdot \overline{T(x)} = \overline{T(x) \cdot a^*} = \overline{T(x \cdot a^*)} = \overline{T}(\overline{x \cdot a^*}) = \overline{T}(a \cdot \overline{x})$$

as well as

$$\overline{T}(\overline{x}) \cdot b = \overline{T(x)} \cdot b = \overline{b^* \cdot T(x)} = \overline{T(b^* \cdot x)} = \overline{T}(\overline{b^* \cdot x}) = \overline{T}(\overline{x} \cdot b)$$

for all $a \in \mathscr{A}$, $b \in \mathscr{B}$ and $x \in \mathscr{E}$. Thus $\overline{T}$ is indeed a bimodule morphism. Moreover, we have

$$\big\langle \overline{x}, \overline{T}(\overline{y})\big\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} = \big\langle \overline{x}, \overline{T(y)}\big\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} = {}_{\mathscr{B}}\langle x, T(y)\rangle^{\mathscr{E}} = {}_{\mathscr{B}}\langle T^*(x), y\rangle^{\mathscr{E}} = \big\langle \overline{T^*(x)}, \overline{y}\big\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} = \big\langle \overline{T^*}(\overline{x}), \overline{y}\big\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}}$$

for all $x, y \in \mathscr{E}$ and analogously for ${}_{\mathscr{A}}\langle \,\cdot\,, \,\cdot\,\rangle^{\overline{\mathscr{E}}}$. Hence $\overline{T}$ is adjointable with adjoint given as in (4.1.18). This shows that $\overline{T}$ is indeed a morphism in $\underline{\mathsf{Pic}}^*(\mathscr{A},\mathscr{B})$ or $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{A},\mathscr{B})$, respectively. The functoriality $\overline{\mathrm{id}_{\mathscr{E}}} = \mathrm{id}_{\overline{\mathscr{E}}}$ and $\overline{S \circ T} = \overline{S} \circ \overline{T}$ is obvious from the definition. Clearly, we can exchange the role of $\mathscr{A}$ and $\mathscr{B}$ to see that the inverse functor (now really an inverse) of the complex conjugation is again the complex conjugation. Thus we have here an isomorphism of categories. For the second part we have to consider two morphisms $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ and $S\colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \longrightarrow {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}}$. Then we have to show that $S \otimes_{\mathscr{B}} T$ is well-defined on the internal tensor product ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. It follows from Lemma 3.1.6 that $S \mathbin{\widetilde{\otimes}_{\mathscr{B}}} T$ coincides with $S \mathbin{\widehat{\otimes}_{\mathscr{B}}} T$ if we just take the $\mathscr{A}$-valued inner product into account. Thus $S \mathbin{\widetilde{\otimes}_{\mathscr{B}}} T$ is adjointable again by Lemma 3.1.6 with adjoint given by $S^* \mathbin{\widehat{\otimes}} T^*$. The argument for the $\mathscr{C}$-valued inner product is analogous. Since both degeneracy spaces coincide by Lemma 4.1.8 the tensor product $S \mathbin{\widetilde{\otimes}_{\mathscr{B}}} T$ is well-defined on the quotient indeed and the two adjoints coincide. Then the functoriality is easy to see. □

As already for the internal tensor product $\widehat{\otimes}$ we shall also simply write $\widetilde{\otimes}$ for $\widetilde{\otimes}_{\mathscr{B}}$ as soon as the algebra is clear from the context.

4.2 The Structure of Equivalence Bimodules

We shall now examine the structure of equivalence bimodules more closely. It will turn out that the fundamental example ${}_{\mathrm{M}_n(\mathscr{A})}\mathscr{A}^n_{\mathscr{A}}$ contains already most of the relevant features.

4.2.1 Ara's Theorem on *-Equivalence Bimodules

We start with the following theorem on *-Morita equivalence slightly adapted to our framework of *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, see Ara's work [1]:

Theorem 4.2.1 (Ara) *Let $\mathscr{A}$ be a non-degenerate and idempotent *-algebra over C.*

i.) If $\mathscr{E}_{\mathscr{A}}$ is a right $\mathscr{A}$-module with full and non-degenerate $\mathscr{A}$-valued inner product such that $\mathscr{E}_{\mathscr{A}} \cdot \mathscr{A} = \mathscr{E}_{\mathscr{A}}$, then ${}_{\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\mathscr{E}_{\mathscr{A}}$ is a *-equivalence bimodule via the canonical $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$-valued inner product

$$
{}_{\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\langle x, y\rangle = \Theta_{x,y}. \tag{4.2.1}
$$

Moreover, $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is idempotent and non-degenerate.

ii.) Conversely, if ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a *-equivalence bimodule then the left action

$$
\mathscr{B} \ni b \;\mapsto\; (x \mapsto b \cdot x) \in \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \tag{4.2.2}
$$

yields a *-isomorphism if $\mathscr{B}$ is non-degenerate. In this case, $\mathscr{B}$ is necessarily idempotent. Under this *-isomorphism, ${}_{\mathscr{B}}\langle \cdot\,, \cdot\rangle$ corresponds to $\Theta_{\cdot,\cdot}$.

Proof: Let $\mathscr{E}_{\mathscr{A}}$ be an inner-product module over $\mathscr{A}$ with $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$ and a full inner product $\langle \cdot\,, \cdot\rangle_{\mathscr{A}}$. Clearly, ${}_{\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\mathscr{E}_{\mathscr{A}}$ is an inner-product bimodule. Since $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \subseteq \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$, also $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ acts canonically from the left in a way compatible with $\langle \cdot\,, \cdot\rangle_{\mathscr{A}}$. Moreover, the definition (4.2.1) is automatically an $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$-valued inner product which has the correct sesquilinearity properties. Indeed, this follows from the computations in the proof of Lemma 2.1.7. Finally, we have $\Theta_{x\cdot a,y} = \Theta_{x,y\cdot a^*}$ by a simple computation. The compatibility of the inner products ${}_{\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\langle \cdot\,, \cdot\rangle$ and $\langle \cdot\,, \cdot\rangle_{\mathscr{A}}$ is precisely the definition of $\Theta_{\cdot,\cdot}$. Next we observe that $\Theta_{\cdot,\cdot}$ is full by the very definition of the finite-rank operators. Note that this is the reason that we have to take $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ instead of $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ in general. To show that $\Theta_{\cdot,\cdot}$ is a non-degenerate inner product, let $y \in \mathscr{E}_{\mathscr{A}}$ such that $\Theta_{x,y} = 0$ for all x. Then $0 = \langle w, \Theta_{x,y}(z)\rangle_{\mathscr{A}} = \langle w, x \cdot \langle y, z\rangle_{\mathscr{A}}\rangle_{\mathscr{A}} = \langle w, x\rangle_{\mathscr{A}} \langle y, z\rangle_{\mathscr{A}}$ for all $x, z, w \in \mathscr{E}$. Since $\langle \cdot\,, \cdot\rangle_{\mathscr{A}}$ is full, we conclude that $a\langle y, z\rangle_{\mathscr{A}} = 0$ for all $a \in \mathscr{A}$. Since $\mathscr{A}$ is non-degenerate this implies $\langle y, z\rangle_{\mathscr{A}} = 0$. Thus $y = 0$ follows from the non-degeneracy of $\langle \cdot\,, \cdot\rangle_{\mathscr{A}}$ which implies that $\Theta_{\cdot,\cdot}$ is non-degenerate, too. Let now $z \in \mathscr{E}_{\mathscr{A}}$ be given. Then we find $a_i \in \mathscr{A}$ and $x_i \in \mathscr{E}_{\mathscr{A}}$ with $z = \sum_i x_i \cdot a_i$ by the assumption $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$. Since $\langle \cdot\,, \cdot\rangle_{\mathscr{A}}$ is full, we can write $a_i = \sum_j \langle y_{ij}, z_{ij}\rangle_{\mathscr{A}}$ with suitably chosen $y_{ij}, z_{ij} \in \mathscr{E}$. Thus

$$
z = \sum\nolimits_i x_i \cdot \langle y_{ij}, z_{ij}\rangle_{\mathscr{A}} = \sum\nolimits_{i,j} \Theta_{x_i, y_{ij}}(z_{ij}),
$$

which proves $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \cdot \mathscr{E} = \mathscr{E}$. Hence ${}_{\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\mathscr{E}_{\mathscr{A}}$ is indeed a *-equivalence bimodule. Next we show that $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is necessarily non-degenerate and idempotent, generalizing the results of Exercise 2.4.5. Let $x, y \in \mathscr{E}_{\mathscr{A}}$ be given. Then we write $y = \sum_i \Theta_{x_i,y_i}(z_i) = x_i \cdot \langle y_i, z_i\rangle_{\mathscr{A}}$ with appropriate $x_i, y_i, z_i \in \mathscr{E}_{\mathscr{A}}$. For all z we have

$$
\begin{aligned}
\Theta_{x,y}(z) &= x \cdot \langle y, z\rangle_{\mathscr{A}} \\
&= \sum\nolimits_i x \cdot \langle x_i \cdot \langle y_i, z_i\rangle_{\mathscr{A}}, z\rangle_{\mathscr{A}} \\
&= \sum\nolimits_i x \cdot (\langle y_i, z_i\rangle_{\mathscr{A}})^* \cdot \langle x_i, z\rangle_{\mathscr{A}} \\
&= \sum\nolimits_i x \cdot (\langle z_i, y_i\rangle_{\mathscr{A}} \langle x_i, z\rangle_{\mathscr{A}})
\end{aligned}
$$

$$= \sum_i x \cdot \langle z_i, y_i \cdot \langle x_i, z \rangle_{\mathscr{A}} \rangle_{\mathscr{A}}$$
$$= \sum_i \Theta_{x,z_i}(\Theta_{y_i,x_i}(z)),$$

showing $\Theta_{x,y} = \sum_i \Theta_{x,z_i} \circ \Theta_{y_i,x_i}$. This implies that $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is idempotent. Now let $A \in \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ with $BA = 0$ for all $B \in \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Then we have $0 = \Theta_{x,y}(A(z)) = \Theta_{x,A^*y}(z)$ for all $x, y, z \in \mathscr{E}_{\mathscr{A}}$. Since this holds for all x we conclude $A^*y = 0$ as $\Theta_{\cdot,\cdot}$ is non-degenerate. Thus $A = 0$ which implies that $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is non-degenerate. This finally proves the first part. So, let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a *-equivalence bimodule. We first have to show that the left multiplication with $b \in \mathscr{B}$ is a finite-rank operator. Thus let $b \in \mathscr{B}$ be given and choose $x_i, y_i \in \mathscr{E}$ with $b = \sum_i {}_{\mathscr{B}}\langle x_i, y_i \rangle$ by fullness of ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle$. Thus

$$b \cdot z = \sum_i {}_{\mathscr{B}}\langle x_i, y_i \rangle \cdot z = \sum_i x_i \cdot \langle y_i, z \rangle_{\mathscr{A}} = \sum_i \Theta_{x_i,y_i}(z),$$

shows that $z \mapsto b \cdot z$ is in $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Clearly, (4.2.2) is a *-homomorphism as the left $\mathscr{B}$-module structure is compatible with the $\mathscr{A}$-valued inner product. Now let $b = {}_{\mathscr{B}}\langle x, y \rangle$ then

$$b \cdot z = {}_{\mathscr{B}}\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{\mathscr{A}} = \Theta_{x,y}(z)$$

shows that the homomorphism (4.2.2) maps the $\mathscr{B}$-valued inner product to $\Theta_{\cdot,\cdot}$. With the linearity of (4.2.2), this immediately shows the surjectivity of (4.2.2). Under the condition that $\mathscr{B}$ is non-degenerate, (4.2.2) is also injective by Lemma 4.1.7, applied to left modules instead of right modules. Hence, we obtain a *-isomorphism. Since by the first part, $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is idempotent, $\mathscr{B}$ is necessarily idempotent, too. □

Corollary 4.2.2 *Within the class of non-degenerate *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, idempotency is preserved under *-Morita equivalence.*

In particular, this shows that the choice of restricting to idempotent *-algebras was a good choice a posteriori.

Example 4.2.3 The property of having a unit is not preserved under *-Morita equivalence: The simplest example is the *-equivalence bimodule $\mathsf{C}_{\mathsf{C}}^{(\infty)}$: the inner product is clearly full and hence $\mathfrak{F}(\mathsf{C}^{(\infty)}) \cong \mathrm{M}_\infty(\mathsf{C})$ is *-Morita equivalent to C. However, $\mathfrak{F}(\mathsf{C}^{(\infty)})$ does not have a unit element and $\mathfrak{F}(\mathsf{C}^{(\infty)}) \neq \mathfrak{B}(\mathsf{C}^{(\infty)})$.

The general strategy to find *-Morita equivalent *-algebras for a given *-algebra $\mathscr{A}$ consist now in finding right $\mathscr{A}$-modules with a *full* non-degenerate $\mathscr{A}$-valued inner product. In this case, the *-algebra $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ will be *-Morita equivalent and, up to *-isomorphism, all *-Morita equivalent *-algebras arise this way.

For strong Morita equivalence, the situation is more complicated: Even if $(\mathscr{E}_{\mathscr{A}}, \langle \cdot\,, \cdot \rangle_{\mathscr{A}})$ is a right $\mathscr{A}$-module with a full, non-degenerate and completely positive inner product we only know that $\mathscr{A}$ is *-Morita equivalent to $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. For strong Morita equivalence we still have to show that the $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$-valued inner product $\Theta_{\cdot,\cdot}$ is *completely positive*, too. Under particular circumstances, one can show this, in general the situation is unclear. We mention that for C^*-algebras the (complete) positivity of $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ implies the (complete) positivity of $\Theta_{\cdot,\cdot}$, see e.g. [79, Lem. 4.1] as well as Exercise 3.3.5. We will come back to this question later.

4.2.2 The Case of Unital *-Algebras

The whole situation simplifies drastically for unital *-algebras.

Proposition 4.2.4 *Let $\mathscr{A}$, $\mathscr{B}$ be unital *-algebras and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a *-equivalence bimodule. Then*

$$\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \cong \mathscr{B} \quad \textit{and} \quad \mathfrak{B}_{\mathscr{B}}({}_{\mathscr{B}}\mathscr{E}) = \mathrm{End}_{\mathscr{B}}({}_{\mathscr{B}}\mathscr{E}) \cong \mathscr{A} \tag{4.2.3}$$

via the left and right multiplications, respectively.

PROOF: We know from Theorem 4.2.1 that $\mathscr{B} \cong \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ via the left multiplications. Since $\mathscr{B}$ has a unit, it follows from $\mathscr{B} \cdot \mathscr{E} = \mathscr{E}$ that $\mathbb{1}_{\mathscr{B}} \cdot x = x$, showing $\mathrm{id}_{\mathscr{E}} \in \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Since $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is a *-ideal in $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ we conclude $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Since $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ is a left ideal in $\mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ by Lemma 2.1.7, we can still conclude that $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$, proving the first statement. The second follows from symmetry in $\mathscr{A}$ and $\mathscr{B}$. ☐

Corollary 4.2.5 *Let $\mathscr{A}, \mathscr{B}$ be unital *-algebras and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a *-equivalence bimodule. Then $\mathscr{E}_{\mathscr{A}}$ as well as ${}_{\mathscr{B}}\mathscr{E}$ are finitely generated projective modules over $\mathscr{A}$ and $\mathscr{B}$, respectively. Moreover, both inner products are strongly non-degenerate and allow Hermitian dual bases.*

PROOF: This is now clear from Proposition 4.2.4 and Proposition 2.3.12. ☐

Remark 4.2.6 The last corollary shows that for a unital *-algebra $\mathscr{A}$ we have to search for *-equivalence bimodules inside $\underline{\mathsf{Proj}}^*(\mathscr{A})$ which we used as starting point for the construction of the K_0^*-theory of the corresponding *-algebras. Thus *-equivalence bimodules correspond to particular elements in $\underline{\mathsf{Proj}}^*(\mathscr{A})$ and strong equivalence bimodules come from $\underline{\mathsf{Proj}}^{\mathrm{str}}(\mathscr{A})$, respectively. In general, however, not every $\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Proj}}^*(\mathscr{A})$ will give a *-equivalence bimodule between $\mathscr{A}$ and $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ since in addition we need a *full* $\mathscr{A}$-valued inner product.

The next proposition shows that the usual quotient procedure in the internal tensor product is unnecessary once we have Hermitian dual bases. In particular, this applies for *-equivalence bimodules between unital *-algebras:

Proposition 4.2.7 *Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras and $\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be inner-product (bi-) modules such that as right modules $\mathscr{F}_{\mathscr{B}}$ and $\mathscr{E}_{\mathscr{A}}$ have finite Hermitian dual bases. Then the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$ on $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ allows for a finite Hermitian dual basis, too. In particular, the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$ is strongly non-degenerate and $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a finitely generated and projective right $\mathscr{A}$-module.*

PROOF: Let $\phi_\alpha, \psi_\alpha \in \mathscr{F}_{\mathscr{B}}$ and $x_i, y_i \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be the finite Hermitian dual bases, i.e. we have

$$\phi = \sum_\alpha \phi_\alpha \cdot \langle \psi_\alpha, \phi \rangle_{\mathscr{B}} \quad \text{and} \quad x = \sum_i x_i \cdot \langle y_i, x \rangle_{\mathscr{A}}$$

for all $\phi \in \mathscr{F}_{\mathscr{B}}$ and $x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. This allows to compute

$$\begin{aligned}
\sum\nolimits_{\alpha,i} (\phi_\alpha \otimes x_i) \cdot \langle \psi_\alpha \otimes y_i, \phi \otimes x \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}} &= \sum\nolimits_{\alpha,i} \phi_\alpha \otimes \Big(x_i \cdot \big\langle y_i, \langle \psi_\alpha, \phi \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x \big\rangle_{\mathscr{A}}^{\mathscr{E}} \Big) \\
&= \sum\nolimits_{\alpha} \phi_\alpha \otimes \big(\langle \psi_\alpha, \phi \rangle_{\mathscr{B}}^{\mathscr{F}} \cdot x \big) \\
&= \sum\nolimits_{\alpha} \big(\phi_\alpha \cdot \langle \psi_\alpha, \phi \rangle_{\mathscr{B}}^{\mathscr{F}} \big) \otimes x \\
&= \phi \otimes x.
\end{aligned}$$

Since the elementary tensors $\phi \otimes x$ span the tensor product, we see that the set $\{\phi_\alpha \otimes x_i, \psi_\alpha \otimes y_i\}$ constitutes a finite Hermitian dual basis for $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$. By Proposition 2.3.12 the inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{F}\otimes\mathscr{E}}$ is strongly non-degenerate and $\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is finitely generated and projective. ☐

Note that the above proof gives an explicit Hermitian dual basis if the Hermitian dual bases of $\mathscr{F}_{\mathscr{B}}$ and $\mathscr{E}_{\mathscr{A}}$ are known. In this case, the internal tensor product $\widehat{\otimes}$ does not require the quotient procedure as in Definition 3.1.3 since the inner product is already non-degenerate.

Corollary 4.2.8 *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be unital *-algebras and let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be *-equivalence bimodules. Then the inner products on ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ are already non-degenerate and hence*

$$
{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \widetilde{\otimes}_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}. \tag{4.2.4}
$$

Proof: This follows now from Corollary 4.2.5 and Proposition 4.2.7. □

Since for unital *-algebras, a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is of the form $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n$ for some idempotent $e = e^2 \in \mathrm{M}_n(\mathscr{A})$, it raises the question which strongly non-degenerate inner products on $e\mathscr{A}^n$ exist and which of them are full. The first question was already discussed and is encoded in the semi-group morphisms

$$
\mathsf{Proj}^*(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A}) \quad \text{and} \quad \mathsf{Proj}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A}), \tag{4.2.5}
$$

respectively. This suggests to formulate the second question about fullness also in term of the idempotent e alone. Note that the (by assumption unital) *-algebra $\mathscr{B}$ is uniquely determined to be $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ up to *-isomorphism by Theorem 4.2.1. We recall the following definition:

Definition 4.2.9 (Full idempotent) *An idempotent element $e = e^2 \in \mathrm{M}_n(\mathscr{A})$ is called full if the two-sided ideal $\mathscr{A}e\mathscr{A}$ in $\mathscr{A}$ generated from the coefficients e_{ij} of e coincides with $\mathscr{A}$.*

In case where $e = P$ is even a projection we have the following characterization of fullness:

Proposition 4.2.10 *Let $\mathscr{A}$ be a unital *-algebra and $P = P^2 = P^* \in \mathrm{M}_n(\mathscr{A})$.*

- *i.) The subset $P\mathrm{M}_n(\mathscr{A})P$ consisting of elements PAP with $A \in \mathrm{M}_n(\mathscr{A})$ is a *-subalgebra of $\mathrm{M}_n(\mathscr{A})$ with unit P.*
- *ii.) One has $\mathrm{End}_{\mathscr{A}}(P\mathscr{A}^n) = \mathfrak{B}_{\mathscr{A}}(P\mathscr{A}^n) = \mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n) \cong P\mathrm{M}_n(\mathscr{A})P$.*
- *iii.) The canonical inner product on $P\mathscr{A}^n$ is full if and only if the projection P is full. In this case, $P\mathrm{M}_n(\mathscr{A})P$ and $\mathscr{A}$ are *-Morita equivalent via the *-equivalence bimodule $P\mathscr{A}^n$.*

Proof: The first part is a simple verification. Since $P\mathscr{A}^n$ has a finite Hermitian dual basis, namely $\{Pe_i, Pe_i\}_{i=1,\ldots,n}$, we have $\mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n) = \mathfrak{B}_{\mathscr{A}}(P\mathscr{A}^n)$ by Proposition 2.3.12. With the same argument as in the proof of Proposition 4.2.4, we have $\mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n) = \mathrm{End}_{\mathscr{A}}(P\mathscr{A}^n)$. Now we want to establish the isomorphism to $P\mathrm{M}_n(\mathscr{A})P$. Suppose $A \in \mathrm{End}_{\mathscr{A}}(P\mathscr{A}^n)$ is given. Then we can extend A to an endomorphism of $\mathscr{A}^n$ by setting A equal to zero on $(\mathbb{1}-P)\mathscr{A}^n$ thanks to the direct sum decomposition $\mathscr{A}^n = P\mathscr{A}^n \oplus (\mathbb{1}-P)\mathscr{A}^n$. Thus we can identify A with an element in $\mathrm{M}_n(\mathscr{A})$. By construction of this identification we have $A = PAP$. Conversely, let $A = PAP \in P\mathrm{M}_n(\mathscr{A})P$ be given. Viewing A as endomorphism of $\mathscr{A}^n$ we have $A\big|_{(\mathbb{1}-P)\mathscr{A}^n} = 0$. Hence the restriction of A to $P\mathscr{A}^n$ is an injective map $P\mathrm{M}_n(\mathscr{A})P \longrightarrow \mathrm{End}_{\mathscr{A}}(P\mathscr{A}^n)$. It is clearly the inverse of the extension described before and an algebra isomorphism. We have to show that this is even a *-isomorphism. Thus let $A = PAP \in P\mathrm{M}_n(\mathscr{A})P$. We have $A^* = PA^*P$ since $P^* = P$. From this it follows immediately, that A^* is indeed the adjoint with respect to the inner product of $P\mathscr{A}^n$ as this is just the restriction of the canonical one on $\mathscr{A}^n$. This shows the second part. For the third we assume that $\langle \cdot, \cdot \rangle$, restricted to $P\mathscr{A}^n$, is still a full inner product. Then we find $x_\alpha, y_\alpha \in P\mathscr{A}^n$ with

$$
\mathbb{1}_{\mathscr{A}} = \sum\nolimits_\alpha \langle x_\alpha, y_\alpha \rangle = \sum\nolimits_\alpha \langle x_\alpha, P y_\alpha \rangle = \sum\nolimits_{\alpha,i,j} x_{\alpha i}^* P_{ij} y_{\alpha j},
$$

where $x_{\alpha i}$ and $y_{\alpha j}$ are the components of x_α and y_α, respectively. Thus $\mathbb{1}_{\mathscr{A}}$ is contained in the two-sided ideal generated by P and hence P is full. Conversely, let P be full and let $x_{\alpha i}, y_{\alpha j} \in \mathscr{A}$ be given such that $\mathbb{1}_{\mathscr{A}} = \sum_{\alpha,i} x^*_{\alpha i} P_{ij} y_{\alpha,j}$. Without restriction, we can assume that the index α runs over the same index set for every $i = 1, \ldots, n$. This defines $x_\alpha, y_\alpha \in \mathscr{A}^n$ via their components $x_{\alpha,i}$ and $y_{\alpha,i}$. Note that in general x_α, y_α are not in $P\mathscr{A}^n$. Nevertheless, we have

$$\mathbb{1}_{\mathscr{A}} = \sum\nolimits_\alpha \langle x_\alpha, P y_\alpha \rangle = \sum\nolimits_\alpha \langle P x_\alpha, P y_\alpha \rangle,$$

since $P^2 = P = P^*$. Thus $\langle \cdot\,, \cdot \rangle$, restricted to $P\mathscr{A}^n$, is still full. By Theorem 4.2.1 the statement on *-Morita equivalence follows. □

The fullness property of an idempotent element $e \in \mathrm{M}_n(\mathscr{A})$ depends only on its equivalence class in $\mathsf{Proj}(\mathscr{A})$:

Lemma 4.2.11 *Let $e, f \in \mathrm{M}_\infty(\mathscr{A})$ be idempotent and equivalent in the sense of Proposition 2.3.5. Then e is full if and only if f is full.*

PROOF: More generally, we show that the ideals $\mathscr{A}e\mathscr{A}$ and $\mathscr{A}f\mathscr{A}$ coincide whenever e and f are equivalent. By assumption and Proposition 2.3.5 we have $e, f \in \mathrm{M}_n(\mathscr{A})$ for a suitable $n \in \mathbb{N}$ such that $e = VfV^{-1}$ with some invertible $V \in \mathrm{M}_n(\mathscr{A})$. Thus $e_{ij} = \sum_{k,\ell} V_{ik} f_{k\ell} (V^{-1})_{\ell j} \in \mathscr{A}f\mathscr{A}$. This implies $\mathscr{A}e\mathscr{A} \subseteq \mathscr{A}f\mathscr{A}$. By exchanging e and f we finally obtain $\mathscr{A}e\mathscr{A} = \mathscr{A}f\mathscr{A}$. □

In particular, this lemma allows us to say that a finitely generated projective module $\mathscr{E}_{\mathscr{A}}$ is *full* if one and hence all idempotents $e \in \mathrm{M}_n(\mathscr{A})$ with $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n$ are full. Thus we see that the question whether or not $e\mathscr{A}^n$ yields a *-equivalence bimodule only depends on the class of e in $\mathsf{Proj}(\mathscr{A})$, but not on the choice of e itself.

4.2.3 Strong Morita Equivalence with (K) and (H)

Let now $\mathscr{A}$ and $\mathscr{B}$ be *-Morita equivalent unital *-algebras with a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. Then we know that $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n$ as right $\mathscr{A}$-modules for some idempotent $e \in \mathrm{M}_n(\mathscr{A})$ and $\mathscr{B} \cong \mathrm{End}_{\mathscr{A}}(e\mathscr{A}^n)$ as associative algebras via the left multiplications. In general,

$$\mathrm{End}_{\mathscr{A}}(e\mathscr{A}^n) \cong e\mathrm{M}_n(\mathscr{A})e, \tag{4.2.6}$$

which follows analogously to Proposition 4.2.10, *ii.)*, ignoring the *-involution. Without further assumptions, $e\mathscr{A}^n$ may admit many non-isometric full inner products which will determine the *-involution of $\mathscr{B}$ via the compatibility between the $\mathscr{B}$-valued and the $\mathscr{A}$-valued inner product. Without further knowledge about these possibilities the analysis stops here and we can not make proper use of Proposition 4.2.10. However, the situation simplifies further, if only one $\mathscr{A}$-valued inner product is possible [29, Thm. 7.3]. This situation is guaranteed by the properties **(K)** and **(H$^-$)**:

Theorem 4.2.12 (Equivalence bimodules with (K) and (H)) *Let $\mathscr{A}$ and $\mathscr{B}$ be unital *-algebras such that $\mathscr{A}$ fulfills* **(K)** *and* **(H$^-$)**. *Moreover, let $({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, \langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}, {}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}})$ be a *-equivalence bimodule such that $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ is completely positive. Then we have:*

i.) There exists a projection $P = P^2 = P^* \in \mathrm{M}_n(\mathscr{A})$ such that $(\mathscr{E}_{\mathscr{A}}, \langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}})$ is isometrically isomorphic to $(P\mathscr{A}^n, \langle \cdot\,, \cdot \rangle_{\mathscr{A}})$ as pre-Hilbert module over $\mathscr{A}$.

ii.) The algebra $\mathscr{B}$ is *-isomorphic to $P\mathrm{M}_n(\mathscr{A})P$ via the left multiplications of $\mathscr{B}$ on $\mathscr{E}_{\mathscr{A}} \cong P\mathscr{A}^n$.

iii.) Under this isomorphism, ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}$ corresponds to the canonical $P\mathrm{M}_n(\mathscr{A})P$-valued inner product $\Theta_{\cdot,\cdot}$ on $P\mathscr{A}^n$.

iv.) The inner product ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}$ is completely positive and hence ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is even a strong Morita equivalence bimodule.

v.) Conversely, if $P \in \mathrm{M}_n(\mathscr{A})$ is a full projection then $P\mathscr{A}^n$ with the canonical inner products is a strong $(P\mathrm{M}_n(\mathscr{A})P, \mathscr{A})$-equivalence bimodule.

PROOF: Since $\mathscr{E}_{\mathscr{A}}$ is a finitely generated and projective module by Corollary 4.2.5, we find a projection $P = P^* = P^2$ with $\mathscr{E}_{\mathscr{A}} \cong P\mathscr{A}^n$ as right $\mathscr{A}$-modules by Theorem 2.3.16. By Corollary 4.2.5 the inner product $\langle \cdot, \cdot \rangle_{\mathscr{A}}$ is strongly non-degenerate and by Proposition 2.3.23 isometrically isomorphic to the canonical inner product on $P\mathscr{A}^n$. This shows the first part. We can assume $\mathscr{E}_{\mathscr{A}} = P\mathscr{A}^n$ and $\langle \cdot, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle \cdot, \cdot \rangle_{\mathscr{A}}$ from now on. Then $\mathscr{B}$ is *-isomorphic to $\mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n)$ by Theorem 4.2.1 via the left multiplications. By Proposition 4.2.10 we have $\mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n) \cong P\mathrm{M}_n(\mathscr{A})P$ and hence $\mathscr{B}$ is *-isomorphic to $P\mathrm{M}_n(\mathscr{A})P$ yielding the second part. The compatibility of ${}_{\mathscr{B}}\langle \cdot, \cdot \rangle^{\mathscr{E}}$ and $\langle \cdot, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle \cdot, \cdot \rangle_{\mathscr{A}}$ fixes ${}_{\mathscr{B}}\langle \cdot, \cdot \rangle^{\mathscr{E}}$ uniquely, hence it corresponds to $\Theta_{\cdot,\cdot}$, thereby showing the third part. For the fourth part it is sufficient to show that $\Theta_{\cdot,\cdot}$ is completely positive. Since $\langle \cdot, \cdot \rangle_{\mathscr{A}}$ is full, we find $Px_1, \ldots, Px_k, Py_1, \ldots, Py_k$ with

$$\mathbb{1}_{\mathscr{A}} = \sum_{r=1}^{k} \langle Px_r, Py_r \rangle_{\mathscr{A}} = \sum_{r=1}^{k} \langle Py_r, Px_r \rangle_{\mathscr{A}},$$

since $\mathbb{1}^*_{\mathscr{A}} = \mathbb{1}_{\mathscr{A}}$. Note however, that the single terms in the sum are not necessarily Hermitian. We obtain

$$\sum_{r=1}^{k} \langle Px_r + Py_r, Px_r + Py_r \rangle_{\mathscr{A}} = \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{A}} + \sum_{r=1}^{k} \langle Px_r, Px_r \rangle_{\mathscr{A}} + \sum_{r=1}^{k} \langle Py_r, Py_r \rangle_{\mathscr{A}} = \mathbb{1}_{\mathscr{A}} + \sum_{\alpha} a^*_\alpha a_\alpha, \quad (*)$$

with certain $a_\alpha \in \mathscr{A}$ which can be computed from the coefficients of Px_r and Py_r. Lemma 2.3.20 shows that $(*)$ is invertible. Moreover, since $(*)$ is algebraically positive we can apply the property $\mathbf{(H^-)}$ in order to find an invertible $u \in \mathscr{A}$ with

$$\sum_{r=1}^{k} \langle Px_r + Py_r, Px_r + Py_r \rangle_{\mathscr{A}} = u^* u.$$

With $Pz_r = P(x_r + y_r)u^{-1} \in P\mathscr{A}^n$ we find

$$\sum_{r=1}^{k} \langle Pz_r, Pz_r \rangle_{\mathscr{A}} = \mathbb{1}_{\mathscr{A}}.$$

We claim that this feature implies the complete positivity of the canonical $\mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n)$-valued inner product $\Theta_{\cdot,\cdot}$. Indeed, let $Px_1, \ldots, Px_N \in P\mathscr{A}^n$ be given. Then we have

$$\begin{aligned}
\big(\Theta_{Px_\alpha, Px_\beta}\big) &= \big(\Theta_{Px_\alpha \cdot \mathbb{1}_{\mathscr{A}}, Px_\beta}\big) \\
&= \sum_{r=1}^{k} \Big(\Theta_{Px_\alpha \cdot \langle Pz_r, Pz_r \rangle_{\mathscr{A}}, Py_\beta}\Big) \\
&= \sum_{r=1}^{k} \Big(\Theta_{\Theta_{Px_\alpha, Pz_r}(Pz_r), Px_\beta}\Big) \\
&= \sum_{r=1}^{k} \big(\Theta_{Px_\alpha, Pz_r} \Theta_{Pz_r, Px_\beta}\big) \\
&= \sum_{r=1}^{k} \big(\Theta^*_{Pz_r, Px_\alpha} \Theta_{Pz_r, Px_\beta}\big) \in \mathrm{M}_N(\mathfrak{F}_{\mathscr{A}}(P\mathscr{A}^n))^{++},
\end{aligned}$$

by Lemma 2.1.12, *i.)*. This shows the complete positivity of $\Theta_{\cdot,\cdot}$ and hence the fourth part. The fifth part follows from this as well. ☐

Since the properties **(K)** and **(H$^-$)** play such a central role both for Hermitian K_0-theory and for strong Morita theory it is interesting to investigate their behaviour under Morita equivalence: The following proposition will provide us a first result when we restrict ourselves to **(H)** or **(H$^+$)**, see also the discussion in [29, Prop. 7.4].

Proposition 4.2.13 *The properties* **(K)** *and* **(H$^+$)** *together as well as the properties* **(K)** *and* **(H)** *together are invariant under strong Morita equivalence.*

Proof: Assume $\mathscr{A}$ and $\mathscr{B}$ are unital *-algebras and $\mathscr{A}$ satisfies **(K)** and **(H$^+$)** or **(K)** and **(H)**, respectively. Moreover, assume that $\mathscr{A}$ and $\mathscr{B}$ are strongly Morita equivalent and hence $\mathscr{B} \cong P\mathrm{M}_n(\mathscr{A})P$ for some full projection $P = P^2 = P^* \in \mathrm{M}_n(\mathscr{A})$ thanks to Theorem 4.2.12. Thus we only have to consider $\mathscr{B} = P\mathrm{M}_n(\mathscr{A})P$. Note that $P \in P\mathrm{M}_n(\mathscr{A})P$ is the unit element of $\mathscr{B}$. Thus let $B \in \mathrm{M}_N(\mathscr{B})$ be given. We have to show that $\mathbb{1}_{\mathrm{M}_N(\mathscr{B})} + B^*B = P^{(N)} + B^*B$ is invertible in $\mathrm{M}_N(\mathscr{B})$ where $P^{(N)}$ is the $N \times N$ matrix in $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ with P's on the diagonal. Since $\mathrm{M}_N(\mathscr{B}) \subseteq \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ we consider first $\mathbb{1}_{\mathrm{M}_N(\mathscr{B})} + B^*B + (\mathbb{1} - P^{(N)}) = \mathbb{1} + B^*B$ which is invertible in $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ by **(K)** for $\mathscr{A}$. Since $B^*B \in \mathrm{M}_N(P\mathrm{M}_n(\mathscr{A})P) = P^{(N)}\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))P^{(N)}$ commutes with $P^{(N)}$, the inverse of $\mathbb{1} + B^*B$ commutes with $P^{(N)}$, too. Thus we consider $P^{(N)}(\mathbb{1} + B^*B)^{-1}P^{(N)} \in \mathrm{M}_N(\mathscr{B})$ and compute

$$
\begin{aligned}
P^{(N)}(\mathbb{1} + B^*B)^{-1}P^{(N)}\Big(P^{(N)} + B^*B\Big) &= P^{(N)}(\mathbb{1} + B^*B)^{-1}P^{(N)}\Big(P^{(N)} + (\mathbb{1} - P^{(N)}) + B^*B\Big) \\
&= P^{(N)}(\mathbb{1} + B^*B)^{-1}(\mathbb{1} + B^*B)P^{(N)} \\
&= P^{(N)},
\end{aligned}
$$

since $P^{(N)}(\mathbb{1} - P^{(N)}) = 0$. This shows that $\mathbb{1}_{\mathrm{M}_N(\mathscr{B})} + B^*B$ is invertible in $\mathrm{M}_N(\mathscr{B})$ with inverse given by $P^{(N)}(\mathbb{1} + B^*B)^{-1}P^{(N)}$. Hence $\mathscr{B}$ satisfies **(K)**, too.

Now let $H \in \mathrm{M}_N(\mathscr{B})^+$ be a positive and invertible matrix. Since $\mathscr{B} = P\mathrm{M}_n(\mathscr{A})P \subseteq \mathrm{M}_n(\mathscr{A})$ is a *-subalgebra, also $\mathrm{M}_N(\mathscr{B}) \subseteq \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ is a *-subalgebra. It follows that H viewed as an element of $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ is still positive by Remark 1.1.13, *iii.)*. However, H is no longer invertible, but $H + (\mathbb{1} - P^{(N)})$ is invertible. In fact, if H^{-1} is the inverse of H in $\mathrm{M}_N(\mathscr{B})$ then $H^{-1} + (\mathbb{1} - P^{(N)})$ is the inverse of $H + (\mathbb{1} - P^{(N)})$ in $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ since $H(\mathbb{1} - P^{(N)}) = 0 = (\mathbb{1} - P^{(N)})H$. Since $\mathbb{1} - P^{(N)} = (\mathbb{1} - P^{(N)})^*(\mathbb{1} - P^{(N)})$ is even algebraically positive, the matrix $H + (\mathbb{1} - P^{(N)})$ is still a positive and now invertible matrix in $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$.

So assume that $\mathscr{A}$ satisfies **(H$^+$)**. Then we find an invertible matrix $V \in \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ with $H + \mathbb{1} - P^{(N)} = V^*V$ and V commutes with any projection which commutes with $H + \mathbb{1} - P^{(N)}$. In particular, V commutes with $P^{(N)}$. Define $U = P^{(N)}VP^{(N)} \in \mathrm{M}_N(\mathscr{B})$ then $H = U^*U$. Moreover, U is invertible in $\mathrm{M}_N(\mathscr{B})$ with inverse given by $U^{-1} = P^{(N)}V^{-1}P^{(N)}$. Indeed, this is a simple computation using the fact that V commutes with $P^{(N)}$. Now let $Q = Q^* = Q^2 \in \mathrm{M}_N(\mathscr{B})$ be a projection with $[H, Q] = 0$. Viewing Q as element in $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ we have $QP^{(N)} = Q = P^{(N)}Q$ and hence $[H + \mathbb{1} - P^{(N)}, Q] = 0$. By **(H$^+$)** for $\mathscr{A}$, the matrix V commutes with Q. But this immediately implies that U also commutes with Q. Thus $\mathscr{B}$ satisfies the property **(H$^+$)**, too.

Alternatively, let us assume that $\mathscr{A}$ satisfies only **(H)**. Then, let $\{Q_\alpha\}_{\alpha=1,\ldots,k}$ be a finite orthogonal partition of unity in $\mathrm{M}_N(\mathscr{B})$ with $[Q_\alpha, H] = 0$. Viewing the Q_α as elements in $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$, we have $Q_\alpha P^{(N)} = Q_\alpha = P^{(N)}Q_\alpha$ and $\sum_\alpha Q_\alpha = P^{(N)}$. It follows that together with $Q_0 = \mathbb{1} - P^{(N)}$ we obtain an orthogonal partition of unity $\{Q_\alpha\}_{\alpha=0,\ldots,k}$ for $\mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$. Clearly, we have $[H + \mathbb{1} - P^{(N)}, Q_\alpha] = 0$ for all $\alpha = 0, \ldots, k$. Thus we can use **(H)** for $\mathscr{A}$ and find an invertible $V \in \mathrm{M}_N(\mathrm{M}_n(\mathscr{A}))$ with $H + \mathbb{1} - P^{(N)} = V^*V$ and $[V, Q_\alpha] = 0$ for $k = 0, \ldots, k$. Analogously to the case of **(H$^+$)** we can use $U = P^{(N)}VP^{(N)}$ and conclude that $\mathscr{B}$ satisfies **(H)**, too. □

Note that the property **(H$^-$)** seems to be more difficult to discuss in this context. Nevertheless the combinations **(K)** & **(H$^+$)** or **(K)** & **(H)** are very adapted to strong Morita equivalence since we will not leave the class of such *-algebras when passing to strongly Morita equivalent ones. From the proof we note that the property **(K)** is already invariant under the slightly weaker notion of *-Morita equivalence:

Corollary 4.2.14 *The property* **(K)** *is invariant under* **-Morita equivalence.*

4.3 The Bicategory Approach

In this section we shall embed the question of Morita equivalence into some bigger context: we want to find categories in which the notion of isomorphism coincides with the notion of Morita equivalence for each of our flavours of Morita equivalence: ring-theoretic, *-equivalence, and strong equivalence. We will present essentially two ways of achieving this goal by either constructing an honest category or a category with relaxed "associativity" for the composition of morphisms, i.e. a bicategory.

4.3.1 The Category Bimod

As warming up, we consider the ring-theoretic framework first without inner products at all, in order to simplify things. Let Ring denote the category of associative unital rings: the objects are associative and unital rings while the morphisms are the usual unital ring homomorphisms.

The main idea is now to enlarge the category Ring drastically by keeping the objects while extending the notion of morphisms. For $\mathscr{A}, \mathscr{B} \in \mathrm{Obj}(\mathsf{Ring})$ we consider as new morphisms from $\mathscr{A}$ to $\mathscr{B}$ *all bimodules* ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ to construct a new category. Here we always use strongly non-degenerate bimodules with respect to *both* rings, i.e. we require $x \cdot \mathbb{1}_{\mathscr{A}} = x = \mathbb{1}_{\mathscr{B}} \cdot x$ for all $x \in {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$. Now we have to justify in which sense this can be understood as a category: we need to define a composition of morphisms and a unit morphism for every object. As composition of ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ and ${}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}}$ we use the tensor product over the ring in the middle, i.e. ${}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ is the composition. As unit morphism for the ring $\mathscr{A}$ we use the canonical bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$. With this definition one obtains *almost* a category, since

i.) the tensor product as composition is not associative but only associative up to the usual isomorphism,

ii.) the tensor product ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ is not equal to ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ but only isomorphic to ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ via the usual isomorphism. The same holds for tensoring with ${}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}$ from the left.

There are now two possible ways out of this difficulty: on one hand one can use isomorphism classes of bimodules instead of the bimodules directly. Then $\otimes$ is indeed associative and the class of ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ is indeed the identity morphism. On the other hand, one can simply ask "so what?" and enlarge the concept of categories to *bicategories*. We shall present both options here, starting with the simpler one.

Definition 4.3.1 (The category Bimod) *The category* Bimod *consists of unital rings as objects and isomorphism classes of strongly non-degenerate bimodules as morphisms. The composition of morphisms is the tensor product*

$$[{}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}}] \circ [{}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}] = [{}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}] \tag{4.3.1}$$

and the unit morphisms are $[{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]$.

Theorem 4.3.2 (The category Bimod) Bimod *is a category.*

Proof: There is a small subtlety here concerning the notion of a category which we have ignored so far: strictly speaking, one should first fix a Grothendieck universe and consider only algebras and bimodules inside this universe to form the equivalence classes needed for the category Bimod. We refer to [72] for a detailed discussion on the set-theoretic aspects of this: in conclusion, we will not mention these considerations in the following anymore. Beside this difficulty we have to show the usual properties: first it is clear that the composition is well-defined. Indeed, isomorphic bimodules yield isomorphic tensor products. This is just the functoriality of the tensor product. But then the associativity of $\circ$ is clear. Moreover, the class of ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ becomes a true identity with respect to $\circ$ since ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ is isomorphic to ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ via the bimodule map $x \otimes a \mapsto x \cdot a$ having the inverse $x \mapsto x \otimes \mathbb{1}_{\mathscr{A}}$. For the multiplication with $[{}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}]$ from the left one argues analogously. Here we need the assumption that the bimodules are strongly non-degenerate. □

For a unital ring homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ we can construct a $(\mathscr{B}, \mathscr{A})$-bimodule ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$ as we did this already in the more particular situation of *-algebras in Section 4.1.2. On $\mathscr{B}$ we consider the usual left $\mathscr{B}$-module structure and set $b \cdot_{\Phi} a = b\Phi(a)$ to get a right $\mathscr{A}$-module structure. Clearly, this will give us a bimodule. Since we require Φ to be unital, it is clear that ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$ is a strongly non-degenerate bimodule and hence defines a morphism $[{}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}]$ from $\mathscr{A}$ to $\mathscr{B}$ in Bimod which we shall denote by

$$\ell(\Phi) = [{}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}]. \tag{4.3.2}$$

Taking $\ell(\mathscr{A}) = \mathscr{A}$ on objects, this gives a functor from Ring to Bimod:

Proposition 4.3.3 *The map ℓ gives a functor*

$$\ell\colon \mathsf{Ring} \longrightarrow \mathsf{Bimod}\,. \tag{4.3.3}$$

PROOF: We already noted that $\ell(\Phi)$ is a valid morphism in Bimod. Hence we only have to check functoriality. First, $\ell(\mathrm{id}_{\mathscr{A}}) = [{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]$ is the identity at $\mathscr{A}$ in Bimod. For unital ring homomorphisms $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $\Psi\colon \mathscr{B} \longrightarrow \mathscr{C}$ we have a bimodule homomorphism

$${}_{\mathscr{C}}\mathscr{C}^{\Psi}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}} \ni c \otimes b \;\mapsto\; c\Psi(b) \in {}_{\mathscr{C}}\mathscr{C}^{\Psi\circ\Phi}_{\mathscr{A}}. \tag{$*$}$$

Indeed, it is easily verified that this is well-defined over the tensor product over $\mathscr{B}$ and gives a $(\mathscr{C}, \mathscr{A})$-bimodule homomorphism. Moreover, since we are in a unital situation, the map $c \mapsto c \otimes \mathbb{1}_{\mathscr{B}}$ gives the inverse bimodule morphism. Thus $(*)$ is a bimodule isomorphism showing that $\ell(\Psi) \circ \ell(\Phi) = \ell(\Psi \circ \Phi)$.□

As a first application we can now define the ring-theoretic notion of Morita equivalence. The following is not the original way of defining Morita equivalence but probably the most clean one:

Definition 4.3.4 (Ring-theoretic Morita equivalence) *Two unital rings $\mathscr{A}$ and $\mathscr{B}$ are called Morita equivalent if they are isomorphic in* Bimod.

From this definition it is immediately clear, that Morita equivalence is indeed an equivalence relation. More explicitly, Morita equivalence means that we find an "invertible" bimodule $[{}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}]$, i.e. there exists a class $[{}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}]$ of bimodules with

$$[{}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}] \circ [{}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}] = [{}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}] \quad \text{and} \quad [{}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}] \circ [{}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}] = [{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]. \tag{4.3.4}$$

As usual for inverses in categories, the class of $[{}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}]$ is uniquely determined by $[{}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}]$ and hence ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ is uniquely determined up to a bimodule isomorphism. Clearly, (4.3.4) is equivalent to the existence of a bimodule ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ and isomorphisms

$${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \cong {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \quad \text{and} \quad {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \cong {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}. \tag{4.3.5}$$

The structure of such bimodules ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ is clarified by the following classical theorem of Morita [88]:

Theorem 4.3.5 (Morita) *Let $\mathscr{A}$, $\mathscr{B}$ be unital rings and let ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ be a strongly non-degenerate bimodule. Then the following statements are equivalent:*

i.) *There exists a strongly non-degenerate bimodule* ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ *with*

$$ {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \cong {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \quad \textit{and} \quad {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \cong {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}, \tag{4.3.6} $$

i.e. $\mathscr{A}$ *and* $\mathscr{B}$ *are Morita equivalent.*

ii.) *The right* $\mathscr{A}$*-module* $\mathscr{M}_{\mathscr{A}}$ *is finitely generated, projective, and full, and one has*

$$ \mathscr{B} \cong \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}) \tag{4.3.7} $$

via the left multiplications.

In this case, ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} = \operatorname{Hom}_{\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}})$ *is an inverse bimodule to* ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ *and the isomorphisms in* (4.3.6) *are the canonical ones.*

Here, $\operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}})$ is viewed as a $(\mathscr{A}, \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}))$-bimodule via

$$ (a \cdot \varphi \cdot B)(x) = a\varphi(B(x)) \tag{4.3.8} $$

for $B \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$, $a \in \mathscr{A}$, $x \in \mathscr{M}_{\mathscr{A}}$ and $\varphi \in \operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}})$. Moreover, we use the canonical bimodule maps

$$ \operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}}) \otimes_{\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})} \mathscr{M}_{\mathscr{A}} \ni \varphi \otimes x \ \mapsto\ \varphi(x) \in {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \tag{4.3.9} $$

and

$$ \mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} \operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}}) \ni x \otimes \varphi \ \mapsto\ (y \mapsto x \cdot \varphi(y)) \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}) \tag{4.3.10} $$

from Exercise 4.4.5 to implement the isomorphisms (4.3.6). The proof of this theorem is omitted here but can be found in many algebra textbooks, see e.g. [78, §18C]. Later, we will present an analogous but more complicated theorem including a detailed proof for *-Morita and strong Morita equivalence from which the above statement can easily be deduced, see also Exercise 4.4.2.

4.3.2 The Bicategory Bimod

After this brief introduction to ring-theoretic Morita theory we come to the second option of enlarging the category Ring. We want to consider all bimodules and relax the axioms of a category in such a way that associativity of morphisms and the properties of the identity morphisms do not have to hold strictly. To construct the bicategory Bimod we first use the same objects as before, i.e.

$$ \operatorname{Obj}(\mathsf{Bimod}) = \operatorname{Obj}(\mathsf{Ring}). \tag{4.3.11} $$

For two given unital rings $\mathscr{A}$ and $\mathscr{B}$ we consider the *category* $\mathsf{Bimod}(\mathscr{B}, \mathscr{A})$ of all strongly non-degenerate $(\mathscr{B}, \mathscr{A})$-bimodules with the usual bimodule morphisms as morphisms. We will view the bimodules as arrows from $\mathscr{A}$ to $\mathscr{B}$: note that we have reversed our notation in order to match with the usual bimodule notation.

For two such bimodules ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \in \mathsf{Bimod}(\mathscr{B}, \mathscr{A})$ and ${}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}} \in \mathsf{Bimod}(\mathscr{C}, \mathscr{B})$ the usual tensor product ${}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \in \mathsf{Bimod}(\mathscr{C}, \mathscr{A})$ gives a *functor*

$$ \otimes_{\mathscr{B}} \colon\ \mathsf{Bimod}(\mathscr{C}, \mathscr{B}) \times \mathsf{Bimod}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Bimod}(\mathscr{C}, \mathscr{A}). \tag{4.3.12} $$

For the following it will be crucial that the tensor product is really functorial in both arguments: not only the bimodules can be tensored but also the bimodule morphisms. The bimodules ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ will now be the morphisms from the object $\mathscr{A}$ to the object $\mathscr{B}$. The bimodule morphisms $T\colon {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}$ are *morphisms between morphisms.* Alternatively one calls the morphisms 1*-morphisms* and the morphisms between morphisms are called 2*-morphisms.* We denote this sometimes by

$$ \mathsf{Bimod}_1(\mathscr{B}, \mathscr{A}) = 1\text{-}\operatorname{Morph}(\mathscr{B}, \mathscr{A}) = \{\text{strongly non-degenerate } (\mathscr{B}, \mathscr{A})\text{-bimodules}\} \tag{4.3.13} $$

$$\mathsf{Bimod}_2\big({}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}\big) = 2\text{-}\operatorname{Morph}\big({}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}\big) = \big\{\text{bimodule morphisms } T\colon {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}\big\}. \tag{4.3.14}$$

Consequently, the objects are sometimes called 0-*morphisms* of $\underline{\mathsf{Bimod}}$. For any object $\mathscr{A}$ we have also a canonical 1-morphism $\mathsf{Id}_{\mathscr{A}} = {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \in 1\text{-}\operatorname{Morph}(\mathscr{A}, \mathscr{A})$ which will be called the *identity* 1-*morphism*. Between $\otimes$ and the identity morphism we have the following compatibilities which make the "up to isomorphism" statements as in Theorem 4.3.2 more precise. To formulate this properly, we introduce some more notation: with $\mathbb{1}$ we denote the category consisting of only one object with one morphism (the identity for this object). Given any other category $\mathfrak{C}$ and an object $c \in \mathfrak{C}$ we have a unique functor $c\colon \mathbb{1} \longrightarrow \mathfrak{C}$ sending the object of $\mathbb{1}$ to c. Finally, we note that there is a unique identification functor

$$\mathsf{i}\colon \mathbb{1} \times \mathfrak{C} \longrightarrow \mathfrak{C}. \tag{4.3.15}$$

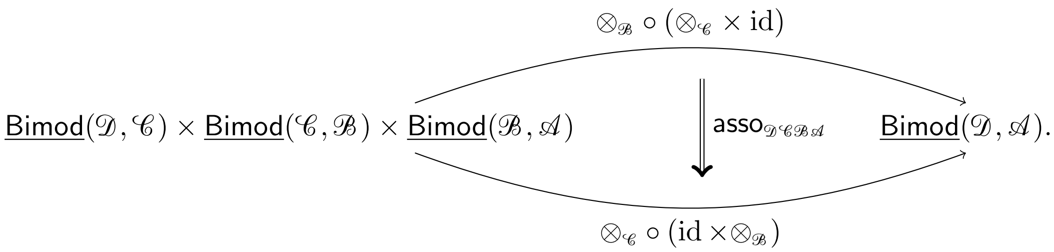
Using this notation we can state the following proposition:

Proposition 4.3.6 *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ be unital rings.*

i.) There is a natural isomorphism, called the associativity,

$$\mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}\colon \otimes_{\mathscr{B}} \circ (\otimes_{\mathscr{C}} \times \mathrm{id}) \longrightarrow \otimes_{\mathscr{C}} \circ (\mathrm{id} \times \otimes_{\mathscr{B}}) \tag{4.3.16}$$

between the functors $\otimes_{\mathscr{B}} \circ (\otimes_{\mathscr{C}} \times \mathrm{id})$ and $\otimes_{\mathscr{C}} \circ (\mathrm{id} \times \otimes_{\mathscr{B}})$ from the Cartesian product category $\underline{\mathsf{Bimod}}(\mathscr{D}, \mathscr{C}) \times \underline{\mathsf{Bimod}}(\mathscr{C}, \mathscr{B}) \times \underline{\mathsf{Bimod}}(\mathscr{B}, \mathscr{A})$ to $\underline{\mathsf{Bimod}}(\mathscr{D}, \mathscr{A})$ visualized by



(4.3.17)

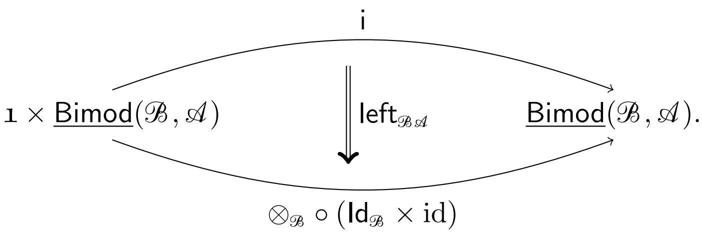
ii.) There is a natural isomorphism, called the left identity,

$$\mathsf{left}_{\mathscr{B}\mathscr{A}}\colon \otimes_{\mathscr{B}} \circ (\mathsf{Id}_{\mathscr{B}} \times \mathrm{id}) \longrightarrow \mathsf{i} \tag{4.3.18}$$

between the functors

$$\mathsf{i} \quad \textit{and} \quad \otimes_{\mathscr{B}} \circ (\mathsf{Id}_{\mathscr{B}} \times \mathrm{id})\colon \mathbb{1} \times \underline{\mathsf{Bimod}}(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Bimod}}(\mathscr{B}, \mathscr{A}), \tag{4.3.19}$$

visualized by



(4.3.20)

iii.) Analogously, there is a natural isomorphism, called the right identity,

$$\mathsf{right}_{\mathscr{B}\mathscr{A}}\colon \otimes_{\mathscr{A}} \circ (\mathrm{id} \times \mathsf{Id}_{\mathscr{A}}) \longrightarrow \mathsf{i} \tag{4.3.21}$$

between the functors

$$\mathsf{i} \quad \textit{and} \quad \otimes_{\mathscr{A}} \circ (\mathrm{id} \times \mathsf{Id}_{\mathscr{A}})\colon \underline{\mathsf{Bimod}}(\mathscr{B}, \mathscr{A}) \times \mathbb{1} \longrightarrow \underline{\mathsf{Bimod}}(\mathscr{B}, \mathscr{A}) \tag{4.3.22}$$

visualized by

$$\underline{\mathsf{Bimod}}(\mathscr{B},\mathscr{A}) \times \mathbb{1} \underset{\otimes_{\mathscr{A}} \circ (\mathrm{id} \times \mathsf{Id}_{\mathscr{A}})}{\overset{\mathsf{i}}{\rightrightarrows}} \underline{\mathsf{Bimod}}(\mathscr{B},\mathscr{A}), \qquad \Downarrow \mathsf{right}_{\mathscr{B}\mathscr{A}} \tag{4.3.23}$$

Proof: The main point of this proof is to re-formulate well-known statements on the tensor product of bimodules in the language of natural transformations. The first part of the proposition makes the statement about the associativity up to isomorphism more precise. Applying the functors $\otimes_{\mathscr{B}} \circ (\otimes_{\mathscr{C}} \times \mathrm{id})$ and $\otimes_{\mathscr{C}} \circ (\mathrm{id} \times \otimes_{\mathscr{B}})$ to objects in $\underline{\mathsf{Bimod}}(\mathscr{B},\mathscr{A})$, i.e. 1-morphisms in $\underline{\mathsf{Bimod}}$, gives

$$(\otimes_{\mathscr{B}} \circ (\otimes_{\mathscr{C}} \times \mathrm{id}))({}_{\mathscr{D}}\mathscr{O}_{\mathscr{C}}, {}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}}, {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}) = (\mathscr{O} \otimes_{\mathscr{C}} \mathscr{N}) \otimes_{\mathscr{B}} \mathscr{M}$$

and

$$(\otimes_{\mathscr{C}} \circ (\mathrm{id} \times \otimes_{\mathscr{B}}))({}_{\mathscr{D}}\mathscr{O}_{\mathscr{C}}, {}_{\mathscr{C}}\mathscr{N}_{\mathscr{B}}, {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}) = \mathscr{O} \otimes_{\mathscr{C}} (\mathscr{N} \otimes_{\mathscr{B}} \mathscr{M}).$$

On morphisms $S\colon \mathscr{M} \longrightarrow \mathscr{M}'$, $T\colon \mathscr{N} \longrightarrow \mathscr{N}'$, and $U\colon \mathscr{O} \longrightarrow \mathscr{O}'$ in $\underline{\mathsf{Bimod}}(\mathscr{B},\mathscr{A})$, i.e. 2-morphism in $\underline{\mathsf{Bimod}}$, we have

$$(\otimes_{\mathscr{B}} \circ (\otimes_{\mathscr{C}} \times \mathrm{id}))(U,T,S) = (U \otimes_{\mathscr{C}} T) \otimes_{\mathscr{B}} S \quad \text{and} \quad (\otimes_{\mathscr{C}} \circ (\mathrm{id} \times \otimes_{\mathscr{B}}))(U,T,S) = U \otimes_{\mathscr{C}} (T \otimes_{\mathscr{B}} S).$$

For the natural transformation $\mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}$ we use the isomorphism determined by

$$\mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{O},\mathscr{N},\mathscr{M})\colon (\mathscr{O} \otimes_{\mathscr{C}} \mathscr{N}) \otimes_{\mathscr{B}} \mathscr{M} \ni (z \otimes y) \otimes x \ \mapsto \ z \otimes (y \otimes x) \in \mathscr{O} \otimes_{\mathscr{C}} (\mathscr{N} \otimes_{\mathscr{B}} \mathscr{M}). \tag{4.3.24}$$

It is clear that $\mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{O},\mathscr{N},\mathscr{M})$ is a well-defined isomorphism of $(\mathscr{D},\mathscr{A})$-bimodules. We have to show that it is also natural, i.e. compatible with the bimodule morphisms T, S, and U. A simple evaluation on elementary tensors yields

$$(U \otimes_{\mathscr{C}} (T \otimes_{\mathscr{B}} S)) \circ \mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{O},\mathscr{N},\mathscr{M}) = \mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{O}',\mathscr{N}',\mathscr{M}') \circ ((U \otimes_{\mathscr{C}} T) \otimes_{\mathscr{B}} S),$$

which shows that $\mathsf{asso}_{\mathscr{D}\mathscr{C}\mathscr{B}\mathscr{A}}$ is natural. For the second part, we have to show that tensoring with $\mathsf{Id}_{\mathscr{B}}$ from the left reproduces the module up to a natural isomorphism. Let ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \in \underline{\mathsf{Bimod}}(\mathscr{B},\mathscr{A})$ then $\otimes_{\mathscr{B}} \circ (\mathsf{Id}_{\mathscr{B}} \times \mathrm{id})(1, {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}) = \mathsf{Id}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ while $\mathsf{i}(1, {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}) = {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$. For a bimodule morphism $T\colon {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}$ and the unique morphism $\mathrm{id}_1 \in \mathrm{Morph}(1,1)$ in $\mathbb{1}$ we have $\otimes_{\mathscr{B}} \circ (\mathsf{Id}_{\mathscr{B}} \times \mathrm{id})(\mathrm{id}_1, T) = \mathrm{id}_{\mathscr{B}} \otimes_{\mathscr{B}} T$ and $\mathsf{i}(\mathrm{id}_1, T) = T$. For the natural transformation $\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}})$ we use the additive extension of

$$\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}})\colon \mathsf{Id}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \ni b \otimes x \ \mapsto \ b \cdot x \in {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}. \tag{4.3.25}$$

Clearly, $\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}})$ is a bimodule isomorphism with inverse determined by $x \mapsto \mathbb{1}_{\mathscr{B}} \otimes x$. We have

$$\big(\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}) \circ (\mathrm{id} \otimes_{\mathscr{B}} T)\big)(b \otimes x) = \mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}})(b \otimes T(x)) = b \cdot T(x),$$

while on the other hand

$$(T \circ \mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}))(b \otimes x) = T(b \cdot x) = b \cdot T(x),$$

since T is a bimodule morphism. This proves

$$\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}'_{\mathscr{A}}) \circ \big((\otimes_{\mathscr{B}} \circ (\mathsf{Id}_{\mathscr{B}} \times \mathrm{id}))(\mathrm{id}_1, T)\big) = (\mathsf{i}(\mathrm{id}_1, T)) \circ \mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}),$$

which means that $\mathsf{left}_{\mathscr{B}\mathscr{A}}$ is natural. Finally, the third part follows analogously to the second part where now we use the natural isomorphism

$$\mathsf{right}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}})\colon {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathsf{Id}_{\mathscr{A}} \ni x \otimes a \ \mapsto \ x \cdot a \in {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \tag{4.3.26}$$

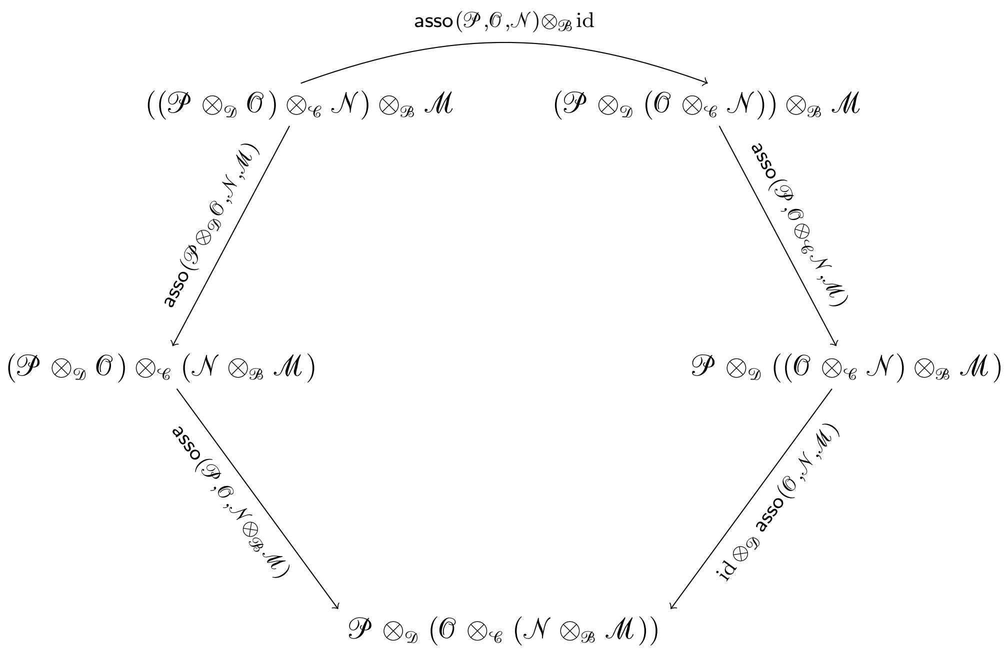
with inverse $x \mapsto x \otimes \mathbb{1}_{\mathscr{A}}$. □

Remark 4.3.7 The proposition makes precise in which sense the tensor product is associative and in which sense the tensoring with the canonical bimodule behaves like a unit element. The main emphasize lies on the word *natural* as of course one could e.g. rescale $\mathsf{asso}_{\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{A}}(\mathcal{O}, \mathcal{N}, \mathcal{M})$ in a way which depends on the particular choices of $\mathcal{O}$, $\mathcal{N}$, and $\mathcal{M}$ and thereby spoiling the compatibility with the morphisms U, T, and S.

In the following we shall drop the subscripts in $\mathsf{asso}_{\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{A}}$, $\mathsf{left}_{\mathcal{B}\mathcal{A}}$, and $\mathsf{right}_{\mathcal{B}\mathcal{A}}$, and write simply asso, left and right, respectively, whenever the context is clear. The natural isomorphisms satisfy now the following compatibility condition:

Proposition 4.3.8 *In* <u>Bimod</u>*, the associativity and identity isomorphisms* asso*,* left*, and* right *satisfy the following coherence conditions:*

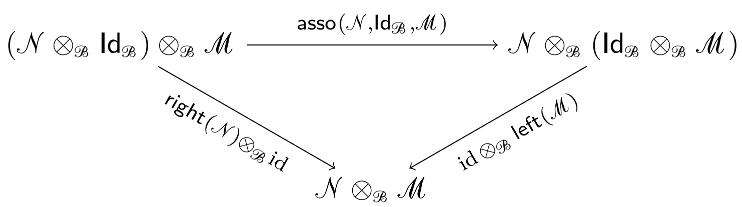
i.) The associativity coherence for asso*: the hexagon diagram*



(4.3.27)

commutes for all bimodules ${}_{\mathcal{E}}\mathcal{P}_{\mathcal{D}}$*,* ${}_{\mathcal{D}}\mathcal{O}_{\mathcal{C}}$*,* ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{B}}$*, and* ${}_{\mathcal{B}}\mathcal{M}_{\mathcal{A}}$*.*

ii.) The identity coherence for asso*,* left*, and* right*: the diagram*



(4.3.28)

commutes for all bimodules ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{B}}$ *and* ${}_{\mathcal{B}}\mathcal{M}_{\mathcal{A}}$*.*

PROOF: Behind these diagrams there are rather trivial properties and computations. Let $p \in \mathcal{P}$, $o \in \mathcal{O}$, $n \in \mathcal{N}$, and $m \in \mathcal{M}$. Then on one hand

$$
\begin{aligned}
&\big((\mathrm{id} \otimes_{\mathcal{D}} \mathsf{asso}(\mathcal{O}, \mathcal{N}, \mathcal{M})) \circ \mathsf{asso}(\mathcal{P}, \mathcal{O} \otimes_{\mathcal{C}} \mathcal{N}, \mathcal{M}) \circ (\mathsf{asso}(\mathcal{P}, \mathcal{O}, \mathcal{N}) \otimes_{\mathcal{B}} \mathrm{id})\big)(((p \otimes o) \otimes n) \otimes m) \\
&\quad = \big((\mathrm{id} \otimes_{\mathcal{D}} \mathsf{asso}(\mathcal{O}, \mathcal{N}, \mathcal{M})) \circ \mathsf{asso}(\mathcal{P}, \mathcal{O} \otimes_{\mathcal{C}} \mathcal{N}, \mathcal{M})\big)((p \otimes (o \otimes n)) \otimes m) \\
&\quad = (\mathrm{id} \otimes_{\mathcal{D}} \mathsf{asso}(\mathcal{O}, \mathcal{N}, \mathcal{M}))(p \otimes ((o \otimes n) \otimes m))
\end{aligned}
$$

$$= p \otimes (o \otimes (n \otimes m)).$$

On the other hand we have

$$\begin{aligned}
&\big(\mathsf{asso}(\mathscr{P}, \mathscr{O}, \mathscr{N} \otimes_{\mathscr{B}} \mathscr{M}) \circ \mathsf{asso}(\mathscr{P} \otimes_{\mathscr{D}} \mathscr{O}, \mathscr{N}, \mathscr{M})\big)(((p \otimes o) \otimes n) \otimes m) \\
&\quad = \mathsf{asso}(\mathscr{P}, \mathscr{O}, \mathscr{N} \otimes_{\mathscr{B}} \mathscr{M})((p \otimes o) \otimes (n \otimes m)) \\
&\quad = p \otimes (o \otimes (n \otimes m)),
\end{aligned}$$

which proves (4.3.27) since every tensor is a sum of such elementary tensors. Analogously, we have for $n \in \mathscr{N}$, $b \in \mathscr{B}$, and $m \in \mathscr{M}$

$$\big((\mathrm{id} \otimes_{\mathscr{B}} \mathsf{left}(\mathscr{M})) \circ \mathsf{asso}(\mathscr{N}, \mathscr{B}, \mathscr{M})\big)((n \otimes b) \otimes m) = (\mathrm{id} \otimes_{\mathscr{B}} \mathsf{left}(\mathscr{M}))(n \otimes (b \otimes m)) = n \otimes (b \cdot m),$$

and on the other hand

$$(\mathsf{right}(\mathscr{N}) \otimes_{\mathscr{B}} \mathrm{id})((n \otimes b) \otimes m) = (n \cdot b) \otimes m.$$

Since the tensor product is formed over $\mathscr{B}$, the two sides coincide and (4.3.28) follows. □

We take now Bimod as a motivating example for the following definition of a bicategory as introduced by Benabou in [7]:

Definition 4.3.9 (Bicategory) *A bicategory $\underline{\mathfrak{B}}$ consists of the following data:*

i.) A class $\mathfrak{B}_0$, the objects of $\underline{\mathfrak{B}}$.

ii.) For each two objects $a, b \in \mathfrak{B}_0$ a category $\underline{\mathfrak{B}}(b, a)$. The objects $\mathfrak{B}_1(b, a) = \mathrm{Obj}(\underline{\mathfrak{B}}(b, a))$ of this category are called 1-morphisms from a to b. The morphisms $T\colon M \longrightarrow M'$ for two 1-morphisms $M, M' \in \mathfrak{B}_1(b, a)$ are called 2-morphisms from M to M'. The set of these 2-morphisms is denoted by $\mathfrak{B}_2(M, M')$.

iii.) For each triple of objects $a, b, c \in \mathfrak{B}_0$ a functor

$$\otimes_b \colon \underline{\mathfrak{B}}(c, b) \times \underline{\mathfrak{B}}(b, a) \longrightarrow \underline{\mathfrak{B}}(c, a), \tag{4.3.29}$$

called the composition of 1-morphisms or the tensor product. If the context is clear we simply write $\otimes$ instead of $\otimes_b$.

iv.) For each object $a \in \mathfrak{B}_0$ a 1-morphism $\mathsf{Id}_a \in \mathfrak{B}_1(a, a)$, called the identity at a.

v.) For each quadruple of objects $a, b, c, d \in \mathfrak{B}_0$ a natural isomorphism

$$\mathsf{asso}_{dcba} \colon \otimes_b \circ (\otimes_c \times \mathrm{id}) \longrightarrow \otimes_c \circ (\mathrm{id} \times \otimes_b), \tag{4.3.30}$$

called the associativity. If the context is clear, we simply write asso instead of asso_{dcba}.

vi.) For each pair of objects $a, b \in \mathfrak{B}_0$ two natural isomorphisms

$$\mathsf{left}_{ba} \colon \otimes_b \circ (\mathsf{Id}_b \times \mathrm{id}) \longrightarrow \mathrm{id} \tag{4.3.31}$$

and

$$\mathsf{right}_{ba} \colon \otimes_a \circ (\mathrm{id} \times \mathsf{Id}_a) \longrightarrow \mathrm{id}, \tag{4.3.32}$$

called the left and right identity, respectively. Again, we write left and right if the context is clear.

These data are required to fulfill the following coherence conditions:

i.) Associativity coherence: the pentagon diagram

$$
\begin{array}{ccc}
((P \otimes_d O) \otimes_c N) \otimes_B M & \xrightarrow{\mathsf{asso}(P,O,N) \otimes_b \mathrm{id}} & (P \otimes_d (O \otimes_c N)) \otimes_b M \\
\downarrow{\scriptstyle \mathsf{asso}(P \otimes_d O, N, M)} & & \downarrow{\scriptstyle \mathsf{asso}(P, O \otimes_c N, M)} \\
(P \otimes_d O) \otimes_c (N \otimes_b M) & & P \otimes_d ((O \otimes_c N) \otimes_b M) \\
& {\scriptstyle \mathsf{asso}(P,O,N \otimes_b M)} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes_d \mathsf{asso}(O,N,M)} & \\
& P \otimes_d (O \otimes_c (N \otimes_b M)) &
\end{array}
\tag{4.3.33}
$$

commutes for all $P \in \mathfrak{B}_1(e,d)$, $O \in \mathfrak{B}_1(d,c)$, $N \in \mathfrak{B}_1(c,b)$, *and* $M \in \mathfrak{B}_1 b, a)$.

ii.) Identity coherence: the diagram

$$
\begin{array}{ccc}
(N \otimes_b \mathsf{Id}_b) \otimes_b M & \xrightarrow{\mathsf{asso}(N, \mathsf{Id}_b, M)} & N \otimes_b (\mathsf{Id}_b \otimes_b M) \\
& {\scriptstyle \mathsf{right}(N) \otimes_b \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes_b \mathsf{left}(M)} & \\
& N \otimes_b M &
\end{array}
\tag{4.3.34}
$$

commutes for all 1*-morphisms* $N \in \mathfrak{B}_1(c,b)$ *and* $M \in \mathfrak{B}_1(b,a)$.

Corollary 4.3.10 *The unital rings as objects, the bimodules between them as* 1*-morphisms and bimodule morphisms as* 2*-morphisms yield a bicategory* $\underline{\mathsf{Bimod}}$ *with respect to the tensor product of bimodules and the natural isomorphisms* asso, left *and* right.

Remark 4.3.11 The coherence conditions are eventually responsible for the fact that the successive use of the functor $\otimes$ and the natural transformations asso, left, and right does not produce new natural isomorphisms. This is part of the statement that "every diagram in a bicategory commutes" provided it is build out of the data of the bicategory, see e.g. [83, Sect. 1.5] for a discussion.

Example 4.3.12 (2-Category) A 2*-category* $\underline{\mathfrak{B}}$ consists of the same data *i.) – iv.)* as for a bicategory with the following difference: the associativity isomorphism and the identity isomorphism are required to be the identity, i.e. one has

$$
\otimes_b \circ (\otimes_c \times \mathrm{id}) = \otimes_c \circ (\mathrm{id} \times \otimes_b) \tag{4.3.35}
$$

as well as

$$
\otimes_b \circ (\mathsf{Id}_b \times \mathrm{id}) = \mathrm{id} = \otimes_a \circ (\mathrm{id} \times \mathsf{Id}_a). \tag{4.3.36}
$$

In this case, the coherence conditions *i.)* and *ii.)* are automatically fulfilled. For this reason, a bicategory is also called a *weak* 2*-category*, see also [83] for additional information and references. However, note also that the newer conventions in the literature refer to a 2-category in the above sense as *strict* 2*-category* while a bicategory is now called 2-category without the attribute *weak*.

Remark 4.3.13 (Monoidal categories) If a bicategory $\underline{\mathfrak{B}}$ has only one object $*$, then one has a category $\mathfrak{B} = \underline{\mathfrak{B}}(*, *)$ where one has a functorial tensor product for the *objects* of $\mathfrak{B}$. This is the structure usually called a *tensor category* or *monoidal category*. If the bicategory is even a 2-category then the monoidal category is called *strict*. More details on monoidal categories can be found in e.g. [73, Chap. XI].

4.3.3 The Bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$

We are now in the position to formulate the main result of this section: we can construct for the class of idempotent and non-degenerate *-algebras over C the *bicategory of pre-Hilbert bimodules* $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ and the *bicategory of inner product bimodules* $\underline{\mathsf{Bimod}}^*$, respectively. The class of objects are in both cases the idempotent and non-degenerate *-algebras over C. For two such *-algebras we define the 1-morphisms to be

$$\mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A}) = \left\{ {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B}) \;\middle|\; \mathscr{E} \cdot \mathscr{A} = \mathscr{E} \right\} \tag{4.3.37}$$

and

$$\mathsf{Bimod}_1^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) = \left\{ {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B}) \;\middle|\; \mathscr{E} \cdot \mathscr{A} = \mathscr{E} \right\} \subseteq \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A}), \tag{4.3.38}$$

respectively. Note that for ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ we already require the strong non-degeneracy $\mathscr{B} \cdot \mathscr{E} = \mathscr{E}$ concerning the algebra $\mathscr{B}$ but now we also need it with respect to the algebra $\mathscr{A}$.

Recall that according to our convention every bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is also a C-module such that all structure maps are C-(anti-)multilinear. The 2-morphisms are in both cases the *adjointable bimodule morphisms*, i.e. the intertwiners, as already in ${}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ and ${}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$, respectively. For the composition of 1-morphisms, i.e. the tensor product functor we use the internal tensor product $\widehat{\otimes}$: for an idempotent and non-degenerate *-algebra $\mathscr{B}$ we set

$$\widehat{\otimes}_{\mathscr{B}} \colon \underline{\mathsf{Bimod}}^*(\mathscr{C}, \mathscr{B}) \times \underline{\mathsf{Bimod}}^*(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Bimod}}^*(\mathscr{C}, \mathscr{A}) \tag{4.3.39}$$

and

$$\widehat{\otimes}_{\mathscr{B}} \colon \underline{\mathsf{Bimod}}^{\mathrm{str}}(\mathscr{C}, \mathscr{B}) \times \underline{\mathsf{Bimod}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Bimod}}^{\mathrm{str}}(\mathscr{C}, \mathscr{A}), \tag{4.3.40}$$

respectively. Note that the property $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$ is clearly preserved by the internal tensor product. Moreover, by Corollary 3.1.9 and Corollary 3.1.12 the internal tensor product is indeed a functor as required. For the natural isomorphism asso as required by Definition 4.3.9 we can rely on the already shown associativity of the internal tensor product according to Proposition 3.1.5. The only point which remains to be shown is the naturalness of asso. We prove this in a slightly more general context:

Lemma 4.3.14 *Let $\mathscr{G}, \mathscr{G}' \in {}^*\text{-}\mathsf{Mod}_{\mathscr{C}}(\mathscr{D})$, $\mathscr{F}, \mathscr{F}' \in {}^*\text{-}\mathsf{Mod}_{\mathscr{B}}(\mathscr{C})$, and $\mathscr{E}, \mathscr{E}' \in {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ be given. Moreover, let $U \colon \mathscr{G} \longrightarrow \mathscr{G}'$, $T \colon \mathscr{F} \longrightarrow \mathscr{F}'$, and $S \colon \mathscr{E} \longrightarrow \mathscr{E}'$ be intertwiners. Then we have*

$$\mathsf{asso}(\mathscr{G}', \mathscr{F}', \mathscr{E}') \circ \big((U \mathbin{\widehat{\otimes}} T) \mathbin{\widehat{\otimes}} S\big) = \big(U \mathbin{\widehat{\otimes}} (T \mathbin{\widehat{\otimes}} S)\big) \circ \mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E}), \tag{4.3.41}$$

showing that asso is a natural transformation.

PROOF: Let $z \in \mathscr{G}$, $y \in \mathscr{F}$, and $x \in \mathscr{E}$. Then we consider $[[z \otimes y] \otimes x] \in (\mathscr{G} \mathbin{\widehat{\otimes}} \mathscr{F}) \mathbin{\widehat{\otimes}} \mathscr{E}$ and compute

$$\begin{aligned}
\big(\mathsf{asso}(\mathscr{G}', \mathscr{F}', \mathscr{E}') \circ ((U \mathbin{\widehat{\otimes}} T) \mathbin{\widehat{\otimes}} S)\big)\big([[z \otimes y] \otimes x]\big) &= \mathsf{asso}(\mathscr{G}', \mathscr{F}', \mathscr{E}')\big([[U(z) \otimes T(y)] \otimes S(x)]\big) \\
&= [U(z) \otimes [T(y) \otimes S(x)]] \\
&= \big((U \mathbin{\widehat{\otimes}} (T \mathbin{\widehat{\otimes}} S)) \circ \mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E})\big)\big([[z \otimes y] \otimes x]\big).
\end{aligned}$$

Since such classes of elementary tensors span the whole internal tensor product, the claim follows. □

Together with the already shown property that $\mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E})$ is isometric we obtain a natural isomorphism consisting of *unitary* intertwiners. For the identity 1-morphisms we use the canonical bimodules

$$\mathsf{Id}_{\mathscr{A}} = {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}, \tag{4.3.42}$$

equipped with the canonical inner product $\langle a, b\rangle_{\mathscr{A}} = a^* b$. Under our assumptions on $\mathscr{A}$ we have ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \in {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{A})$ and ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \cdot \mathscr{A} = {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$. Thus ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \in \mathsf{Bimod}_1^{\mathrm{str}}(\mathscr{A}, \mathscr{A})$ as wanted. For the left and right identity transformations we use the bimodule morphisms arising from

$$\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})\colon \mathsf{Id}_{\mathscr{B}} \mathbin{\widehat{\otimes}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \ni [b \otimes x] \;\mapsto\; b \cdot x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \tag{4.3.43}$$

and

$$\mathsf{right}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}} \mathsf{Id}_{\mathscr{A}} \ni [x \otimes a] \;\mapsto\; x \cdot a \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}. \tag{4.3.44}$$

They turn out to be indeed the required identity isomorphisms. Note that for this to be true for right we need the additional requirement in the definitions (4.3.37) and (4.3.38): for ${}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ or ${}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$ the following lemma would only be true for left but not for right.

Lemma 4.3.15 *Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \in \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A})$ and let $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ be a 2-morphism. Then*

$$\mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})\colon \mathsf{Id}_{\mathscr{B}} \mathbin{\widehat{\otimes}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \quad \text{and} \quad \mathsf{right}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}} \mathsf{Id}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \tag{4.3.45}$$

are unitary intertwiners with

$$T \circ \mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}) = \mathsf{left}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}) \circ (\mathrm{id} \mathbin{\widehat{\otimes}} T) \quad \text{and} \quad T \circ \mathsf{right}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}) = \mathsf{right}_{\mathscr{B}\mathscr{A}}({}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}) \circ (T \mathbin{\widehat{\otimes}} \mathrm{id}). \tag{4.3.46}$$

PROOF: First we have to show that $\mathsf{left}\colon \mathscr{B} \mathbin{\widehat{\otimes}} \mathscr{E} \longrightarrow \mathscr{E}$ is well-defined. To this end we compute

$$\langle b \otimes x, b' \otimes y\rangle_{\mathscr{A}}^{\mathscr{B}\otimes\mathscr{E}} = \big\langle x, \langle b, b'\rangle_{\mathscr{B}} \cdot y\big\rangle_{\mathscr{A}}^{\mathscr{E}} = \langle x, (b^* b') \cdot y\rangle_{\mathscr{A}}^{\mathscr{E}} = \langle b \cdot x, b' \cdot y\rangle_{\mathscr{A}}^{\mathscr{E}}.$$

Since elementary tensors span everything, left is isometric on the level of $\otimes_{\mathscr{B}}$. Thus it passes to an isometric and hence injective map on the quotient $\mathscr{B} \mathbin{\widehat{\otimes}} \mathscr{E}$. The surjectivity of left is clear from $\mathscr{E} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ since by definition $\mathscr{B} \cdot \mathscr{E} = \mathscr{E}$. Thus left is bijective and isometric, hence adjointable and unitary. It follows that left is right $\mathscr{A}$-linear, too, which can also be seen directly. Moreover, left is clearly left $\mathscr{B}$-linear showing that left is indeed a unitary intertwiner. For elementary tensors we obtain

$$T(\mathsf{left}(\mathscr{E})(b \otimes x)) = T(b \cdot x) = b \cdot T(x) = \mathsf{left}(\mathscr{E}')(b \otimes T(x)),$$

which proves (4.3.46) for left. Using $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$, which is granted by $\mathscr{E} \in \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A})$, the case of right is treated analogously. □

To obtain a bicategory we finally have to show the associativity and identity coherence. This is easy and can be done exactly the same way as for the bicategory $\underline{\mathsf{Bimod}}$ in Proposition 4.3.8 by considering (equivalence classes of) elementary tensors, see Exercise 4.4.6. Thus we can summarize the results of our discussion:

Theorem 4.3.16 (The bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$) *The inner-product bimodules $\underline{\mathsf{Bimod}}^*$ over idempotent and non-degenerate *-algebras form a bicategory with respect to the internal tensor product $\widehat{\otimes}$, the identity bimodules $\mathsf{Id}_{\mathscr{A}}$, and asso, left, and right as above. The pre-Hilbert bimodules $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ over idempotent and non-degenerate *-algebras form a sub-bicategory of $\underline{\mathsf{Bimod}}^*$.*

The bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ have an additional structure: for two given 1-morphisms ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \in \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A})$ the corresponding 2-morphisms $\mathsf{Bimod}_2^*({}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$ form a C-module. This is clear since C-linear combinations of intertwiners are again intertwiners. Moreover, we have the adjoint $T \mapsto T^*$ of an intertwiner $T \in \mathsf{Bimod}_2^*({}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$ yielding again an intertwiner $T^* \in \mathsf{Bimod}_2^*({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}})$ in the opposite direction. This *-involution allows to speak of *unitary* intertwiners, i.e. of 2-isomorphisms which fulfill $T^{-1} = T^*$ in addition. We take these features as motivation to define a *-bicategory over C:

Definition 4.3.17 (*-Bicategory over C) *A bicategory $\underline{\mathfrak{B}}$ is called *-bicategory over C if for any two objects $a, b \in \mathfrak{B}_0$ and two 1-morphisms $M, M' \in \mathfrak{B}_1(b, a)$ the 2-morphisms $\mathfrak{B}_2(M', M)$ are a C-module and if there is a map*

$$^*\colon \mathfrak{B}_2(M', M) \longrightarrow \mathfrak{B}_2(M, M'), \tag{4.3.47}$$

called the *-involution, such that the following properties are fulfilled:

i.) The composition of 2-morphisms is C-bilinear.

ii.) The *-involution * is C-antilinear, involutive, and

$$(T \circ S)^* = S^* \circ T^* \tag{4.3.48}$$

for $S \in \mathfrak{B}_2(M', M)$ and $T \in \mathfrak{B}_2(M'', M')$.

iii.) The tensor product $\otimes$ of 2-morphisms is C-bilinear.

iv.) For $S \in \mathfrak{B}_2(N', N)$ and $T \in \mathfrak{B}_2(M', M)$ with $N, N' \in \mathfrak{B}_1(c, b)$ and $M, M' \in \mathfrak{B}_1(b, a)$ we have

$$(S \otimes T)^* = S^* \otimes T^*. \tag{4.3.49}$$

v.) The natural isomorphisms $\mathsf{asso}(O, N, M)$ as well as $\mathsf{left}(M)$ and $\mathsf{right}(M)$ are unitary for all $O \in \mathfrak{B}_1(d, c)$, $N \in \mathfrak{B}_1(c, b)$, and $M \in \mathfrak{B}_1(b, a)$, i.e. one has

$$\mathsf{asso}(O, N, M)^* = \mathsf{asso}(O, N, M)^{-1} \tag{4.3.50}$$

$$\mathsf{left}(M)^* = \mathsf{left}(M)^{-1} \tag{4.3.51}$$

and

$$\mathsf{right}(M)^* = \mathsf{right}(M)^{-1}. \tag{4.3.52}$$

Remark 4.3.18 (*-Bicategory over C) It follows immediately that for a *-bicategory $\underline{\mathfrak{B}}$ over C the 2-*endomorphisms* $2\text{-}\operatorname{End}(M) = \mathfrak{B}_2(M, M)$ of a 1-morphism $M \in \mathfrak{B}_1(b, a)$ are a unital *-algebra over C with unit element $\mathbb{1} = \mathrm{id}_M$. Indeed, the composition of 2-endomorphisms is associative and C-bilinear and id_M is the unit element for this composition. The *-involution of $\underline{\mathfrak{B}}$ gives the *-involution of the algebra $2\text{-}\operatorname{End}(M)$. In particular,

$$\mathrm{id}_M^* = \mathrm{id}_M \tag{4.3.53}$$

follows for any *-bicategory over C and any 1-morphism M.

Analogously to Definition 4.3.17 one can also define a *-*category over* C as a category $\mathfrak{C}$ such that for any two objects $a, b \in \operatorname{Obj}(\mathfrak{C})$ the morphisms $\operatorname{Morph}(b, a)$ are a C-module and such that there is a map

$$^*\colon \operatorname{Morph}(b, a) \longrightarrow \operatorname{Morph}(a, b) \tag{4.3.54}$$

with the properties that the composition of morphisms is C-bilinear and * has the properties of a *-involution, i.e. * is C-antilinear, involutive, and $(T \circ S)^* = S^* \circ T^*$. For two such *-categories $\mathfrak{C}$ and $\mathfrak{D}$ over C one defines a *-*functor* to be a functor $\mathsf{F}\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ such that for any two objects $a, b \in \operatorname{Obj}(\mathfrak{C})$ the corresponding map

$$\mathsf{F}\colon \operatorname{Morph}(b, a) \longrightarrow \operatorname{Morph}(\mathsf{F}(b), \mathsf{F}(a)) \tag{4.3.55}$$

is C-linear and satisfies $\mathsf{F}(T^*) = \mathsf{F}(T)^*$ for all $T \in \operatorname{Morph}(b, a)$. Finally, for two *-categories $\mathfrak{C}$ and $\mathfrak{D}$ and two *-functors $\mathsf{F}, \mathsf{G}\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ a *natural unitary equivalence* from F to G is a natural transformation

$$\mathsf{u}\colon \mathsf{F} \longrightarrow \mathsf{G}, \tag{4.3.56}$$

such that for all objects $c \in \mathfrak{C}$ the morphism $\mathsf{u}(c)\colon \mathsf{F}(c) \longrightarrow \mathsf{G}(c)$ is unitary in the sense of the *-category $\mathfrak{D}$. Two *-categories $\mathfrak{C}$ and $\mathfrak{D}$ are called *unitarily equivalent* if there are *-functors $\mathsf{F}\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ and $\mathsf{G}\colon \mathfrak{D} \longrightarrow \mathfrak{C}$ such that $\mathsf{F} \circ \mathsf{G}$ and $\mathsf{G} \circ \mathsf{F}$ are naturally unitarily equivalent to the identity functors. If $\mathsf{s}, \mathsf{t}\colon \mathsf{F} \longrightarrow \mathsf{G}$ are arbitrary natural transformation then we can define new natural transformations

$$z\mathsf{s} + w\mathsf{t}\colon \mathsf{F} \longrightarrow \mathsf{G} \tag{4.3.57}$$

for $z, w \in \mathsf{C}$ and

$$\mathsf{s}^*\colon \mathsf{G} \longrightarrow \mathsf{F} \tag{4.3.58}$$

by using the C-module structure and the *-involution for the morphism spaces of $\mathfrak{D}$, see also Exercise 4.4.7.

Example 4.3.19 (*-Categories and *-functors) The category $^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ is a *-category for all *-algebras $\mathscr{A}$ and $\mathscr{B}$, where we use the fact that the intertwiners $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ between *-representations of $\mathscr{B}$ on inner-product modules over $\mathscr{A}$ form a C-module, see Remark 2.1.22. This additional structure was the ultimate reason for the choice of adjointable intertwiners as morphisms in the representation theories. Then $^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{B})$, $^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$, and $^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$ are *-subcategories of $^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$. The Rieffel induction functor $\mathsf{R}_{\mathscr{E}}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{B})$ for a bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ and an auxiliary *-algebra $\mathscr{D}$ is then an example of a *-functor, see also Exercise 4.4.8.

With these notions we can re-interpret the definition of a *-bicategory $\underline{\mathfrak{B}}$ as follows: for any two objects $a, b \in \mathfrak{B}_0$ the category $\underline{\mathfrak{B}}(b, a)$ is a *-category over C, the tensor product $\otimes$ is a *-functor, and the natural equivalences asso, left, and right are unitary equivalences.

Theorem 4.3.20 ($\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ are *-bicategories) *With respect to the usual C-module structure of intertwiners and the adjoint as *-involution the bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ are *-bicategories over C.*

Remark 4.3.21 (Isomorphisms in *-bicategories) Let $\underline{\mathfrak{B}}$ be a bicategory. Two 1-morphisms $M, M' \in \mathfrak{B}_1(b, a)$ are called *equivalent* if there is a 2-isomorphism $T\colon M \longrightarrow M'$. Two objects $a, b \in \mathfrak{B}_0$ are called *isomorphic* if there are 1-morphisms $M \in \mathfrak{B}_1(b, a)$ and $M' \in \mathfrak{B}_1(a, b)$ such that $M \otimes M'$ is equivalent to Id_b and $M' \otimes M$ is equivalent to Id_a. This is the general idea behind the usage of bicategories. If $\underline{\mathfrak{B}}$ is now even a *-bicategory over C, we can refine the notion of equivalence and isomorphism: Two 1-morphisms $M, M' \in \mathfrak{B}_1(b, a)$ are called *unitarily equivalent* if there is a unitary 2-isomorphism $T\colon M \longrightarrow M'$. This gives also two notions of isomorphism for the objects: We can either use the bicategory version or we can demand that $M \otimes M'$ and $M' \otimes M$, are *unitarily* equivalent to Id_b and Id_a, respectively. In general, the second implies the first but not vice versa. In the following we shall exclusively use the second notion of isomorphism based on unitary equivalence for a *-bicategory over C. Sometimes we will emphasize this by calling the isomorphic objects *-isomorphic in this case.

We conclude this section with a general construction which brings us from a bicategory back to an ordinary category by passing to isomorphism classes of 1-morphisms [7]:

Theorem 4.3.22 (Classifying category) *Let $\underline{\mathfrak{B}}$ be a bicategory. Then a category $\mathfrak{B}$, the classifying category of $\underline{\mathfrak{B}}$, is obtained as follows:*

i.) The objects of $\mathfrak{B}$ are the objects of $\underline{\mathfrak{B}}$.

ii.) For $a, b \in \mathrm{Obj}(\mathfrak{B})$ one defines the morphisms

$$\mathrm{Morph}(b, a) = \big\{[M] \bigm| M \in 1\text{-}\mathrm{Morph}(b, a)\big\}, \tag{4.3.59}$$

where $[M]$ denotes the equivalence class of M with respect to the 2-morphisms of $\underline{\mathfrak{B}}$.

iii.) *The composition of morphisms and the identity morphisms are defined by*

$$[N] \circ [M] = [N \otimes M] \quad \textit{and} \quad \mathrm{id}_a = [\mathsf{Id}_a]. \tag{4.3.60}$$

PROOF: First we have to show that the composition (4.3.60) is well-defined. Since $\otimes$ is functorial it follows that for two 2-isomorphisms $T\colon M \longrightarrow M'$ and $S\colon N \longrightarrow N'$ also $S \otimes T\colon N \otimes M \longrightarrow N' \otimes M'$ is a 2-isomorphism. Thus (4.3.60) is well-defined. The associativity of $\circ$ follows now from the associativity of $\otimes$ up to asso. For the same reason, id_a is the unit element since Id_a is the unit up to the natural isomorphisms coming from left and right. ☐

Remark 4.3.23 (Classifying category of a *-bicategory) For a *-bicategory over C we use instead of equivalence classes of 1-morphisms the stronger version of unitary equivalence classes of 1-morphisms according to our convention in Remark 4.3.21. Clearly, an analogous argument shows that also in this case we obtain a category, the classifying category of a *-bicategory over C.

Example 4.3.24 (Classifying category)

i.) The category Bimod from Definition 4.3.1 is by construction the classifying category of the bicategory $\underline{\mathsf{Bimod}}$.

ii.) Analogously, one obtains the classifying categories Bimod^* and $\mathsf{Bimod}^{\mathrm{str}}$ for the *-bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively. Here the objects are idempotent and non-degenerate *-algebras over C and the morphisms are the *unitary* isomorphism classes of inner-product bimodules and pre-Hilbert bimodules, respectively, according to Remark 4.3.23. Now $\mathsf{Bimod}^{\mathrm{str}}$ is a sub-category of Bimod^* with the same objects but less morphisms.

4.3.4 Invertible Bimodules in $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$

Having the bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ we want to understand the notions of isomorphism arising from them. In particular, we shall derive an analogous statement to Theorem 4.3.5 in this context. Thus, let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A})$ be an *invertible* bimodule. According to our convention in Remark 4.3.21, ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is invertible if there is a ${}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \in \mathsf{Bimod}_1^*(\mathscr{A}, \mathscr{B})$ such that there exist bimodule isomorphisms

$$\phi\colon {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} = \mathsf{Id}_{\mathscr{A}} \tag{4.3.61}$$

$$\psi\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \longrightarrow {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} = \mathsf{Id}_{\mathscr{B}}, \tag{4.3.62}$$

which are in addition *isometric*. This is the notion of *unitary equivalence* which we agreed to use for *-bicategories. The following proposition gives a first example of such pairs of bimodules:

Proposition 4.3.25 *Let* ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ *be a *-equivalence bimodule.*

i.) *The maps*

$$\phi_{\mathrm{can}}\colon {}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \ni \overline{x} \otimes y \;\mapsto\; \langle x, y\rangle^{\mathscr{E}}_{\mathscr{A}} \in {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \tag{4.3.63}$$

$$\psi_{\mathrm{can}}\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widetilde{\otimes}_{\mathscr{A}}} {}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}} \ni x \otimes \overline{y} \;\mapsto\; {}_{\mathscr{B}}\langle x, y\rangle^{\mathscr{E}} \in {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \tag{4.3.64}$$

are isometric bimodule isomorphisms, even with respect to both inner products on the *-equivalence bimodules.

ii.) *We have* ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A})$ *and* ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ *is invertible.*

iii.) *One has the compatibility*

$$\psi_{\mathrm{can}}(x \otimes \overline{y}) \cdot z = x \cdot \phi_{\mathrm{can}}(\overline{y} \otimes z). \tag{4.3.65}$$

Proof: Clearly, ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Bimod}_1^*(\mathscr{B}, \mathscr{A})$ since $\mathscr{E} \cdot \mathscr{A} = \mathscr{E}$ by definition of a *-equivalence bimodule. Observe that we have written $\widetilde{\otimes}$ instead of $\widehat{\otimes}$ in (4.3.63) and (4.3.64) emphasizing that these isomorphisms are isometric for *both* inner products. First it is clear that ϕ_{can} is well-defined over $\otimes_{\mathscr{B}}$ since $\overline{x} \cdot b \otimes y = \overline{b^* \cdot x} \otimes y \mapsto \langle b^* \cdot x, y\rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x, b \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}}$ for all $x, y \in \mathscr{E}$ and $b \in \mathscr{B}$. Moreover, ϕ_{can} is isometric with respect to the right-linear inner products since

$$
\begin{aligned}
\big\langle \overline{x} \otimes y, \overline{x}' \otimes y' \big\rangle^{\overline{\mathscr{E}} \otimes \mathscr{E}}_{\mathscr{A}} &= \big\langle y, \langle \overline{x}, \overline{x}'\rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} \cdot y' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle y, {}_{\mathscr{B}}\langle x, x'\rangle^{\mathscr{E}} \cdot y' \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle y, x \cdot \langle x', y'\rangle^{\mathscr{E}}_{\mathscr{A}} \big\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \langle y, x\rangle^{\mathscr{E}}_{\mathscr{A}} \langle x', y'\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big(\langle x, y\rangle^{\mathscr{E}}_{\mathscr{A}}\big)^* \langle x', y'\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \big\langle \phi_{\mathrm{can}}(\overline{x} \otimes y), \phi_{\mathrm{can}}(\overline{x}' \otimes y') \big\rangle_{\mathscr{A}}
\end{aligned}
$$

for all $x, x', y, y' \in \mathscr{E}$. Similarly, an explicit computation shows that ϕ_{can} is also isometric with respect to the left-linear inner products. Thus ϕ_{can} is well-defined over $\widetilde{\otimes}_{\mathscr{B}}$, too, where it becomes injective, since now all inner products are non-degenerate. The surjectivity of ϕ_{can} follows at once from the fullness of $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$. This shows that ϕ_{can} is bijective and isometric, hence unitary and adjointable (for both inner products). Hence we proved the first part since the statement for ψ_{can} is obtained from symmetry: we have to exchange the roles of $\mathscr{A}$ and $\mathscr{B}$ as well as ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$. The compatibility (4.3.65) follows from the definition of a *-equivalence bimodule, proving the last part. □

We will now show that these are indeed the only invertible bimodules. To this end we need some more technical lemmas. The first shows that already under slightly milder assumptions the inner products turn out to be full:

Lemma 4.3.26 *Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ and ${}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{B}}(\mathscr{A})$ such that there is a isometric isomorphism*

$$
\phi\colon {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}. \tag{4.3.66}
$$

Then $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ is full.

Proof: Let $a \in \mathscr{A}$ be given. Since $\mathscr{A}$ is idempotent we find $b_i, c_i \in \mathscr{A}$ with $a = \sum_i b_i^* c_i = \sum_i \langle b_i, c_i\rangle_{\mathscr{A}}$. Since ϕ is surjective, we have $x_{ij}, y_{ij} \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and $x'_{ij}, y'_{ij} \in {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}}$ with $b_i = \phi(\sum_j x'_{ij} \otimes x_{ij})$ and $c_i = \phi(\sum_k y'_{ik} \otimes y_{ik})$. Since ϕ is isometric we compute

$$
\begin{aligned}
a &= \sum\nolimits_i \langle b_i, c_i\rangle_{\mathscr{A}} \\
&= \sum\nolimits_{i,j,k} \big\langle \phi(x'_{ij} \otimes x_{ij}), \phi(y'_{ik} \otimes y_{ik}) \big\rangle_{\mathscr{A}} \\
&= \sum\nolimits_{i,j,k} \big\langle x'_{ij} \otimes x_{ij}, y'_{ik} \otimes y_{ik} \big\rangle^{\mathscr{E}' \otimes \mathscr{E}}_{\mathscr{A}} \\
&= \sum\nolimits_{i,j,k} \big\langle x_{ij}, \langle x'_{ij}, y'_{ik}\rangle^{\mathscr{E}'}_{\mathscr{B}} \cdot y_{ik} \big\rangle^{\mathscr{E}}_{\mathscr{A}},
\end{aligned}
$$

which shows that $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ is full. □

The proof of the lemma only uses the surjectivity and isometry of ϕ. However, since $\mathscr{A}$ is assumed to be non-degenerate throughout this section, the canonical inner products are non-degenerate and hence an isometric map is necessarily injective.

The next lemma generalizes the result of Lemma 4.3.15 to arbitrary inner-product modules:

Lemma 4.3.27 *Let $\mathscr{E}_{\mathscr{A}}$ be an inner-product right $\mathscr{A}$-module. Then the restriction of $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ to the submodule $\mathscr{E} \cdot \mathscr{A} \subseteq \mathscr{E}$ is non-degenerate and the linear extension of*

$$\mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{A} \ni [x \otimes a] \;\mapsto\; x \cdot a \in \mathscr{E} \cdot \mathscr{A} \tag{4.3.67}$$

is an isometric isomorphism of inner-product modules over $\mathscr{A}$.

PROOF: First note that $\mathscr{E} \cdot \mathscr{A} \subseteq \mathscr{E}$ is a sub-module of $\mathscr{E}$ which now satisfies $(\mathscr{E} \cdot \mathscr{A}) \cdot \mathscr{A} = \mathscr{E} \cdot \mathscr{A}$ since $\mathscr{A}$ is idempotent. Let $\phi \in \mathscr{E} \cdot \mathscr{A}$ be such that $\langle \phi, x \cdot a \rangle^{\mathscr{E}}_{\mathscr{A}} = 0$ for all $x \cdot a \in \mathscr{E} \cdot \mathscr{A}$. Since $0 = \langle \phi, x \cdot a \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle \phi, x \rangle^{\mathscr{E}}_{\mathscr{A}} a$ and since $\mathscr{A}$ is assumed to be non-degenerate it follows that $\langle \phi, x \rangle^{\mathscr{E}}_{\mathscr{A}} = 0$ for all $x \in \mathscr{E}$ implying $\phi = 0$. Thus the restriction of $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ to the sub-module is still non-degenerate. Now, (4.3.67) is clearly well-defined over $\otimes_{\mathscr{A}}$ and surjective. Moreover,

$$\langle x \otimes a, y \otimes b \rangle^{\mathscr{E} \otimes \mathscr{A}}_{\mathscr{A}} = \left\langle a, \langle x, y \rangle^{\mathscr{E}}_{\mathscr{A}} \cdot b \right\rangle_{\mathscr{A}} = a^* \langle x, y \rangle^{\mathscr{E}}_{\mathscr{A}} b = \langle x \cdot a, y \cdot b \rangle^{\mathscr{E}}_{\mathscr{A}}$$

shows that (4.3.67) is isometric. Thus, (4.3.67) is well-defined over $\widehat{\otimes}_{\mathscr{A}}$ and becomes injective after passing to the quotient possibly needed for $\widehat{\otimes}_{\mathscr{A}}$. ☐

Lemma 4.3.28 *Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$. Then $\mathscr{B} \cdot \mathscr{E} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$ and the linear extension of*

$$\mathscr{B} \mathbin{\widehat{\otimes}_{\mathscr{B}}} \mathscr{E} \ni [b \otimes x] \;\mapsto\; b \cdot x \in \mathscr{B} \cdot \mathscr{E} \tag{4.3.68}$$

is a unitary intertwiner.

PROOF: Here the argumentation is slightly different. First it is clear that $\mathscr{B} \cdot \mathscr{E}$ is a sub-bimodule which now satisfies $\mathscr{B} \cdot (\mathscr{B} \cdot \mathscr{E}) = \mathscr{B} \cdot \mathscr{E}$ since $\mathscr{B}$ is idempotent. Moreover, (4.3.68) is well-defined over $\otimes_{\mathscr{B}}$ and surjective. Next we compute

$$\langle b \otimes x, c \otimes y \rangle^{\mathscr{B} \otimes \mathscr{E}}_{\mathscr{A}} = \left\langle x, \langle b, c \rangle_{\mathscr{B}} \cdot y \right\rangle^{\mathscr{E}}_{\mathscr{A}} = \left\langle x, (b^* c) \cdot y \right\rangle^{\mathscr{E}}_{\mathscr{A}} = \langle b \cdot x, c \cdot y \rangle^{\mathscr{E}}_{\mathscr{A}},$$

which means that (4.3.68) is isometric. Thus it is also well-defined on the quotient (4.3.68). Since now the inner product on $\mathscr{B} \mathbin{\widehat{\otimes}_{\mathscr{B}}} \mathscr{E}$ is non-degenerate, it follows that (4.3.68) is injective (whether the inner product on the target side is non-degenerate or not). Thus we have an isometric and bijective map which implies that the inner product on the target $\mathscr{B} \cdot \mathscr{E}$ is non-degenerate, too. ☐

For the invertibility of a 1-morphism in a bicategory one has the following general result. We formulate this for the case of a *-bicategory, the case of a bicategory can be deduced easily.

Proposition 4.3.29 *Let $\underline{\mathfrak{B}}$ be a *-bicategory over C.*

i.) Let $E \in \mathfrak{B}_1(b, a)$ be a 1-morphism from a to b. Then the tensor product with E from the right defines a *-functor

$$\mathsf{S}_E \colon \mathfrak{B}(c, b) \longrightarrow \mathfrak{B}(c, a) \tag{4.3.69}$$

for all c where

$$\mathsf{S}_E(M) = M \otimes_b E \quad \text{and} \quad \mathsf{S}_E(T \colon M \longrightarrow M') = (T \otimes_b \mathrm{id}_E \colon M \otimes_b E \longrightarrow M' \otimes_b E). \tag{4.3.70}$$

ii.) If E has a right inverse, i.e. a 1-morphism $E' \in \mathfrak{B}_1(a, b)$ with a unitary isomorphism $\psi \colon E \otimes_a E' \longrightarrow \mathsf{Id}_a$ then the map

$$\mathsf{S}_E \colon \mathfrak{B}_2(M', M) \longrightarrow \mathfrak{B}_2(M' \otimes_b E, M \otimes_b E) \tag{4.3.71}$$

is injective for all 1-morphisms $M, M' \in \mathfrak{B}_1(c, b)$.

iii.) *Let* $E \in \mathfrak{B}_1(b, a)$ *be invertible with inverse* $E' \in \mathfrak{B}_1(a, b)$ *and unitary isomorphisms*

$$\phi\colon E' \otimes_b E \longrightarrow \mathsf{Id}_a \quad \textit{and} \quad \psi\colon E \otimes_a E' \longrightarrow \mathsf{Id}_b. \tag{4.3.72}$$

Then for a given ϕ *with* (4.3.72) *there exists a unique* ψ *with* (4.3.72) *such that in addition*

$$\mathsf{left}(E) \circ (\psi \otimes_b \mathrm{id}_E) = \mathsf{right}(E) \circ (\mathrm{id}_E \otimes_a \phi) \circ \mathsf{asso}(E, E', E), \tag{4.3.73}$$

i.e. the diagram

$$\begin{array}{ccccc} & (E \otimes_a E') \otimes_b E & \xrightarrow{\mathsf{asso}(E,E',E)} & E \otimes_a (E' \otimes_b E) & \\ \swarrow^{\psi \otimes_b \mathrm{id}_E} & & & & \searrow^{\mathrm{id}_E \oplus_a \phi} \\ \mathsf{Id}_b \otimes_b E & & & & E \otimes_a \mathsf{Id}_a \\ & \searrow_{\mathsf{left}(E)} & & \swarrow_{\mathsf{right}(E)} & \\ & & E & & \end{array} \tag{4.3.74}$$

commutes.

PROOF: First it is clear that $M \otimes_b E \in \mathfrak{B}_1(c, a)$ and $T \otimes_b \mathrm{id}_E \in \mathfrak{B}_2(M' \otimes_b E, M \otimes_b E)$ since $\otimes_b$ is a functor. By the functoriality of $\otimes_b$ we have $S \circ T \mapsto (S \circ T) \otimes_b \mathrm{id}_E = (S \otimes_b \mathrm{id}_E) \circ (T \otimes_b \mathrm{id}_E)$ since $\mathrm{id}_E \circ \mathrm{id}_E = \mathrm{id}_E$. Moreover, $\mathrm{id}_M \mapsto \mathrm{id}_M \otimes_b \mathrm{id}_E = \mathrm{id}_{M \otimes_b E}$, showing that S_E is a functor indeed. Finally, the C-bilinearity of $\otimes_b$ shows that $T \mapsto T \otimes_b \mathrm{id}_E$ is C-linear. Together with $T^* \mapsto T^* \otimes_b \mathrm{id}_E = (T \otimes_b \mathrm{id}_E)^*$ this shows that S_E is a *-functor. For the second part, let E' be a right inverse to E with a unitary isomorphism $\psi\colon E \otimes_a E' \longrightarrow \mathsf{Id}_b$. Then $\mathrm{id}_M \otimes_b \psi\colon M \otimes_b (E \otimes_a E') \longrightarrow M \otimes_b \mathsf{Id}_b$ is again a unitary isomorphism by the functoriality of $\otimes_b$. The computation

$$\begin{aligned} &\mathsf{right}(M) \circ (\mathrm{id}_M \otimes_b \psi) \circ \mathsf{asso} \circ ((T \otimes_b \mathrm{id}_E) \otimes_a \mathrm{id}_{E'}) \circ \mathsf{asso}^{-1} \circ (\mathrm{id}_M \otimes_b \psi)^{-1} \circ \mathsf{right}(M)^{-1} \\ &= \mathsf{right}(M) \circ (\mathrm{id}_M \otimes_b \psi) \circ (T \otimes_b (\mathrm{id}_E \otimes_a \mathrm{id}_{E'})) \circ (\mathrm{id}_M \otimes_b \psi)^{-1} \circ \mathsf{right}(M)^{-1} \\ &= \mathsf{right}(M) \circ (\mathrm{id}_M \otimes_b \psi) \circ (T \otimes_b \mathrm{id}_{E \otimes_a E'}) \circ (\mathrm{id}_M \otimes_b \psi)^{-1} \circ \mathsf{right}(M)^{-1} \\ &= \mathsf{right}(M) \circ \big(T \otimes_b \big(\psi \circ \mathrm{id}_{E \otimes_a E'} \circ \psi^{-1}\big)\big) \circ \mathsf{right}(M)^{-1} \\ &= \mathsf{right}(M) \circ (T \otimes_b \mathrm{id}_{\mathsf{Id}_b}) \circ \mathsf{right}(M)^{-1} \\ &= T \end{aligned}$$

then shows that $T \mapsto T \otimes_b \mathrm{id}_E$ is injective. Here we used the properties of the natural isomorphism asso and right intensely. For the third part we note that (4.3.73) implies

$$\psi \otimes_b \mathrm{id}_E = \mathsf{left}(E)^{-1} \circ \mathsf{right}(E) \circ (\mathrm{id}_E \otimes_a \phi) \circ \mathsf{asso}(E, E', E),$$

and hence $\psi \otimes_b \mathrm{id}_E$ is determined uniquely. By the second part, this determines ψ uniquely. To show the existence of ψ satisfying the additional requirement (4.3.73) we assume to have an arbitrary $\tilde{\psi}$ with (4.3.72). Then define

$$\Theta = \mathsf{right}(E) \circ (\mathrm{id}_E \otimes_a \phi) \circ \mathsf{asso}(E, E', E) \circ (\tilde{\psi} \otimes_b \mathrm{id}_E)^{-1} \circ \mathsf{left}(E)^{-1},$$

which is the unitary automorphism $\Theta\colon E \longrightarrow E$ obtained from running through the diagram (4.3.74) clockwise. This is possible since all arrows are unitary isomorphisms by assumption. It allows to define

$$\psi = \tilde{\psi} \circ (\Theta \otimes_a \mathrm{id}_{E'})\colon E \otimes_a E' \longrightarrow \mathsf{Id}_b,$$

which is still a unitary isomorphism by *-functoriality of $\otimes$. A similar computation as above then shows that this new ψ has the property (4.3.73). ☐

Back to bimodules, we can assume for invertible bimodules without restrictions that the additional property (4.3.73) is already fulfilled for the isomorphisms ϕ and ψ. More explicitly, (4.3.73) reads for $x, y \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and $x' \in {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}}$

$$\psi(x \otimes x') \cdot y = x \cdot \phi(x' \otimes y), \tag{4.3.75}$$

which reminds already on the compatibility of the inner products for a *-equivalence bimodule. In particular, for a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} = {}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$ the canonical isomorphisms as in Proposition 4.3.25 have this property. Note however, that for a given invertible bimodule a priori there may be non-trivial unitary automorphisms Θ as in the proof of Proposition 4.3.29 which can be used to spoil the additional requirement (4.3.73).

Example 4.3.30 We consider a non-compact manifold M and the idempotent and non-degenerate commutative *-algebra $\mathscr{A} = \mathscr{C}^\infty_0(M)$. Then ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ has non-trivial unitary bimodule automorphisms like e.g. $\Theta\colon f \mapsto uf$ for some function $u \in \mathscr{C}^\infty(M)$ with $u\overline{u} = 1$. Thus in this case the compatibility between the morphisms ϕ and ψ can actually spoiled by introducing such an automorphisms of ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$.

We consider now again the change of the base ring functor from Example 3.1.14, see [29, Lem. 6.4] and [79, Prop. 4.7] for the C^*-algebraic version:

Lemma 4.3.31 *Let $\mathscr{F}_{\mathscr{B}}$ and $\mathscr{F}'_{\mathscr{B}}$ be inner-product modules over $\mathscr{B}$ with $\mathscr{F}'_{\mathscr{B}} \cdot \mathscr{B} = \mathscr{F}'_{\mathscr{B}}$. Moreover, let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ have the additional property that the left $\mathscr{B}$-multiplications $x \mapsto b \cdot x$ are elements of $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ for all $b \in \mathscr{B}$. Then*

$$\mathsf{S}_{\mathscr{E}}\big(\mathfrak{F}(\mathscr{F}_{\mathscr{B}}, \mathscr{F}'_{\mathscr{B}})\big) \subseteq \mathfrak{F}\big(\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, \mathscr{F}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big). \tag{4.3.76}$$

Proof: Since by assumption $\mathscr{F}'_{\mathscr{B}} \cdot \mathscr{B} = \mathscr{F}'_{\mathscr{B}}$, every operator on $\mathfrak{F}_{\mathscr{B}}(\mathscr{F}_{\mathscr{B}}, \mathscr{F}'_{\mathscr{B}})$ is a linear combination of operators of the form $\Theta_{y' \cdot b, y}$ with $y' \in \mathscr{F}'_{\mathscr{B}}$, $y \in \mathscr{F}_{\mathscr{B}}$ and $b \in \mathscr{B}$. Thus it suffices to show $\mathsf{S}_{\mathscr{E}}(\Theta_{y' \cdot b, y}) \in \mathfrak{F}_{\mathscr{A}}(\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, \mathscr{F}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$. For a fixed $y \in$
FB we consider the right $\mathscr{A}$-linear operator $t_y\colon \mathscr{E} \longrightarrow \mathscr{F} \mathbin{\widehat{\otimes}} \mathscr{E}$ defined by $t_y(x) = y \otimes x$. This operator is adjointable since

$$\big\langle y' \otimes x', t_y(x)\big\rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}} = \big\langle y' \otimes x', y \otimes x\big\rangle^{\mathscr{F} \otimes \mathscr{E}}_{\mathscr{A}} = \big\langle x', \langle y', y\rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x\big\rangle^{\mathscr{E}}_{\mathscr{A}} = \big\langle \langle y, y'\rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x', x\big\rangle^{\mathscr{E}}_{\mathscr{A}}.$$

Note that it suffices to consider factorizing tensors $y' \otimes x'$ since the right hand side has indeed the correct $\mathscr{B}$-bilinearity properties in y' and x' to make this well-defined on the whole tensor product. Thus the adjoint is $t^*_y(y' \otimes x') = \langle y, y'\rangle^{\mathscr{F}}_{\mathscr{B}} \cdot x'$. For $x \in \mathscr{E}$ and $z \in \mathscr{F}$ we find

$$\mathsf{S}_{\mathscr{E}}(\Theta_{y' \cdot b, y})(z \otimes x) = (\Theta_{y' \cdot b, y}(z)) \otimes x = \big(y' \cdot (b\langle y, z\rangle^{\mathscr{F}}_{\mathscr{B}})\big) \otimes x = y' \otimes \big(b\langle y, z\rangle^{\mathscr{F}}_{\mathscr{B}}\big) \cdot x = t_{y'}\big(b \cdot t^*_y(z \otimes x)\big).$$

Since the left multiplication $x \mapsto b \cdot x$ is a finite rank operator by assumption, the claim follows from the ideal properties (2.1.11). □

We are now in the position to formulate the following result characterizing the invertible bimodules completely [29, Thm. 6.1], see also [82, 106] for the particular case of C^*-algebras:

Theorem 4.3.32 (Invertible bimodules in $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$) *A bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Bimod}^*_1(\mathscr{B}, \mathscr{A})$ (or ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Bimod}^{\mathrm{str}}_1(\mathscr{B}, \mathscr{A})$, respectively) is invertible in $\underline{\mathsf{Bimod}}^*$ (or in $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively) if and only if there exists a $\mathscr{B}$-valued inner product ${}_{\mathscr{B}}\langle\,\cdot\,,\,\cdot\,\rangle^{\mathscr{E}}$ on ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ such that with this inner product ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ becomes a *-equivalence (or strong equivalence, respectively) bimodule. The $\mathscr{B}$-valued inner product is uniquely determined by this requirement.*

Proof: That an equivalence bimodule is invertible in the sense of $\underline{\mathsf{Bimod}}^*$ or $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively, was already shown in Proposition 4.3.25, an inverse bimodule is given by the complex conjugate. Thus consider an invertible ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Bimod}^*_1(\mathscr{B},\mathscr{A})$ with ${}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \in \mathsf{Bimod}^*_1(\mathscr{A},\mathscr{B})$ being an inverse bimodule. Moreover, let ϕ and ψ be the isometric isomorphisms as in (4.3.61) and (4.3.62). After choosing ϕ we can assume that ψ is the unique isomorphism with the additional requirement

$$\psi(x \otimes x') \cdot y = x \cdot \phi(x' \otimes y)$$

according to Proposition 4.3.29. We consider now the change of base ring functor $\mathsf{S}_{\mathscr{E}}$ applied to the $(\mathsf{C},\mathscr{B})$-bimodule ${}_{\mathsf{C}}\mathscr{B}_{\mathscr{B}} \in \mathsf{Bimod}^{\mathrm{str}}_1(\mathsf{C},\mathscr{B})$. Indeed, since all bimodule operations are in addition C-multilinear, we can view $\mathscr{B}$ as such a bimodule. Moreover, the adjointable intertwiners $2\text{-}\operatorname{Morph}({}_{\mathsf{C}}\mathscr{B}_{\mathscr{B}}, {}_{\mathsf{C}}\mathscr{B}_{\mathscr{B}})$ in the sense of $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ or $\underline{\mathsf{Bimod}}^*$ are just *all* adjointable endomorphisms $\mathfrak{B}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}})$. By Proposition 4.3.29, the functor $\mathsf{S}_{\mathscr{E}}$ gives an injective *-homomorphism

$$\mathsf{S}_{\mathscr{E}} \colon \mathfrak{B}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}}) \longrightarrow \mathfrak{B}_{\mathscr{A}}\big(\mathscr{B}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\big),$$

where again $\mathfrak{B}_{\mathscr{A}}(\mathscr{B}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}) = 2\text{-}\operatorname{Morph}({}_{\mathsf{C}}\mathscr{B}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathsf{C}}\mathscr{B}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$ in the bicategory sense of $\underline{\mathsf{Bimod}}^*$ or $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively. Viewing $\mathscr{E}_{\mathscr{A}}$ as $(\mathsf{C},\mathscr{A})$-bimodule we obtain analogously an injective *-homomorphism

$$\mathsf{S}_{\mathscr{E}'} \colon \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \longrightarrow \mathfrak{B}_{\mathscr{B}}(\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}}).$$

Moreover, we consider the unitary isomorphisms $\mathsf{left}(\mathscr{E}) \colon \mathscr{B}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and $\psi \colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \mathbin{\widehat{\otimes}_{\mathscr{A}}} {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}} \longrightarrow \mathsf{Id}_{\mathscr{B}}$ which induce *-isomorphisms for the corresponding *-algebras of adjointable operators on these modules by conjugation. Composing things, we eventually obtain *-homomorphisms

$$\mathsf{s}_{\mathscr{E}} \colon \mathfrak{B}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}}) \ni T \;\mapsto\; \mathsf{left}(\mathscr{E}) \circ (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{E}}) \circ \mathsf{left}(\mathscr{E})^{-1} \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$$

as well as

$$\mathsf{s}_{\mathscr{E}'} \colon \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \ni A \;\mapsto\; \psi \circ (A \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{E}'}) \circ \psi^{-1} \in \mathfrak{B}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}}).$$

Exchanging the role of $\mathscr{A}$ and $\mathscr{B}$, as well as the role of $\mathscr{E}$ and $\mathscr{E}'$ we obtain analogously the *-homomorphisms

$$\mathsf{t}_{\mathscr{E}'} \colon \mathfrak{B}_{\mathscr{A}}(\mathscr{A}_{\mathscr{A}}) \ni T \;\mapsto\; \mathsf{left}(\mathscr{E}') \circ (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{E}'}) \circ \mathsf{left}(\mathscr{E}')^{-1} \in \mathfrak{B}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})$$

as well as

$$\mathsf{t}_{\mathscr{E}} \colon \mathfrak{B}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}}) \ni B \;\mapsto\; \phi \circ (B \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{E}}) \circ \phi^{-1} \in \mathfrak{B}_{\mathscr{A}}(\mathscr{A}_{\mathscr{A}}).$$

In a first step, we want to show that $\mathsf{s}_{\mathscr{E}}$ and $\mathsf{s}_{\mathscr{E}'}$ are inverse to each other (and hence by symmetry also for $\mathsf{t}_{\mathscr{E}}$ and $\mathsf{t}_{\mathscr{E}'}$).

Thus let $T \in \mathfrak{B}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}})$ be given. Since $\mathscr{E} = \mathscr{B} \cdot \mathscr{E}$ and since ψ is surjective, we can write every element in $\mathscr{B}$ as linear combination of elements of the form $\psi(b \cdot x \otimes x')$. On these elements we compute

$$\begin{aligned}
((\mathsf{s}_{\mathscr{E}'} \circ \mathsf{s}_{\mathscr{E}})(T))(\psi(b \cdot x \otimes x')) &= \psi\big((\mathsf{s}_E(T)(b \cdot x)) \otimes x'\big) \\
&= \psi\Big(\big(\big(\mathsf{left}(\mathscr{E}) \circ (T \otimes \mathrm{id}_{\mathscr{E}}) \circ \mathsf{left}(\mathscr{E})^{-1}\big)(b \cdot x)\big) \otimes x'\Big) \\
&= \psi\big((T(b) \cdot x) \otimes x'\big) \\
&= T(b)\psi(x \otimes x') \\
&= T(b\psi(x \otimes x')) \\
&= T(\psi((b \cdot x) \otimes x')),
\end{aligned}$$

showing $(\mathsf{s}_{\mathscr{E}'} \circ \mathsf{s}_{\mathscr{E}})(T) = T$. Conversely, let $A \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ be given. Moreover, let $\Theta \in 2\text{-}\operatorname{End}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$ be the unitary automorphism as in the proof of Proposition 4.3.29, i.e. $\Theta(\psi(x \otimes x') \cdot y) = x \cdot \phi(x' \otimes y)$

for all $x, y \in \mathscr{E}$ and $x' \in \mathscr{E}'$. We know that we can choose ψ in such a way that $\Theta = \mathrm{id}$. However, we shall not make use of this at the moment. Again, we write every element of $\mathscr{E}$ as linear combination of elements of the form $\psi(x \otimes x') \cdot y$. Using this, we compute

$$\begin{aligned}
((\mathsf{s}_{\mathscr{E}} \circ \mathsf{s}_{\mathscr{E}'})(A))(\psi(x \otimes x') \cdot y) &= \big(\mathsf{left}(\mathscr{E}) \circ (\mathsf{s}_{\mathscr{E}'}(A) \otimes \mathrm{id}_{\mathscr{E}}) \circ \mathsf{left}(\mathscr{E})^{-1}\big)(\psi(x \otimes x') \cdot y) \\
&= \big(\mathsf{left}(\mathscr{E}) \circ \big((\psi \circ (A \otimes \mathrm{id}_{\mathscr{E}'}) \circ \psi^{-1}) \otimes \mathrm{id}_{\mathscr{E}}\big)\big)(\psi(x \otimes x') \otimes y) \\
&= \mathsf{left}(\mathscr{E})\big(\psi\big(A(x) \otimes x'\big) \otimes y\big) \\
&= \psi(A(x) \otimes x') \cdot y \\
&= \Theta^{-1}\big(A(x) \cdot \phi(x' \otimes y)\big) \\
&= \Theta^{-1}\big(A(x \cdot \phi(x' \otimes y))\big) \\
&= \Theta^{-1}\big(A\big(\Theta(\psi(x \otimes x') \cdot y)\big)\big) \\
&= (\Theta^{-1} \circ A \circ \Theta)(\psi(x \otimes x') \cdot y),
\end{aligned}$$

and thus

$$(\mathsf{s}_{\mathscr{E}} \circ \mathsf{s}_{\mathscr{E}'})(A) = \Theta^{-1} A \Theta.$$

This shows that $\mathsf{s}_{\mathscr{E}}$ and also $\mathsf{s}_{\mathscr{E}'}$ are bijective. Since the inverse is unique we conclude that $\Theta^{-1} A \Theta = A$ for *all* $A \in \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. This shows that Θ is necessarily a central element in all adjointable endomorphisms of $\mathscr{E}_{\mathscr{A}}$. By Proposition 4.3.29 we know that we can achieve $\Theta = \mathrm{id}$, which would simplify the above computation slightly. In any case, $\mathsf{s}_{\mathscr{E}}$ and $\mathsf{s}_{\mathscr{E}'}$ are inverse to each other.

In a second step we recall that $\mathfrak{F}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}}) \cong \mathscr{B}$ via the left multiplications. Indeed, this follows from the fact that ${}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}$ is a strong equivalence bimodule, see Lemma 4.1.5, and Theorem 4.2.1. We want to determine the image of $\mathfrak{F}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}})$ under the *-isomorphism $\mathsf{s}_{\mathscr{E}}$ and similar for $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. To this end we compute for $b \in \mathscr{B}$, $x, y, z \in \mathscr{E}$, and $x' \in \mathscr{E}'$

$$\begin{aligned}
(\mathsf{S}_{\mathscr{E}'}(\Theta_{b \cdot x, y}))(z \otimes x') &= \Theta_{b \cdot x, y}(z) \otimes x' \\
&= \big(b \cdot x \cdot \langle y, z \rangle^{\mathscr{E}}_{\mathscr{A}}\big) \otimes x' \\
&= (b \cdot \Theta_{x,y}(z)) \otimes x' \\
&= (b \cdot \mathsf{S}_{\mathscr{E}'}(\Theta_{x,y}))(z \otimes x'),
\end{aligned}$$

yielding $\mathsf{S}_{\mathscr{E}'}(\Theta_{b \cdot x, y}) = b \cdot \mathsf{S}_{\mathscr{E}'}(\Theta_{x,y})$. Applying the bimodule morphism ψ gives

$$\mathsf{s}_{\mathscr{E}'}(\Theta_{b \cdot x, y}) = \psi \circ (b \cdot \mathsf{S}_{\mathscr{E}'}(\Theta_{x,y})) \circ \psi^{-1} = b \cdot \big(\psi \circ \mathsf{S}_{\mathscr{E}'}(\Theta_{x,y}) \circ \psi^{-1}\big) = b \cdot \mathsf{s}_{\mathscr{E}'}(\Theta_{x,y}) \in \mathfrak{F}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}}),$$

since the left multiplication with b is in $\mathfrak{F}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}})$ and the finite-rank operators are a *-ideal. Since finally $\mathscr{B} \cdot \mathscr{E} = \mathscr{E}$ we conclude that the rank one operators $\Theta_{b \cdot x, y}$ span all finite rank operators and hence

$$\mathsf{s}_{\mathscr{E}'} \colon \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \longrightarrow \mathfrak{F}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}}) \cong \mathscr{B}. \tag{$*$}$$

In a third step, we want to show that $\mathsf{s}_{\mathscr{E}'}$, restricted to $\mathfrak{F}_{A}(\mathscr{E}_{\mathscr{A}})$, is surjective onto $\mathfrak{F}_{\mathscr{B}}(\mathscr{B}_{\mathscr{B}})$. To this end we use $(*)$ to turn $\mathscr{B}$ into a $(\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}), \mathscr{B})$-bimodule. By symmetry we see that

$$\mathsf{t}_{\mathscr{E}} \colon \mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}}) \longrightarrow \mathfrak{F}_{\mathscr{A}}(\mathscr{A}_{\mathscr{A}}) \cong \mathscr{A}, \tag{$**$}$$

which allows to turn $\mathscr{A}$ into a $(\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}}), \mathscr{A})$-bimodule. Note that these bimodules are even *-representations of the finite-rank operators since $\mathsf{s}_{\mathscr{E}}$ and $\mathsf{s}_{\mathscr{E}'}$ are *-homomorphisms. By Lemma 4.3.26 and Theorem 4.2.1 it follows that the bimodules ${}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{E}'_{\mathscr{B}}$ as well as ${}_{\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\mathscr{E}_{\mathscr{A}}$ are *-equivalence bimodules if one uses the canonical inner product $\Theta_{\cdot,\cdot}$. Thus we know that the complex conjugate bimodule provides an inverse. In particular, we have

$${}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \cong {}_{\mathscr{B}}\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \mathbin{\widehat{\otimes}}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} {}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{E}'_{\mathscr{B}}$$

via the isometric isomorphism ψ_{can} from Proposition 4.3.25. Collecting these results yields the following isometric isomorphisms (in the sense of $\underline{\mathsf{Bimod}}^*$)

$$\begin{aligned}
{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} &\cong {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \\
&\cong \left({}_{\mathscr{B}}\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \mathbin{\widehat{\otimes}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}} {}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{E}'_{\mathscr{B}} \right) \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \\
&\cong {}_{\mathscr{B}}\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \mathbin{\widehat{\otimes}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}} \left({}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{E}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \right) \\
&\cong {}_{\mathscr{B}}\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \mathbin{\widehat{\otimes}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}} {}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{A}_{\mathscr{A}},
\end{aligned} \tag{$\star$}$$

where we first use $\mathsf{left}(\mathscr{E})$, then Proposition 4.3.25, after that the associativity $\mathsf{asso}(\overline{\mathscr{E}}', \mathscr{E}', \mathscr{E})$, and finally the isomorphism ϕ turning $\mathscr{A}$ into a $(\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}}), \mathscr{A})$-bimodule: indeed, transporting the left module structure of ${}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{E}'_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ to $\mathscr{A}$ via ϕ gives precisely the left module structure induced by $\mathsf{t}_{\mathscr{E}}$ according to $(**)$. This follows from the explicit definition of $\mathsf{t}_{\mathscr{E}}$. Since ${}_{\mathscr{B}}\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}$ is a *-equivalence bimodule we know that via the left multiplications we have

$$\mathscr{B} \cong \mathfrak{F}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\left(\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \right), \tag{$\star\star$}$$

according the Theorem 4.2.1. Moreover, we know that $\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} = \overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \cdot \mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})$. Furthermore, since via $(**)$ the finite-rank operators $\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})$ act on $\mathscr{A}_{\mathscr{A}}$ via finite-rank operators, we can apply Lemma 4.3.31 for $\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}$ and ${}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{A}_{\mathscr{A}}$. This gives that the change of base ring functor $\mathsf{S}_{\mathscr{A}}$ preserves the finite-rank operators, i.e.

$$\mathsf{S}_{\mathscr{A}} \colon \mathfrak{F}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\left(\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \right) \longrightarrow \mathfrak{F}_{\mathscr{A}}\left(\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \mathbin{\widehat{\otimes}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}} {}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{A}_{\mathscr{A}} \right).$$

On the other hand we know that the tensor product on the right hand side is isometrically isomorphic to $\mathscr{E}_{\mathscr{A}}$ according to $(\star)$ while the finite-rank operators on the left hand side are *-isomorphic to $\mathscr{B}$ via the left multiplications according to $(\star\star)$. This yields eventually the chain of isomorphisms

$$\mathscr{B} \cong \mathfrak{F}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\left(\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \right) \xrightarrow{\mathsf{S}_{\mathscr{A}}} \mathfrak{F}_{\mathscr{A}}\left(\overline{\mathscr{E}}'_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})} \mathbin{\widehat{\otimes}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}} {}_{\mathfrak{F}_{\mathscr{B}}(\mathscr{E}'_{\mathscr{B}})}\mathscr{A}_{\mathscr{A}} \right) \cong \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}).$$

Since the first and the last isomorphisms are implemented by the left multiplications we see that this chain of isomorphisms *is* the left multiplication of $\mathscr{B}$ on $\mathscr{E}_{\mathscr{A}}$ since $\mathsf{S}_{\mathscr{A}}$ is a *bimodule* morphism. So finally we have the desired surjectivity and hence the *-isomorphism

$$\mathscr{B} \cong \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \tag{☺}$$

via $\mathsf{s}_{\mathscr{E}'}$.

By Lemma 4.3.26 and Theorem 4.2.1 we know that ${}_{\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})}\mathscr{E}_{\mathscr{A}}$ is a *-equivalence bimodule. So pulling back the canonical inner product $\Theta_{\cdot,\cdot}$ to $\mathscr{B}$ via (☺) shows the existence of a compatible $\mathscr{B}$-valued inner product making ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ a *-equivalence bimodule. This shows the first part of the theorem.

Now assume in addition that ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{E}'_{\mathscr{B}}$ are equipped with *completely positive* inner products. Since inverses in $\underline{\mathsf{Bimod}}^*$ of 1-morphisms are again unique up to isometric isomorphisms we conclude that

$$\left(\mathscr{E}'_{\mathscr{B}}, \langle \cdot\,, \cdot \rangle'_{\mathscr{B}}\right) \cong \left(\overline{\mathscr{E}}_{\mathscr{B}}, \langle \cdot\,, \cdot \rangle^{\overline{\mathscr{E}}}_{\mathscr{B}} \right),$$

where $\langle \cdot\,, \cdot \rangle^{\overline{\mathscr{E}}}_{\mathscr{B}}$ is the canonical inner product $\Theta_{\cdot,\cdot}$ under the *-isomorphism (☺). But since the inner product on the left hand side is completely positive, the inner product $\langle \cdot\,, \cdot \rangle^{\overline{\mathscr{E}}}_{\mathscr{B}}$ is completely positive, too. But then also the $\mathscr{B}$-valued inner product on ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is completely positive, as this notion behaves well with respect to complex conjugating the module, see Exercise 2.4.9. This shows that ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ carries the structure of a strong equivalence bimodule in this case.

Finally, the uniqueness of the $\mathscr{B}$-valued inner product is easy: this follows directly from Theorem 4.2.1 once we know that ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a *-equivalence bimodule. □

Remark 4.3.33 The above technical proof simplifies drastically if one considers unital *-algebras only. In this case $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ and hence most of the above proof becomes trivial. Form this case one also deduces easily the proof of Theorem 4.3.5, see also Exercise 4.4.10.

4.4 Exercises

Exercise 4.4.1 (Full idempotents) Show that a finitely generated projective module $\mathscr{M}_{\mathscr{A}}$ over a unital ring $\mathscr{A}$ is full iff the span of elements of the form $\varphi(x) \in \mathscr{A}$ is the whole ring $\mathscr{A}$, where $\varphi \in \operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A})$ is from the dual module and $x \in \mathscr{M}_{\mathscr{A}}$. With other words, fullness is a measure on how non-trivial the dual module is.

Hint: Show first that for module $\mathscr{M}_{\mathscr{A}}$ the elements of the form $\varphi(x)$ constitute a two-sided ideal in $\mathscr{A}$. If now $\mathscr{M}_{\mathscr{A}} = e\mathscr{A}^n$ with an idempotent $e \in \mathrm{M}_n(\mathscr{A})$ it is easy to see, using Exercise 2.4.15, that $\mathbb{1}$ belongs to this ideal iff e is full.

Exercise 4.4.2 (Morita's Theorem) A rather down-to-earth proof of Theorem 4.3.5 can be obtained as follows. Let $\mathscr{A}$ and $\mathscr{B}$ be unital rings and let ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ be bimodules satisfying (4.3.5) with bimodule isomorphisms $\psi\colon {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \longrightarrow {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}$ and $\phi\colon {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$.

i.) Show that ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \ni x \mapsto \mathbb{1} \otimes x \in \mathscr{B} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ and $\psi \otimes \mathrm{id}\colon {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ are $(\mathscr{B}, \mathscr{A})$-bimodule isomorphisms. Give explicit formulas for their inverses.

ii.) Show that for a given choice of ϕ there is a (unique) choice of ψ such that the diagram

$$\begin{array}{ccc} & {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} & \\ \swarrow^{\psi \otimes \mathrm{id}} & & \searrow^{\mathrm{id} \otimes \phi} \\ \mathscr{B} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} & & {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathscr{A} \\ \searrow & & \swarrow \\ & {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}} & \end{array} \tag{4.4.1}$$

commutes. In the following we assume to have made such a choice.

iii.) Let $y \in {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ be fixed. Show that the map $x \mapsto \phi(y \otimes x)$ is right $\mathscr{A}$-linear.

iv.) Show that there is a $k \in \mathbb{N}$ and $y_i \in {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ and $x_i \in {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ for $i = 1, \ldots, k$ such that $\mathbb{1}_{\mathscr{B}} = \sum_i \psi(x_i \otimes y_i)$. Conclude that

$$x = \sum_{i=1}^{k} x_i \cdot \phi(y_i \otimes x) \tag{4.4.2}$$

holds for all $x \in {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$. This gives a dual basis for $\mathscr{M}_{\mathscr{A}}$.

v.) Prove that the map ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}} \ni y \mapsto (x \mapsto \phi(y \otimes x)) \in \operatorname{Hom}_{\mathscr{A}}({}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}, \mathscr{A})$ is a left $\mathscr{A}$-linear isomorphism between ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ and the dual module of $\mathscr{M}_{\mathscr{A}}$.

Hint: Use the above dual basis to show that this map is surjective. For the injectivity, suppose $\phi(y \otimes x) = 0$ and hence $y \otimes x = 0$ for all $x \in \mathscr{M}_{\mathscr{A}}$ since ϕ is an isomorphism. Exchanging the roles of ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ gives an analogous diagram to (4.4.1). Use this to conclude that $y = 0$.

vi.) Now let $B \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ be a right $\mathscr{A}$-linear endomorphism. Show that there exists a unique $b \in \mathscr{B}$ with $B(x) = b \cdot x$.

Hint: Use the dual basis to write $B(x) = \sum_i x_i \cdot \phi(y_i \otimes B(x))$. Since $x \mapsto \phi(y_i \otimes B(x))$ is right $\mathscr{A}$-linear there exists a (unique) $z_i \in {}_{\mathscr{A}}\mathscr{M}'_{\mathscr{B}}$ with $\phi(y_i \otimes B(x)) = \phi(z_i \otimes x)$. From this one can deduce the existence of b. For uniqueness, suppose $b \cdot x = 0$ for all $x \in {}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$. Then also $\sum_i b \cdot x_i \otimes y_i = 0$. Apply now ψ.

vii.) Show that $\mathscr{M}_{\mathscr{A}}$ is full.

Hint: Exercise 4.4.1.

This clarifies the more complicated direction in Theorem 4.3.5. If one has a finitely generated, projective and full right $\mathscr{A}$-module, it is fairly easy to verify that this gives an equivalence bimodule between $\mathscr{A}$ and $\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ by using a dual basis.

Exercise 4.4.3 (Full projections of $\mathscr{C}^\infty(M)$) Consider a connected smooth manifold M and a projection $P \in \mathrm{M}_n(\mathscr{C}^\infty(M))$. Show that $P \neq 0$ iff P is full using the pointwise trace. How does the situation look like in the non-connected case? Generalize this to continuous functions on a reasonable topological space.

Exercise 4.4.4 (Complete positivity of $\Theta_{\cdot,\cdot}$) Let $\mathscr{A}$ be a unital *-algebra and let $\mathscr{E}_{\mathscr{A}}$ be an inner-product right $\mathscr{A}$-module. Suppose that there exist vectors $x_1, \ldots, x_N \in \mathscr{E}_{\mathscr{A}}$ with

$$\mathbb{1}_{\mathscr{A}} = \sum_{r=1}^{N} \langle x_r, x_r \rangle_{\mathscr{A}}. \tag{4.4.3}$$

Show that the canonical $\mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$-valued inner product $\Theta_{\cdot,\cdot}$ is completely positive.

Exercise 4.4.5 (The bimodule structure of $\operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}})$) Let $\mathscr{A}$ be a unital ring and let $\mathscr{M}_{\mathscr{A}}$ be a right $\mathscr{A}$-module.

i.) Show that the dual module $\operatorname{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{A}_{\mathscr{A}})$ becomes a $(\mathscr{A}, \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}))$-bimodule via (4.3.8) which is always strongly non-degenerate for both module structures.

ii.) Show that the maps (4.3.9) and (4.3.10) are well-defined bimodule morphisms.

Exercise 4.4.6 (Coherence in $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$) Formulate and prove the coherence properties to show that $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$ are bicategories.

Exercise 4.4.7 (Linear combinations of natural transformations) Let $\mathfrak{C}$ and $\mathfrak{D}$ be *-categories over C with two *-functors $\mathsf{F}, \mathsf{G}\colon \mathfrak{C} \longrightarrow \mathfrak{D}$. Show that for natural transformations $\mathsf{s}, \mathsf{t}\colon \mathsf{F} \longrightarrow \mathsf{G}$ the linear combination $z\mathsf{s}+w\mathsf{t}$ with $z, w \in \mathsf{C}$ as well as the adjoint s^* are again natural transformations.

Exercise 4.4.8 (Rieffel induction is a *-functor) Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras over C. Verify in detail that the Rieffel induction $\mathsf{R}_{\mathscr{E}}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{B})$ for a bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ and an auxiliary *-algebra $\mathscr{D}$ is indeed a *-functor.

Exercise 4.4.9 (Passing to strongly non-degenerate *-representations) Let $\mathscr{B}$ be an idempotent and non-degenerate *-algebra and let $\mathscr{A}$ be an arbitrary *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ and let $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ be an intertwiner. Define

$$\mathsf{NonDeg}({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}) = \mathscr{B} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \quad \text{and} \quad \mathsf{NonDeg}(T) = \mathrm{id}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} T. \tag{4.4.4}$$

Show that this defines a functor $\mathsf{NonDeg}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B})$. Is this functor an equivalence of categories?

Exercise 4.4.10 (Invertible bimodules: unital case) Suppose that $\mathscr{A}$ and $\mathscr{B}$ are unital *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Give a more direct proof of Theorem 4.3.32 under this simplifying assumption along the same lines as for the classical Morita theorem.

Hint: Exercise 4.4.2 gives enough inspiration.

Chapter 5

The Picard Groupoids and Morita Invariants

We come back to our original goal to compare the representation theories of two given *-algebras $\mathscr{A}$ and $\mathscr{B}$. Using Rieffel induction with respect to a bimodule $\mathscr{E} \in {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B})$ we have a functor

$$\mathsf{R}_{\mathscr{E}} \colon {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{B}) \tag{5.0.1}$$

for any auxiliary *-algebra $\mathscr{D}$, in particular for $\mathscr{D} = \mathsf{C}$. This gives us a very powerful tool for comparing the representation theories. We want to understand under which conditions on the bimodule $\mathscr{E}$ we obtain an "inverse" functor to $\mathsf{R}_{\mathscr{E}}$. Of course, we can ask the same question for *-Rep being replaced by *-Mod or Mod in the purely ring-theoretic situation.

First we recall that for categories the good notion of isomorphism is *equivalence*: Two categories $\mathfrak{C}$ and $\mathfrak{D}$ are called *equivalent* if there exist functors $\mathsf{F} \colon \mathfrak{C} \longrightarrow \mathfrak{D}$ and $\mathsf{G} \colon \mathfrak{D} \longrightarrow \mathfrak{C}$ such that $\mathsf{G} \circ \mathsf{F} \colon \mathfrak{C} \longrightarrow \mathfrak{C}$ and $\mathsf{F} \circ \mathsf{G} \colon \mathfrak{D} \longrightarrow \mathfrak{D}$ are naturally isomorphic to the identity functors $\mathrm{id}_{\mathfrak{C}}$ and $\mathrm{id}_{\mathfrak{D}}$, respectively. Note that in general this is a *weaker* condition than having isomorphisms, i.e. $\mathsf{G} \circ \mathsf{F} = \mathrm{id}_{\mathfrak{C}}$ and $\mathsf{F} \circ \mathsf{G} = \mathrm{id}_{\mathfrak{D}}$. The notion of isomorphism in category theory is usually rather pointless and occurs typically only in rare situations.

Thus we are looking for a functor which is inverse to $\mathsf{R}_{\mathscr{E}}$ only up to a natural isomorphism. With our present machinery on equivalence bimodules it is now fairly easy to see that a strong equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ will provide an equivalence of the categories *-$\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})$ and *-$\mathsf{Rep}_{\mathscr{D}}(\mathscr{B})$ via $\mathsf{R}_{\mathscr{E}}$ with the "inverse" functor given by the Rieffel induction with the complex conjugate ${}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$. Similarly, *-equivalence bimodules implement an equivalence between *-$\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ and *-$\mathsf{Mod}_{\mathscr{D}}(\mathscr{B})$ while the ring-theoretic equivalence bimodules give an equivalence of $\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ and $\mathsf{Mod}_{\mathscr{D}}(\mathscr{B})$. In this sense, the representation theories become *Morita invariants.* While this can be done directly and very explicitly, we postpone the proof and obtain it as a corollary of a much more detailed construction. This provides additional insights beyond the mere equivalence of the representations.

In order to uncover this additional structure of the representation theories (and many other Morita invariants), we take a little excursion into the realm of (bi-) groupoids and (bi-) groupoid actions: the invertible bimodules form a groupoid, the so-called Picard groupoid, for which we have several flavours as usual. We will investigate this groupoid and the corresponding isotropy groups, the *Picard groups.* It will turn out that a Morita invariant is always equipped with a group action of the Picard group which is also part of the invariant.

5.1 The Picard Bigroupoids

In order to define the Picard (bi-) groupoids we first remind on some basic notions about groupoids and bigroupoids. Then the Picard bigroupoids will be the bigroupoids of invertible 1-morphisms in the bicategories $\underline{\mathsf{Bimod}}$, $\underline{\mathsf{Bimod}}^*$, and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively.

5.1.1 Groupoids and Bigroupoids

First we recall the definition of a groupoid:

Definition 5.1.1 (Groupoid) *A groupoid $\mathfrak{G}$ is a category where all morphisms are invertible.*

To visualize a groupoid, one identifies the objects $a \in \mathrm{Obj}(\mathfrak{G})$ of $\mathfrak{G}$ with the local unit elements $\mathrm{id}_a \in \mathrm{Morph}(a, a)$ and views the morphisms $g\colon a \longrightarrow b$ as arrows from a to b. One calls $a = \mathrm{source}(g)$ the *source* of g while $b = \mathrm{target}(g)$ is the *target*. For every arrow $g\colon a \longrightarrow b$ there is an inverse arrow $\mathrm{inv}(g) = g^{-1}\colon b \longrightarrow a$ with $\mathrm{source}(g^{-1}) = \mathrm{target}(g)$ and $\mathrm{target}(g^{-1}) = \mathrm{source}(g)$. The composition of arrows g and h is denoted as usual by $h \circ g$ provided we can compose at all, i.e. $\mathrm{target}(g) = \mathrm{source}(h)$. Thus a groupoid is determined by the local unit elements $\mathfrak{G}_0$, the arrows $\mathfrak{G}_1$ as well as the structure maps $\mathrm{target}, \mathrm{source}\colon \mathfrak{G}_1 \longrightarrow \mathfrak{G}_0$, $\mathrm{inv}\colon \mathfrak{G}_1 \longrightarrow \mathfrak{G}_1$, $\mathrm{id}\colon \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$. Usually, the composable arrows are denoted by

$$\mathfrak{G}_{(2)} = \left\{(h, g) \in \mathfrak{G}_1 \times \mathfrak{G}_1 \;\middle|\; \mathrm{source}(h) = \mathrm{target}(g)\right\} \subseteq \mathfrak{G}_1 \times \mathfrak{G}_1, \tag{5.1.1}$$

such that the composition is a map $\circ\colon \mathfrak{G}_{(2)} \longrightarrow \mathfrak{G}_1$. All the structure maps can be combined in one picture

$$\mathfrak{G}_0 \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} \xrightarrow{\;\text{id}\;} \mathfrak{G}_1 \xrightarrow{\;\text{inv}\;} \mathfrak{G}_1 \xleftarrow{\;\circ\;} \mathfrak{G}_{(2)}. \tag{5.1.2}$$

The following simple example of a groupoid is of fundamental importance:

Proposition 5.1.2 (Isomorphism groupoid) *Let $\mathfrak{C}$ be a category. Then the invertible morphisms of $\mathfrak{C}$ constitute a groupoid, the isomorphism groupoid* $\mathrm{Iso}(\mathfrak{C})$ *of $\mathfrak{C}$.*

The proof is trivial and a simple consequence of the axioms of a category, see Exercise 5.4.1.

Example 5.1.3 The isomorphism groupoid of the category of *-algebras over C consists of all *-algebras over C as objects and the *-isomorphisms $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ as morphisms $\mathrm{Iso}^*(\mathscr{B}, \mathscr{A})$ from $\mathscr{A}$ to $\mathscr{B}$. Note that for unital *-algebras a *-isomorphism satisfies necessarily $\Phi(\mathbb{1}_{\mathscr{A}}) = \mathbb{1}_{\mathscr{B}}$. Thus the difference in the definition of morphisms in *-alg and *-Alg disappears on the level of isomorphisms. This gives a subgroupoid $\mathrm{Iso}(^*\text{-}\mathsf{Alg}) \subseteq \mathrm{Iso}(^*\text{-}\mathsf{alg})$. For this special situation we denote the isomorphism groupoid also simply by Iso^*. Analogously, one has the isomorphism groupoid $\mathrm{Iso}(\mathsf{Ring})$ of the category Ring consisting of the ring isomorphisms, simply denoted by Iso if there is no possibility of confusion.

Proposition 5.1.4 *Let $\mathfrak{G}$ be a groupoid.*

i.) For every object $a \in \mathfrak{G}_0$ the set

$$\mathfrak{G}(a) = \{g\colon a \longrightarrow a\} = \mathrm{Morph}(a, a) \tag{5.1.3}$$

is a group with respect to the unit element id_a *and the composition $\circ$.*

ii.) If $\mathrm{Morph}(b, a) \neq \emptyset$ *then $\mathfrak{G}(a) \cong \mathfrak{G}(b)$ and every morphism $g\colon a \longrightarrow b$ yields a group isomorphism*

$$\mathfrak{G}(a) \ni h \;\mapsto\; ghg^{-1} \in \mathfrak{G}(b). \tag{5.1.4}$$

iii.) If $\mathrm{Morph}(b, a) \neq \emptyset$ *then the map*

$$\mathfrak{G}(a) \ni g \;\mapsto\; hg \in \mathrm{Morph}(b, a) \tag{5.1.5}$$

is a bijection for every fixed choice of $h \in \mathrm{Morph}(b, a)$.

Again, the statements are simple consequences of the definition of a groupoid, see Exercise 5.4.2. Nevertheless, we listed them here as they will have important applications later on. The group $\mathfrak{G}(a)$ is called the *isotropy group* of a and the class

$$\mathfrak{G} \cdot a = \left\{ b \in \mathrm{Obj}(\mathfrak{G}) \;\middle|\; \mathrm{Morph}(b, a) \neq \emptyset \right\} \tag{5.1.6}$$

is called the *orbit* of a in $\mathfrak{G}$. Obviously, we have $b \in \mathfrak{G} \cdot a$ if and only if $a \in \mathfrak{G} \cdot b$ which is the case if and only if there is a morphism between a and b. Along an orbit, all the isotropy groups are isomorphic. In particular, a groupoid with one object is just a group.

We come now to the more interesting notion of a bigroupoid. As already for a bicategory, we want to relax the associativity as well as the invertibility of 1-morphisms, described in a controlled way by 2-morphisms. This is accomplished with the following definition:

Definition 5.1.5 (Bigroupoid) *A bigroupoid $\underline{\mathfrak{G}}$ is a bicategory such that all 1-morphisms are invertible in the sense of a bicategory.*

Let us unwind the definition: for every 1-morphism $E \in \mathfrak{G}_1(b, a)$ from a to b there is a 1-morphism $E' \in \mathfrak{G}_1(a, b)$ from b to a together with 2-isomorphisms

$$\phi\colon E' \otimes_b E \longrightarrow \mathsf{Id}_a \quad \text{and} \quad \psi\colon E \otimes_b E' \longrightarrow \mathsf{Id}_b, \tag{5.1.7}$$

where E' is uniquely determined up to 2-isomorphisms. The 2-isomorphisms ϕ and ψ with (5.1.7) are not unique in general, only there existence is required. In general, there is no unique map to construct E' from E, we only require the existence of E'. Note, however, that the definition of a bigroupoid varies in the literature: sometimes it is required that there is a functor

$$\mathrm{inv}\colon \underline{\mathfrak{G}}(a, b) \longrightarrow \underline{\mathfrak{G}}(b, a), \tag{5.1.8}$$

with the obvious properties such that $E' = \mathrm{inv}(E)$ is an inverse to E. Though we do not require this from the beginning, in our examples we always have such an inversion functor. Moreover, sometimes it is also required that the 2-morphisms in a bigroupoid are always *isomorphisms*. From the point of view of Morita theory this requirement seems to be unnecessarily strong, hence we do not follow this convention.

We note the following general result which can be obtained immediately from the definitions:

Proposition 5.1.6 *Let $\underline{\mathfrak{B}}$ be a bicategory.*

i.) The invertible 1-morphisms in $\underline{\mathfrak{B}}$ constitute a bigroupoid, called the isomorphism bigroupoid of $\underline{\mathfrak{B}}$, where as objects one takes the same objects as of $\underline{\mathfrak{B}}$, as 1-morphisms the invertible 1-morphisms of $\underline{\mathfrak{B}}$, and as 2-morphisms all corresponding 2-morphisms of $\underline{\mathfrak{B}}$.

ii.) The classifying category of a bigroupoid $\underline{\mathfrak{G}}$ is a groupoid $\mathfrak{G}$, called the classifying groupoid of $\underline{\mathfrak{G}}$.

iii.) The isomorphism groupoid of the classifying category of $\underline{\mathfrak{B}}$ is the classifying groupoid of the isomorphism bigroupoid of $\underline{\mathfrak{G}}$.

In case of a *-bicategory over C we have of course an analogous construction of a classifying category: now we base the notion of isomorphic 1-morphisms on the *unitary* 2-morphisms as before.

5.1.2 The Definition of the Picard Bigroupoids

After this general discussion on (bi-) groupoids we come now to our main example of the Picard (bi-) groupoids in various flavours. To this end, we start with the following lemma:

Lemma 5.1.7 *Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \in \underline{\mathsf{Bimod}}^*(\mathscr{B}, \mathscr{A})$ be invertible.*

i.) *Every morphism* $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ *in the sense of* $\underline{\mathsf{Pic}}^*(\mathscr{B},\mathscr{A})$ *is a* 2*-morphism* $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ *in the sense of* $\underline{\mathsf{Bimod}}^*$.

ii.) *An isometric* 2*-isomorphism* $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ *in the sense of* $\underline{\mathsf{Bimod}}^*$ *is an isomorphism* $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ *in the sense of* $\underline{\mathsf{Pic}}^*(\mathscr{B},\mathscr{A})$.

iii.) *Every* 2*-endomorphism* $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ *in the sense of* $\underline{\mathsf{Bimod}}^*$ *is an endomorphism* $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ *in the sense of* $\underline{\mathsf{Pic}}^*(\mathscr{B},\mathscr{A})$.

PROOF: The subtleties of this lemma arise from the fact that in $\underline{\mathsf{Pic}}^*$ we have to take care of *both* inner products while in $\underline{\mathsf{Bimod}}^*$ only *one* inner product is part of the structure. Then the first part is clear, as in $\underline{\mathsf{Pic}}^*(\mathscr{B},\mathscr{A})$ one requires additionally that T^* is also the adjoint of T with respect to the $\mathscr{B}$-valued inner product. For the second part we consider a unitary T. Then $T^* = T^{-1}$ where the adjoint refers to the $\mathscr{A}$-valued inner products. Now for $x, y \in \mathscr{E}$ and $z', u' \in \mathscr{E}'$ we have

$$\begin{aligned}
\left\langle {}_{\mathscr{B}}\langle Tx, Ty\rangle^{\mathscr{E}'} \cdot z', u' \right\rangle^{\mathscr{E}'}_{\mathscr{A}}
&= \left\langle (Tx) \cdot \langle Ty, z'\rangle^{\mathscr{E}'}_{\mathscr{A}}, u' \right\rangle^{\mathscr{E}'}_{\mathscr{A}} \\
&= \left\langle (Tx) \cdot \langle y, T^*z'\rangle^{\mathscr{E}'}_{\mathscr{A}}, u' \right\rangle^{\mathscr{E}'}_{\mathscr{A}} \\
&= \left\langle T\big(x \cdot \langle y, T^*z'\rangle^{\mathscr{E}}_{\mathscr{A}}\big), u' \right\rangle^{\mathscr{E}'}_{\mathscr{A}} \\
&= \left\langle x \cdot \langle y, T^*z'\rangle^{\mathscr{E}}_{\mathscr{A}}, T^*u' \right\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \left\langle {}_{\mathscr{B}}\langle x, y\rangle^{\mathscr{E}} \cdot T^*z', T^*u' \right\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \left\langle T^*\big({}_{\mathscr{B}}\langle x, y\rangle^{\mathscr{E}} \cdot z'\big), T^*u' \right\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \left\langle {}_{\mathscr{B}}\langle x, y\rangle^{\mathscr{E}} \cdot z', u' \right\rangle^{\mathscr{E}'}_{\mathscr{A}},
\end{aligned}$$

where we used the left $\mathscr{B}$-linearity of $T^* = T^{-1}$ as well as the right $\mathscr{A}$-linearity of T together with the unitarity of T with respect to the $\mathscr{A}$-valued inner products. Since all inner products are non-degenerate and $b \mapsto (x' \mapsto b \cdot x')$ is injective, we conclude that T is also isometric for the $\mathscr{B}$-valued inner product. As T is invertible, this means that T is also adjointable with respect to the $\mathscr{B}$-valued inner product and both adjoints coincide since they are simply given by T^{-1}, proving the second part. For the third part, let $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be an endomorphism, then we have

$$\begin{aligned}
{}_{\mathscr{B}}\langle x, Ty\rangle^{\mathscr{E}} \cdot z &= x \cdot \langle Ty, z\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= x \cdot \langle y, T^*z\rangle^{\mathscr{E}}_{\mathscr{A}} \\
&= \Theta_{x,y}(T^*z) \\
&= T^*(\Theta_{x,y}(z)) \\
&= \Theta_{T^*x,y}(z) \\
&= {}_{\mathscr{B}}\langle T^*x, y\rangle^{\mathscr{E}} \cdot z,
\end{aligned}$$

as T^* is left $\mathscr{B}$-linear and $\Theta_{x,y}$ is a left multiplication with some element in $\mathscr{B}$. Note that the above computation does not work for a morphism $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ in general as then $\Theta_{x,y'}$ is not necessarily given by the left multiplication of an element in $\mathscr{B}$. □

The consequence of the lemma is now that despite the difficulties for general morphisms, the two notions of *isomorphisms* coincide.

Definition 5.1.8 (*-Picard and strong Picard (bi-) groupoids) *For a given ordered ring* R *with* $\mathsf{C} = \mathsf{R}(\mathrm{i})$ *one defines:*

i.) *The* **-Picard groupoid* Pic^* *is the groupoid of invertible morphisms of* Bimod^**. The isotropy groups are called the* **-Picard groups.*

ii.) *The* **-Picard bigroupoid* $\underline{\mathsf{Pic}}^*$ *is the bigroupoid of* **-invertible* 1*-morphisms of* $\underline{\mathsf{Bimod}}^*$.

iii.) *The strong Picard groupoid* $\mathsf{Pic}^{\mathrm{str}}$ *is the groupoid of invertible morphisms in* $\mathsf{Bimod}^{\mathrm{str}}$*. The isotropy groups are called the strong Picard groups.*

iv.) *The strong Picard bigroupoid* $\underline{\mathsf{Pic}}^{\mathrm{str}}$ *is the bigroupoid of invertible* 1*-morphisms in* $\underline{\mathsf{Bimod}}^{\mathrm{str}}$*.*

Note that the notion of *-Picard groupoids and strong Picard groupoids depends implicitly on the chosen ring $\mathsf{C} = \mathsf{R}(\mathrm{i})$ of scalars. As long as we consider this ring to be fixed, we shall not indicate this dependence in our notation. Nevertheless, when we come to deformation theory in Chapter 7, we will have to be more careful as then the relation between the Picard groupoids for C and $\mathsf{C}[\![\lambda]\!]$ will be investigated, see e.g. Section 7.1.1.

Remark 5.1.9 (Picard groupoids)

i.) In Definition 4.1.14 we defined the morphisms of $\underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ (as well as those in $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$) to be those bimodule morphisms $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ which are adjointable with respect to both inner products and for which the two adjoints agree. By Lemma 5.1.7 this needs not to be consistent with the definition of the Picard bigroupoid as above: However, for *isomorphisms* and *endomorphisms* both definitions agree. In particular, the classifying groupoids do not depend on this subtleties. At the present stage there seems to be no true reason to prefer one definition over the other, therefore we shall no longer take these subtleties into account and postpone a final decision.

ii.) The classifying groupoid of $\underline{\mathsf{Pic}}^*$ is Pic^* and the classifying groupoid of $\underline{\mathsf{Pic}}^{\mathrm{str}}$ is $\mathsf{Pic}^{\mathrm{str}}$. This follows immediately from the definition and Proposition 5.1.6. From the same Proposition 5.1.6 it also follows that $\underline{\mathsf{Pic}}^*$ and $\underline{\mathsf{Pic}}^{\mathrm{str}}$ are bigroupoids while Pic^* and $\mathsf{Pic}^{\mathrm{str}}$ are groupoids by Proposition 5.1.2.

iii.) With Theorem 4.3.32 we obtain the following interpretation of Morita equivalence: two *-algebras (idempotent and non-degenerate as usual) are strongly Morita equivalent or *-Morita equivalent, respectively, if and only if they are in the same $\mathsf{Pic}^{\mathrm{str}}$-orbit or Pic^*-orbit, respectively.

iv.) From Proposition 4.1.15 it follows that not only the composition $\widetilde{\otimes}$ in $\underline{\mathsf{Pic}}^*$ and $\underline{\mathsf{Pic}}^{\mathrm{str}}$ is functorial but we also have inversion functors

$$\mathrm{inv}\colon \underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^*(\mathscr{A}, \mathscr{B}) \tag{5.1.9}$$

as well as

$$\mathrm{inv}\colon \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{A}, \mathscr{B}), \tag{5.1.10}$$

explicitly given by complex conjugation of the bimodule as well as the corresponding bimodule morphisms, see also Exercise 5.4.4. In particular, this proposition could also be used to construct the Picard bigroupoids directly, without using the ambient bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively. Then, however, the bimodules loose their interpretation of being the invertible ones among general bimodules.

In the ring-theoretic version of Morita theory there is of course also a Picard groupoid Pic as well as a corresponding Picard bigroupoid $\underline{\mathsf{Pic}}$:

Definition 5.1.10 (Picard (bi-)groupoid) *The bigroupoid of invertible* 1*-morphisms in* $\underline{\mathsf{Bimod}}$ *is called the Picard bigroupoid* $\underline{\mathsf{Pic}}$ *and the groupoid of invertible morphisms in* Bimod *is called the Picard groupoid* Pic*. The isotropy groups of* Pic *are called the Picard groups.*

Remark 5.1.11 Note that the objects of Pic are rings with unit while the objects of Pic^* and $\mathsf{Pic}^{\mathrm{str}}$ are idempotent and non-degenerate *-algebras over C. On one hand this is more particular due to the *-algebra structure. On the other hand, we do not necessarily require units. Thus a meaningful comparison between the various Picard groupoids is only possible after restricting to *unital* *-algebras over C as objects. When comparing the different Picard groupoids we shall always implicitly assume that this restriction has been done.

As a first application of our general considerations on groupoids in Proposition 5.1.4 we obtain another Morita invariant:

Corollary 5.1.12 (Morita invariance of the Picard groups) *Let* $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

i.) *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are Morita equivalent unital rings then*

$$\mathsf{Pic}(\mathscr{A}) \cong \mathsf{Pic}(\mathscr{B}). \tag{5.1.11}$$

ii.) *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are* **-Morita equivalent idempotent and non-degenerate* **-algebras over* C *then*

$$\mathsf{Pic}^*(\mathscr{A}) \cong \mathsf{Pic}^*(\mathscr{B}). \tag{5.1.12}$$

iii.) *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are strongly Morita equivalent idempotent and non-degenerate* **-algebras over* C *then*

$$\mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \cong \mathsf{Pic}^{\mathrm{str}}(\mathscr{B}). \tag{5.1.13}$$

In all three cases any equivalence bimodule implements a group isomorphism according to (5.1.4).

Remark 5.1.13 (Morita theory) The whole Morita theory is now encoded in the corresponding Picard (bi-) groupoids. The objects encode which type of *-algebras (rings) are under consideration. The orbits encode which *-algebras are Morita equivalent. Finally, the Picard groups encode how many "self-equivalences" an object has. Moreover, the Picard groups yield also information in how many "different" ways two *-algebras can be Morita equivalent, by Proposition 5.1.4, *iii.)*. Thus for Morita theory we obtain the following two basic tasks:

i.) Determine the orbits of the Picard groupoid.

ii.) Determine the Picard groups.

The additional structure of the bigroupoid approach is not necessary to understand the above two questions. However, we will meet situations where the Picard bigroupoid approach is advantageous for formulating certain Morita invariants.

5.2 The Structure of the Picard Groupoids

In this section we shall discuss some general properties of the Picard groupoids Pic, Pic^*, and $\mathsf{Pic}^{\mathrm{str}}$ and the corresponding Picard groups. The bigroupoid aspects will not be needed here. The statements on the ring-theoretic version are classical, see e.g. the monograph of Bass [4, Chap. 2, §5]. Nevertheless, we will recall them together with the *-version and the strong versions which can be found in [29].

5.2.1 The Canonical Groupoid Morphisms between the Picard Groupoids

The main tool in studying the strong and *-Picard groupoids will be to relate them to the underlying ring-theoretic Picard groupoid: there are canonical groupoid morphisms relating the three Picard groupoids whose images and kernels will contain the relevant information how the strong and *-equivalence differs from the purely ring-theoretic Morita equivalence.

First we remind on the definition of a groupoid morphism:

Definition 5.2.1 (Groupoid morphism) *A groupoid morphism* $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{H}$ *is a covariant functor.*

Indeed, this generalizes the notion of a group morphism in the following sense: If $\mathfrak{G} = G$ and $\mathfrak{H} = H$ are groupoids with just one object, i.e. their morphism spaces are just groups, then a functor

$\Phi\colon G \longrightarrow H$ maps the unique object of G to the unique object of H and is a group morphism on the level of morphisms of G and H, i.e.

$$\Phi(g \circ g') = \Phi(g) \circ \Phi(g') \quad \text{and} \quad \Phi(\mathrm{id}_G) = \mathrm{id}_H\,. \tag{5.2.1}$$

In the general groupoid case we obtain from a groupoid morphism the group morphisms

$$\Phi\colon \mathfrak{G}(a) \longrightarrow \mathfrak{H}(\Phi(a)) \tag{5.2.2}$$

for each object $a \in \mathfrak{G}$. Moreover, Φ maps the orbit $\mathfrak{G} \cdot a$ into (but not necessarily onto) the orbit $\mathfrak{H} \cdot \Phi(a)$.

Back to the Picard groupoids we have the first example of a groupoid morphism: By forgetting the complete positivity of the inner products we obtain a groupoid morphism

$$\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}^{*}, \tag{5.2.3}$$

which is the identity on the objects. Moreover, on morphisms it is *injective* since in $\underline{\mathsf{Pic}}^{\mathrm{str}}$ and $\underline{\mathsf{Pic}}^{*}$ we have the same 2-morphisms and hence the notions of isomorphism between 1-morphisms of $\underline{\mathsf{Pic}}^{\mathrm{str}}$, viewed as 1-morphism of $\underline{\mathsf{Pic}}^{*}$ coincides. Clearly, forgetting the complete positivity is compatible with taking the internal tensor products as in the quotient procedure needed for $\widehat{\otimes}$ we only have to take care of the degeneracy space, a notion which does not refer to positivity. Thus (5.2.3) is indeed functorial. As we shall see, the groupoid morphism (5.2.3) is not surjective in general.

For *unital* *-algebras we can forget the inner products completely. Then, for equivalence bimodules we know from Corollary 4.2.8 that we do *not* need the quotient for the construction of $\widehat{\otimes}$. Finally, isometrically isomorphic equivalence bimodules are in particular isomorphic as bimodules and thus the isomorphism classes of bimodules in $\mathsf{Pic}^{\mathrm{str}}$ or Pic^{*} are mapped to isomorphism classes in Pic in a well-defined way. This shows that we obtain groupoid morphisms

$$\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic} \quad \text{and} \quad \mathsf{Pic}^{*} \longrightarrow \mathsf{Pic}\,. \tag{5.2.4}$$

On objects both are again the identity. However, in general, both of them are neither surjective nor injective: On a ring-theoretic equivalence bimodule in Pic there might be more than one inner product up to isometry or even none. This problematic parallels very much the situation in K_0-theory as indicated in (2.3.11). We summarize the results of this discussion as follows:

Theorem 5.2.2 (Canonical forgetful functors) *Over the class of unital *-algebras, we have canonical forgetful groupoid morphisms such that the diagram*

$$\begin{array}{ccc} \mathsf{Pic}^{\mathrm{str}} & \longrightarrow & \mathsf{Pic}^{*} \\ & \searrow \quad \swarrow & \\ & \mathsf{Pic} & \end{array} \tag{5.2.5}$$

commutes.

In the following we shall mainly study the groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ and discuss its kernel and image.

Remark 5.2.3 Note that since in general the internal tensor product $\widehat{\otimes}$ requires a non-trivial quotient procedure, there is *no* forgetful functor $\mathsf{Bimod}^{\mathrm{str}} \longrightarrow \mathsf{Bimod}$: the tensor products simply do not match. This is only true for equivalence bimodules. However, forgetting complete positivity is still possible leading to a functor

$$\mathsf{Bimod}^{\mathrm{str}} \longrightarrow \mathsf{Bimod}^{*}\,. \tag{5.2.6}$$

From this, we recover the groupoid morphism (5.2.3) by restricting to invertible arrows.

5.2.2 Isomorphisms and Equivalences

Beside relating the different types of the Picard groupoids we can also relate the notion of Morita equivalence to the notion of isomorphism: since Bimod generalizes the category Ring by means of the functor ℓ from Proposition 4.3.3, we can compare the corresponding isomorphism groupoids. For Ring, the groupoid of isomorphisms is just Iso, consisting of arrows which are the usual unital ring isomorphisms. Since ℓ is a functor, it restricts to a functor

$$\ell\colon \mathrm{Iso} \longrightarrow \mathsf{Pic}, \tag{5.2.7}$$

i.e. a groupoid morphism between the isomorphism groupoid of unital rings and the ring-theoretic Picard groupoid.

For *-algebras we can not expect to have an analog of the functor ℓ from *-alg or *-Alg to Bimod^* or $\mathsf{Bimod}^{\mathrm{str}}$: the reason is that if the *-homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is not invertible, then there is no hope to construct a reasonable $\mathscr{A}$-valued inner product on the $(\mathscr{B},\mathscr{A})$-bimodule ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$. Thus we have to proceed differently when we want to find an analog of (5.2.7) in the *-algebra case: we have to consider *-*isomorphisms* only.

In Theorem 4.1.11 we have constructed a strong equivalence bimodule out of a *-isomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$. Indeed, on ${}_{\mathscr{B}}\mathscr{B}$ we considered the right $\mathscr{A}$-module structure defined by $b \cdot_{\Phi} a = b\Phi(a)$ together with the canonical $\mathscr{B}$-valued inner product ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle$ as well as the $\mathscr{A}$-valued inner product $\langle b, b' \rangle^{\mathscr{B}^{\Phi}}_{\mathscr{A}} = \Phi^{-1}(b)^*\Phi^{-1}(b')$. Alternatively, one can endow $\mathscr{A}_{\mathscr{A}}$ with a left $\mathscr{B}$-module structure via

$$b \cdot_{\Phi} a = \Phi^{-1}(b)a. \tag{5.2.8}$$

Together with the canonical $\mathscr{A}$-valued inner product and the $\mathscr{B}$-valued inner product

$${}^{\Phi}_{\mathscr{A}}\langle a, a' \rangle_{\mathscr{B}} = \Phi(a)\Phi(a')^* \tag{5.2.9}$$

the resulting $(\mathscr{B},\mathscr{A})$-bimodule ${}^{\Phi}_{\mathscr{B}}\mathscr{A}_{\mathscr{A}}$ is again a strong equivalence bimodule. This follows analogously to Theorem 4.1.11. Note that for either way, the existence of the inverse of Φ is crucial to obtain an inner-product bimodule.

More generally, for a *-equivalence $(\mathscr{C},\mathscr{B})$-bimodule ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ we can construct a *-equivalence $(\mathscr{C},\mathscr{A})$-bimodule ${}_{\mathscr{C}}\mathscr{F}^{\Phi}_{\mathscr{A}}$ as follows. On $\mathscr{F}$ we keep the left $\mathscr{C}$-module structure and the $\mathscr{C}$-valued inner product. Then we set

$$x \cdot_{\Phi} a = x \cdot \Phi(a) \quad \text{and} \quad \langle x, y \rangle^{\mathscr{F}^{\Phi}}_{\mathscr{A}} = \Phi^{-1}\big(\langle x, y \rangle^{\mathscr{F}}_{\mathscr{B}}\big), \tag{5.2.10}$$

for $x, y \in \mathscr{F}$ and $a \in \mathscr{A}$. Analogously, for a *-isomorphism $\Psi\colon \mathscr{B} \longrightarrow \mathscr{C}$ and a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ we construct ${}^{\Psi}_{\mathscr{C}}\mathscr{E}_{\mathscr{A}}$ by keeping the right $\mathscr{A}$-module structure and the $\mathscr{A}$-valued inner product and setting

$$c \cdot_{\Psi} x = \Psi^{-1}(c) \cdot x \quad \text{and} \quad {}_{\mathscr{C}}\langle x, y \rangle^{\Psi\mathscr{E}} = \Psi\big({}_{\mathscr{B}}\langle x, y \rangle^{\mathscr{E}}\big). \tag{5.2.11}$$

Obviously, this generalizes the construction of ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$ and ${}^{\Phi}_{\mathscr{B}}\mathscr{A}_{\mathscr{A}}$. The properties of this construction are summarized in the following theorem:

Theorem 5.2.4 (The groupoid morphism ℓ) *Let $\Phi \in \mathrm{Iso}^*(\mathscr{B},\mathscr{A})$ and $\Psi \in \mathrm{Iso}^*(\mathscr{C},\mathscr{B})$ and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ be strong equivalence bimodules.*

i.) The map $\Phi^{-1}\colon {}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}} \longrightarrow {}^{\Phi}_{\mathscr{B}}\mathscr{A}_{\mathscr{A}}$ is an isometric isomorphism of strong equivalence bimodules.

ii.) We have ${}^{\Psi}_{\mathscr{C}}\mathscr{E}_{\mathscr{A}}, {}_{\mathscr{C}}\mathscr{F}^{\Phi}_{\mathscr{A}} \in \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{C},\mathscr{A})$.

iii.) The maps

$${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}} \ni x \otimes b \;\mapsto\; x \cdot b \in {}_{\mathscr{C}}\mathscr{F}^{\Phi}_{\mathscr{A}} \tag{5.2.12}$$

and

$${}_{\mathscr{C}}\mathscr{C}^{\Psi}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \ni c \otimes x \;\mapsto\; c \cdot_{\Psi} x = \Psi^{-1}(c) \cdot x \in {}^{\Psi}_{\mathscr{C}}\mathscr{E}_{\mathscr{A}} \tag{5.2.13}$$

are isometric isomorphisms.

iv.) The map $\ell(\mathscr{A}) = \mathscr{A}$ *for objects and*

$$\ell(\Phi) = \left[{}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}\right] \in \mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \tag{5.2.14}$$

for **-isomorphisms* $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ *yields a groupoid morphism*

$$\ell\colon \mathrm{Iso}^* \longrightarrow \mathsf{Pic}^{\mathrm{str}}. \tag{5.2.15}$$

PROOF: First it is clear that Φ^{-1} yields a $(\mathscr{B}, \mathscr{A})$-bimodule morphism since $\Phi^{-1}(b \cdot b') = \Phi^{-1}(b) \cdot \Phi^{-1}(b') = b \cdot_\Phi \Phi^{-1}(b')$ and $\Phi^{-1}(b' \cdot_\Phi a) = \Phi^{-1}(b'\Phi(a)) = \Phi^{-1}(b')a$. Moreover,

$$\left\langle \Phi^{-1}(b), \Phi^{-1}(b')\right\rangle^{\Phi_{\mathscr{A}}}_{\mathscr{A}} = \Phi^{-1}(b)^*\Phi^{-1}(b') = \Phi^{-1}(b^*b') = \left\langle b, b'\right\rangle^{\mathscr{B}^\Phi}_{\mathscr{A}}$$

shows that Φ^{-1} is isometric. Since Φ^{-1} is clearly bijective, we have a bijective isometric bimodule isomorphism, hence an adjointable and unitary one, with respect to the $\mathscr{A}$-valued inner products. By Lemma 5.1.7, *iii.)*, it follows that Φ^{-1} is isometric with respect to the $\mathscr{B}$-valued inner products, too. Of course, this can also be seen in a more elementary way. This shows the first part. For the second part, the required properties of the bimodules are easily checked. For the third part, we show that (5.2.12) is isometric, since

$$\begin{aligned}
\left\langle x \otimes b, x' \otimes b'\right\rangle^{\mathscr{F}\otimes\mathscr{B}^\Phi}_{\mathscr{A}} &= \left\langle b, \langle x, x'\rangle^{\mathscr{F}}_{\mathscr{B}} \cdot b'\right\rangle^{\mathscr{B}^\Phi}_{\mathscr{A}} \\
&= \Phi^{-1}\left(b^* \langle x, x'\rangle^{\mathscr{F}}_{\mathscr{B}} b'\right) \\
&= \Phi^{-1}\left(\langle x \cdot b, x' \cdot b'\rangle^{\mathscr{F}}_{\mathscr{B}}\right) \\
&= \left\langle x \cdot b, x' \cdot b'\right\rangle^{\mathscr{F}^\Phi}_{\mathscr{A}}.
\end{aligned}$$

Now (5.2.12) is clearly a bimodule morphism which is surjective by $\mathscr{F} \cdot \mathscr{B} = \mathscr{B}$ and isometric, hence on the quotient $\mathscr{F} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} \mathscr{B}$ it is also injective. Thus (5.2.12) is bijective and isometric hence adjointable with respect to the $\mathscr{A}$-valued inner product. Thanks to Lemma 5.1.7, *i.)*, or again by an elementary computation we conclude that (5.2.12) is also adjointable with respect to the $\mathscr{C}$-valued inner product. The case of (5.2.13) is treated the same. For the fourth part, we first observe $\ell(\mathrm{id}_{\mathscr{A}}) = [{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]$ and hence ℓ preserves the units. Moreover,

$${}_{\mathscr{C}}\mathscr{C}^{\Psi}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}} \ni c \otimes b \;\mapsto\; c\Psi(b) \in {}_{\mathscr{C}}\mathscr{C}^{\Psi\circ\Phi}_{\mathscr{A}}$$

is an isometric isomorphism as a simple computation confirms. From this, $\ell(\Psi \circ \Phi) = \ell(\Psi) \mathbin{\widetilde{\otimes}} \ell(\Phi)$ follows immediately. ☐

Remark 5.2.5 Analogous statements hold for *-equivalence bimodules and the *-Picard groupoid Pic^* instead. In fact, the corresponding groupoid morphism $\ell\colon \mathrm{Iso}^* \longrightarrow \mathsf{Pic}^*$ is just the composition of ℓ as in (5.2.15) and the canonical groupoid morphism (5.2.3). Moreover, Theorem 5.2.4 has a well-known ring-theoretic analog: for unital rings instead of *-algebras as well as for Pic instead of $\mathsf{Pic}^{\mathrm{str}}$ or Pic^* we obtain that twisting the equivalence bimodule with an automorphism yields a groupoid morphism

$$\ell\colon \mathrm{Iso} \longrightarrow \mathsf{Pic}, \tag{5.2.16}$$

see e.g. [4, Chap. 2, §5]. In fact, here ℓ is just the restriction of the functor $\ell\colon \mathsf{Ring} \longrightarrow \mathsf{Bimod}$ as we have already mentioned. Moreover, ℓ is clearly compatible with the inclusion $\mathrm{Iso}^* \longrightarrow \mathrm{Iso}$.

In general, ℓ is neither injective nor surjective. The lack of surjectivity is clear since there are *-algebras which are strongly or *-Morita equivalent without being *-isomorphic, see the example from Theorem 4.1.12. Thus $\mathsf{Pic}^*(\mathscr{B}, \mathscr{A})$ or $\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$ may be non-empty while $\mathrm{Iso}^*(\mathscr{B}, \mathscr{A})$ is empty. But even for $\mathscr{B} = \mathscr{A}$, the strong Picard group $\mathsf{Pic}^{\mathrm{str}}(\mathscr{A})$ may contain elements which are not of the simple form $\ell(\Phi)$ as we shall see later in examples. Thus the non-surjectivity encodes where strong Morita equivalence goes beyond the notion of *-isomorphism. For $\mathsf{Pic}^*(\mathscr{A})$ there is always a trivial reason as we can replace a completely positive inner product by the corresponding completely negative one which are certainly not isometric. Thus we expect $\mathsf{Pic}^*(\mathscr{A})$ to be "at least" $\mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \times \mathbb{Z}_2$. However, we will see examples where $\mathsf{Pic}^*(\mathscr{A})$ is strictly larger than $\mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \times \mathbb{Z}_2$, see Remark 5.2.16, *ii.)*. Finally, the same lack of surjectivity is expected in the ring-theoretic situation as well.

The lack of injectivity can be explained as follows: for particular *-isomorphisms $\Phi \neq \Psi$ the bimodules ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$ and ${}_{\mathscr{B}}\mathscr{B}^{\Psi}_{\mathscr{A}}$ are different but may still be isomorphic. To investigate the (non-) injectivity we first note that for isomorphic rings the question of injectivity of

$$\ell\colon \operatorname{Iso}^*(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \tag{5.2.17}$$

is equivalent to the question of injectivity of the group morphism

$$\ell\colon \operatorname{Aut}^*(\mathscr{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}). \tag{5.2.18}$$

Indeed, this follows immediately from the groupoid structure and is true for every morphism between groupoids. Replacing $\mathsf{Pic}^{\mathrm{str}}$ by Pic^* does not yield anything new here, as (5.2.3) is injective. In particular, we have the injective group homomorphism

$$\mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Pic}^*(\mathscr{A}). \tag{5.2.19}$$

Restricting to the unital situation and replacing Iso^* by Iso and $\mathsf{Pic}^{\mathrm{str}}$ by Pic gives again an expected non-injectivity for the same reasons.

To actually compute the kernel of ℓ restricted to the isotropy group $\operatorname{Aut}^*(\mathscr{A})$ of Iso^* at $\mathscr{A}$, i.e. the *-automorphism group, we consider *unital* *-algebras from now on. This will allow to proceed analogously to the ring-theoretic situation.

Definition 5.2.6 (Inner *-automorphisms) *For a unital *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ we define the inner *-automorphisms to be*

$$\operatorname{InnAut}^*(\mathscr{A}) = \left\{ \Phi \in \operatorname{Aut}^*(\mathscr{A}) \;\middle|\; \Phi(a) = uau^* \text{ with some } u^{-1} = u^* \in \mathscr{A} \right\}. \tag{5.2.20}$$

Note that in general this is a proper subgroup of $\operatorname{InnAut}(\mathscr{A}) \cap \operatorname{Aut}^*(\mathscr{A})$ as we explicitly require that Φ is the conjugation with some *unitary* element u of $\mathscr{A}$. Clearly, $\operatorname{InnAut}^*(\mathscr{A})$ is a normal subgroup of $\operatorname{Aut}^*(\mathscr{A})$, see also Exercise 5.4.5. Thus we can define the group of *outer *-automorphisms* to be

$$\operatorname{OutAut}^*(\mathscr{A}) = \frac{\operatorname{Aut}^*(\mathscr{A})}{\operatorname{InnAut}^*(\mathscr{A})} \tag{5.2.21}$$

in analogy to the group of outer automorphisms $\operatorname{OutAut}(\mathscr{A}) = \operatorname{Aut}(\mathscr{A}) \big/ \operatorname{InnAut}(\mathscr{A})$ as usual.

For unital *-algebras we have the following general result, where we only have to take care of the strong Picard group thanks to the injective inclusion (5.2.19).

Theorem 5.2.7 (Strong Picard group) *Let $\mathscr{A}$ and $\mathscr{B}$ be unital *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) For $\Phi \in \operatorname{Aut}^*(\mathscr{B})$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$ one has $[{}^{\Phi}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}] = [{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]$ if and only if $\Phi \in \operatorname{InnAut}^*(\mathscr{B})$.

ii.) The sequence of group morphisms

$$1 \longrightarrow \operatorname{InnAut}^*(\mathscr{A}) \longrightarrow \operatorname{Aut}^*(\mathscr{A}) \stackrel{\ell}{\longrightarrow} \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \tag{5.2.22}$$

is exact.

Analogously, in the ring-theoretic framework we have the exact sequence of groups

$$1 \longrightarrow \operatorname{InnAut}(\mathscr{A}) \longrightarrow \operatorname{Aut}(\mathscr{A}) \xrightarrow{\ell} \mathsf{Pic}(\mathscr{A}). \tag{5.2.23}$$

PROOF: The ring-theoretic version can be found in [4, Chap. 2, §5] and serves as motivation for the *-algebra framework: assume that $U\colon {}^{\Phi}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is an isometric isomorphism. Then we have $U(x \cdot a) = U(x) \cdot a$ and hence $U \in \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. Since in the unital case

$$\mathscr{B} \cong \mathfrak{F}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$$

via the left action according to Proposition 4.2.4, there is a unique invertible $u \in \mathscr{B}$ with $U(x) = u \cdot x$ and hence $U^{-1}(x) = u^{-1} \cdot x$. Since U is also left $\mathscr{B}$-linear we obtain for $b \in \mathscr{B}$

$$(bu) \cdot x = b \cdot (u \cdot x) = b \cdot U(x) = U(b \cdot_{\Phi} x) = U(\Phi^{-1}(b) \cdot x) = (u\Phi^{-1}(b)) \cdot x.$$

As the map $b \mapsto (x \mapsto b \cdot x)$ is injective for an equivalence bimodule we conclude $bu = u\Phi^{-1}(b)$ and thus $\Phi(b) = ubu^{-1}$. This shows $\Phi \in \operatorname{InnAut}(\mathscr{B})$. Since U is also isometric we have

$$u\,{}_{\mathscr{B}}\langle x, y\rangle^{\mathscr{E}} u^* = {}_{\mathscr{B}}\langle u \cdot x, u \cdot y\rangle^{\mathscr{E}} = {}_{\mathscr{B}}\langle U(x), U(y)\rangle^{\mathscr{E}} = {}_{\mathscr{B}}\langle x, y\rangle^{\Phi_{\mathscr{E}}} = \Phi\big({}_{\mathscr{B}}\langle x, y\rangle^{\mathscr{E}}\big).$$

Since ${}_{\mathscr{B}}\langle \cdot\,, \cdot\rangle^{\mathscr{E}}$ is full we conclude that $\Phi(b) = ubu^*$ for all $b \in \mathscr{B}$, implying $u^{-1} = u^*$. Thus $\Phi \in \operatorname{InnAut}^*(\mathscr{B})$ as desired. Conversely, let $\Phi(b) = ubu^{-1}$ with some unitary $u \in \mathscr{B}$. Then it is an easy computation that $U(x) = u \cdot x$ provides the isometric isomorphism $U\colon {}^{\Phi}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. This shows the first part. For the second part we consider $\mathscr{B} = \mathscr{A}$ and the two equivalence bimodules ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ and ${}^{\Phi^{-1}}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$. The kernel of the group morphism ℓ consist of those $\Phi \in \operatorname{Aut}^*(\mathscr{A})$ for which these two bimodules are isometrically isomorphic. By the first part, we get (5.2.22). Note that in the proof the positivity was never needed. ☐

Remark 5.2.8 (Outer automorphisms and the Picard group) In the particular case of a *commutative* unital *-algebra (or ring, respectively) $\mathscr{A}$ we have $\operatorname{InnAut}(\mathscr{A}) = \{\mathrm{id}\} = \operatorname{InnAut}^*(\mathscr{A})$. Hence

$$\ell\colon \operatorname{Aut}^*(\mathscr{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \tag{5.2.24}$$

in the *-algebra case as well as

$$\ell\colon \operatorname{Aut}(\mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{A}) \tag{5.2.25}$$

in the ring case are *injective*. Thus the automorphism group becomes a subgroup of the Picard group. In the general case, one can use (5.2.22) and (5.2.23), respectively, to embed the outer *-automorphisms into the strong Picard group and the outer automorphisms into the Picard group, respectively. Thus we obtain injective group morphisms

$$\ell\colon \operatorname{OutAut}^*(\mathscr{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \quad \text{and} \quad \ell\colon \operatorname{OutAut}(\mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{A}), \tag{5.2.26}$$

respectively. In both cases, the interesting part of the Picard groups is the “rest”, not reached by the outer automorphisms.

5.2.3 The Role of the Center

To clarify the structure of the Picard groups further we consider the following construction. Let

$$\mathscr{Z}(\mathscr{A}) = \big\{a \in \mathscr{A} \;\big|\; [a, b] = 0 \text{ for all } b \in \mathscr{A}\big\} \tag{5.2.27}$$

be the *center* of $\mathscr{A}$. Clearly, $\mathscr{Z}(\mathscr{A})$ is a *-subalgebra of $\mathscr{A}$, see also Exercise 5.4.6, *i.)*.

Proposition 5.2.9 *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be unital *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ be *-equivalence bimodules.*

i.) For every $a \in \mathscr{Z}(\mathscr{A})$ there exists a unique $h_{\mathscr{E}}(a) \in \mathscr{Z}(\mathscr{B})$ such that for all $x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$

$$h_{\mathscr{E}}(a) \cdot x = x \cdot a. \tag{5.2.28}$$

ii.) The map $h_{\mathscr{E}} \colon \mathscr{Z}(\mathscr{A}) \longrightarrow \mathscr{Z}(\mathscr{B})$ is a *-isomorphism.

iii.) We have

$$h_{\mathscr{A}} = \mathrm{id}_{\mathscr{Z}(\mathscr{A})} \quad \text{and} \quad h_{\mathscr{F}} \circ h_{\mathscr{E}} = h_{\mathscr{F} \widetilde{\otimes} \mathscr{E}}. \tag{5.2.29}$$

iv.) The map $h_{\mathscr{E}}$ only depends on the isometric isomorphism class $[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]$ of ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$.

Proof: Let $a \in \mathscr{Z}(\mathscr{A})$ then $x \mapsto x \cdot a$ is clearly right $\mathscr{A}$-linear. By Proposition 4.2.4 there is a unique $h_{\mathscr{E}}(a) \in \mathscr{B}$ with (5.2.28) for all $x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. Now let $b \in \mathscr{B}$ be arbitrary then

$$(b h_{\mathscr{E}}(a)) \cdot x = b \cdot (h_{\mathscr{E}}(a) \cdot x) = b \cdot (x \cdot a) = (b \cdot x) \cdot a = h_{\mathscr{E}}(a) \cdot (b \cdot x) = (h_{\mathscr{E}}(a) b) \cdot x$$

for all $x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and $a \in \mathscr{Z}(\mathscr{A})$. This shows that $h_{\mathscr{E}}(a)$ is central. For the second part we first observe that $h_{\mathscr{E}} \colon \mathscr{Z}(\mathscr{A}) \longrightarrow \mathscr{Z}(\mathscr{B})$ is clearly linear. For $a, a' \in \mathscr{Z}(\mathscr{A})$ we have

$$h_{\mathscr{E}}(aa') \cdot x = x \cdot (aa') = (x \cdot a) \cdot a' = h_{\mathscr{E}}(a') \cdot (h_{\mathscr{E}}(a) \cdot x) = (h_{\mathscr{E}}(a') h_{\mathscr{E}}(a)) \cdot x$$

for all $x \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. Hence $h_{\mathscr{E}}(aa') = h_{\mathscr{E}}(a') h_{\mathscr{E}}(a)$ and since $\mathscr{Z}(\mathscr{B})$ is commutative, $h_{\mathscr{E}}$ is an algebra homomorphism. For $x, y \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ we find

$$\langle h_{\mathscr{E}}(a^*) \cdot x, y \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x \cdot a^*, y \rangle^{\mathscr{E}}_{\mathscr{A}} = a \langle x, y \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x, y \rangle^{\mathscr{E}}_{\mathscr{A}} a = \langle x, y \cdot a \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x, h_{\mathscr{E}}(a) \cdot y \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle h_{\mathscr{E}}(a)^* \cdot x, y \rangle^{\mathscr{E}}_{\mathscr{A}}.$$

It follows that $h_{\mathscr{E}}(a^*) = h_{\mathscr{E}}(a)^*$ meaning that $h_{\mathscr{E}}$ is a *-homomorphism. Before showing that $h_{\mathscr{E}}$ is a *-isomorphism we show the third and fourth part. Clearly, $h_{\mathscr{A}} = \mathrm{id}_{\mathscr{Z}(\mathscr{A})}$, and for $x \in \mathscr{E}$ and $y \in \mathscr{F}$ we have for all $a \in \mathscr{Z}(\mathscr{A})$

$$(y \otimes x) \cdot a = y \otimes (h_{\mathscr{E}}(a) \cdot x) = (y \cdot h_{\mathscr{E}}(a)) \otimes x = (h_{\mathscr{F}}(h_{\mathscr{E}}(a))) \cdot (y \otimes x).$$

From this we immediately find (5.2.29). Finally, let $\Phi \colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ be an isometric isomorphism. Then for $x \in \mathscr{E}$ and $a \in \mathscr{Z}(\mathscr{A})$ we have

$$h_{\mathscr{E}'}(a) \cdot \Phi(x) = \Phi(x) \cdot a = \Phi(x \cdot a) = \Phi(h_{\mathscr{E}}(a) \cdot x) = h_{\mathscr{E}}(a) \cdot \Phi(x),$$

since Φ is a $(\mathscr{B}, \mathscr{A})$-bimodule morphism. Thus $h_{\mathscr{E}'} = h_{\mathscr{E}}$ follows. From this we finally conclude $h_{\overline{\mathscr{E}}} \circ h_{\mathscr{E}} = \mathrm{id}_{\mathscr{Z}(\mathscr{A})}$ as well as $h_{\mathscr{E}} \circ h_{\overline{\mathscr{E}}} = \mathrm{id}_{\mathscr{Z}(\mathscr{B})}$ when applying (5.2.29) for $\mathscr{F} = \overline{\mathscr{E}}$. Thus $h_{\mathscr{E}}$ is a *-isomorphism as claimed in the second part. □

Again, we have an analogous statement for the ring-theoretic situation yielding an isomorphism

$$h_{\mathscr{E}} \colon \mathscr{Z}(\mathscr{A}) \longrightarrow \mathscr{Z}(\mathscr{B}) \tag{5.2.30}$$

for every equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ between unital rings $\mathscr{A}$ and $\mathscr{B}$. We also have

$$h_{\mathscr{A}} = \mathrm{id}_{\mathscr{Z}(\mathscr{A})} \quad \text{and} \quad h_{\mathscr{F}} \circ h_{\mathscr{E}} = h_{\mathscr{F} \otimes \mathscr{E}}, \tag{5.2.31}$$

and $h_{\mathscr{E}}$ depends only on the isomorphism class $[\mathscr{E}] \in \mathsf{Pic}(\mathscr{B}, \mathscr{A})$ of $\mathscr{E}$, see again [4, Chap. 2, §5] as well as Exercise 5.4.8. In both situations we have an important corollary stating that Morita theory is essentially a "noncommutative theory", see [1] for the *-equivalence version:

Corollary 5.2.10 *Let $\mathscr{A}$ and $\mathscr{B}$ be two commutative unital* **-algebras. Then $\mathscr{A}$ and $\mathscr{B}$ are* **-Morita equivalent iff they are* **-isomorphic in which case they are also strongly Morita equivalent. Analogously, two commutative unital rings are Morita equivalent iff they are isomorphic.*

Proof: This follows from $\mathscr{A} = \mathscr{Z}(\mathscr{A})$ and $\mathscr{B} = \mathscr{Z}(\mathscr{B})$ as well as Proposition 5.2.9 and Theorem 4.1.11. □

The above corollary tells us that Morita theory of commutative unital *-algebras is non-interesting in so far, as we do not get new "isomorphic" *-algebras in the enlarged context of Bimod^*. Note however, that a commutative unital *-algebra may well be strongly Morita equivalent to a non-commutative one: $\mathscr{A}$ and $\mathrm{M}_n(\mathscr{A})$ are basic examples. While the orbits of the Picard groupoid Pic^* (as well as the ones of $\mathsf{Pic}^{\mathrm{str}}$ and Pic) along commutative unital algebras are the same as the orbits of Iso^* (and Iso, respectively) the isotropy groups may still change: when using Proposition 5.2.9 in this way, commutative *-algebras become interesting again.

To see this, we first consider those *-equivalence or equivalence $(\mathscr{A}, \mathscr{A})$-bimodules ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ for which in addition

$$a \cdot x = x \cdot a \tag{5.2.32}$$

for all $x \in \mathscr{E}$ and $a \in \mathscr{Z}(\mathscr{A})$. Such a bimodule is called *central* or *static*, following a suggestion of [32, Remark 3.4].

Definition 5.2.11 (Static Picard group) *For a unital ring $\mathscr{A}$ one defines the static Picard group*

$$\mathsf{SPic}(\mathscr{A}) = \left\{ [{}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{Pic}^*(\mathscr{A}) \;\middle|\; {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \textit{ is central} \right\}, \tag{5.2.33}$$

and for a unital **-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ one defines the static* **-Picard group*

$$\mathsf{SPic}^*(\mathscr{A}) = \left\{ [{}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{Pic}^*(\mathscr{A}) \;\middle|\; {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \textit{ is central} \right\} \tag{5.2.34}$$

as well as the static strong Picard group

$$\mathsf{SPic}^{\mathrm{str}}(\mathscr{A}) = \left\{ [{}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \;\middle|\; {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \textit{ is central} \right\}. \tag{5.2.35}$$

The map $[\mathscr{E}] \mapsto h_{\mathscr{E}}$ from Proposition 5.2.9 and from (5.2.30) can now be interpreted as a group morphism

$$h\colon \mathsf{Pic}^*(\mathscr{A}) \longrightarrow \mathrm{Aut}^*(\mathscr{Z}(\mathscr{A})) \quad \text{and} \quad h\colon \mathsf{Pic}(\mathscr{A}) \longrightarrow \mathrm{Aut}(\mathscr{Z}(\mathscr{A})), \tag{5.2.36}$$

respectively. With this interpretation, the central self-equivalences give now the following result:

Proposition 5.2.12 *Let $\mathscr{A}$ be a unital ring or a unital* **-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, respectively.*

i.) The sequences of group morphisms

$$1 \longrightarrow \mathsf{SPic}(\mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{A}) \xrightarrow{\;h\;} \mathrm{Aut}(\mathscr{Z}(\mathscr{A})), \tag{5.2.37}$$

$$1 \longrightarrow \mathsf{SPic}^*(\mathscr{A}) \longrightarrow \mathsf{Pic}^*(\mathscr{A}) \xrightarrow{\;h\;} \mathrm{Aut}^*(\mathscr{Z}(\mathscr{A})), \tag{5.2.38}$$

and

$$1 \longrightarrow \mathsf{SPic}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \xrightarrow{\;h\;} \mathrm{Aut}^*(\mathscr{Z}(\mathscr{A})) \tag{5.2.39}$$

are exact.

ii.) If in addition $\mathscr{A}$ is commutative then the sequences

$$1 \longrightarrow \mathsf{SPic}(\mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{A}) \underset{\ell}{\overset{h}{\rightleftarrows}} \mathrm{Aut}(\mathscr{A}) \longrightarrow 1, \tag{5.2.40}$$

$$1 \longrightarrow \mathsf{SPic}^*(\mathscr{A}) \longrightarrow \mathsf{Pic}^*(\mathscr{A}) \underset{\ell}{\overset{h}{\rightleftarrows}} \mathrm{Aut}^*(\mathscr{A}) \longrightarrow 1, \tag{5.2.41}$$

and

$$1 \longrightarrow \mathsf{SPic}^{\mathrm{str}}(\mathscr{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) \underset{\ell}{\overset{h}{\rightleftarrows}} \mathrm{Aut}^*(\mathscr{A}) \longrightarrow 1 \tag{5.2.42}$$

are split exact via the group morphism ℓ.

iii.) If $\mathscr{A}$ is commutative then one has

$$\mathsf{Pic}(\mathscr{A}) = \mathrm{Aut}(\mathscr{A}) \ltimes \mathsf{SPic}(\mathscr{A}), \tag{5.2.43}$$

$$\mathsf{Pic}^*(\mathscr{A}) = \mathrm{Aut}^*(\mathscr{A}) \ltimes \mathsf{SPic}^*(\mathscr{A}), \tag{5.2.44}$$

and

$$\mathsf{Pic}^{\mathrm{str}}(\mathscr{A}) = \mathrm{Aut}^*(\mathscr{A}) \ltimes \mathsf{SPic}^{\mathrm{str}}(\mathscr{A}), \tag{5.2.45}$$

respectively. In the first case, the left action of $\mathrm{Aut}(\mathscr{A})$ on $\mathsf{Pic}(\mathscr{A})$ is given by $[\mathscr{E}] \mapsto [{}^{\Phi}\mathscr{E}^{\Phi^{-1}}]$, while in the second and third case, the induced product structure on the right hand side is explicitly given by

$$(\Phi, [\mathscr{E}]) \cdot (\Psi, [\mathscr{F}]) = \Big(\Phi \circ \Psi, \Big[\mathscr{E} \mathbin{\widetilde{\otimes}} {}^{\Phi}\mathscr{F}^{\Phi^{-1}}\Big]\Big), \tag{5.2.46}$$

where ${}^{\Phi}\mathscr{F}^{\Phi^{-1}}$ is the C-module $\mathscr{F}$ equipped with the new $(\mathscr{A}, \mathscr{A})$-bimodule structure

$$a \cdot_{\Phi} x = \Phi^{-1}(a) \cdot x \quad \textit{and} \quad x \cdot_{\Phi^{-1}} a = x \cdot \Phi^{-1}(a) \tag{5.2.47}$$

and the new $\mathscr{A}$-valued inner products

$$\langle x, y\rangle^{\Phi}_{\mathscr{A}} = \Phi\Big(\langle x, y\rangle^{\mathscr{F}}_{\mathscr{A}}\Big) \quad \textit{and} \quad {}_{\mathscr{A}}\langle x, y\rangle^{\Phi} = \Phi\Big({}_{\mathscr{A}}\langle x, y\rangle^{\mathscr{F}}\Big). \tag{5.2.48}$$

PROOF: The ring-theoretic case is discussed in [4, Chap. 2, §5] and can be reconstructed from the *-algebra case easily, see Exercise 5.4.8. Note that the strong case is completely covered by the *-case. By the definition of h we have $[{}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{SPic}^*(\mathscr{A})$ if and only if $h_{\mathscr{E}} = \mathrm{id}_{\mathscr{Z}(\mathscr{A})}$. Since h is a group morphism the static Picard group $\mathsf{SPic}^*(\mathscr{A})$ is precisely the kernel of h. This proves the exactness of (5.2.38) and analogously the exactness of (5.2.39). If $\mathscr{A}$ is commutative and $\Phi \in \mathrm{Aut}^*(\mathscr{A})$ then $\ell(\Phi)$ is represented by the bimodule ${}_{\mathscr{A}}\mathscr{A}^{\Phi}_{\mathscr{A}}$. For this bimodule we have

$$h_{\mathscr{A}^{\Phi}}(a) \cdot x = x \cdot_{\Phi} a = x \cdot \Phi(a) = \Phi(a) \cdot x,$$

and hence $h(\ell(\Phi)) = \Phi$. Since ℓ is a group morphism as well, the exact sequence (5.2.41) splits. The same argument goes through for the strong case (5.2.42). Thus the Picard group $\mathsf{Pic}^*(\mathscr{A}) \cong \mathrm{Aut}^*(\mathscr{A}) \ltimes \mathsf{SPic}^*(\mathscr{A})$ is a semi-direct product. Using ℓ as split, the semi-direct product structure is explicitly given by $(\Phi, [\mathscr{E}]) \cdot (\Psi, [\mathscr{F}]) = (\Phi \circ \Psi, [\mathscr{E}][\mathscr{A}^{\Phi}][\mathscr{F}][\mathscr{A}^{\Phi}]^{-1})$. Now by Theorem 5.2.4 we have

$$[\mathscr{E}][\mathscr{A}^{\Phi}][\mathscr{F}][\mathscr{A}^{\Phi}]^{-1} = \Big[\mathscr{E} \mathbin{\widetilde{\otimes}} \mathscr{A}^{\Phi} \mathbin{\widetilde{\otimes}} \mathscr{F} \mathbin{\widetilde{\otimes}} \mathscr{A}^{\Phi^{-1}}\Big] = \Big[\mathscr{E} \mathbin{\widetilde{\otimes}} {}^{\Phi}\mathscr{F}^{\Phi^{-1}}\Big]$$

with the bimodule structure and inner products as in (5.2.47) and (5.2.48). □

In all the three cases, the static Picard group describes the interesting new aspects of Morita theory compared to automorphisms in the usual sense. We see that even for a commutative unital *-algebra or ring the structure of the Picard group can be interesting though the question of Morita equivalences within the commutative framework is not. Thus a major task will be to determine the static Picard groups.

5.2.4 The Picard Groups for $\mathscr{C}^\infty(M)$

As a first class of examples where the static Picard group can be identified, we consider again the algebra of smooth functions on a manifold M. As usual, there are several variations of this example for function spaces on various geometric spaces, which we will not discuss. We start with the following definition, well-known from complex and algebraic geometry:

Definition 5.2.13 (Geometric Picard group) *Let M be a manifold. Then the geometric Picard group $\mathsf{Pic}(M)$ of M is the group of isomorphism classes of complex line bundles. The unit element is the class of the trivial bundle, the multiplication comes from the tensor product and the inverse comes from the dual bundle.*

Remark 5.2.14 (Geometric Picard group) Here isomorphism of line bundles $L \cong L'$ means that there is a vector bundle isomorphism $\Phi\colon L \longrightarrow L'$ over the *identity* of M. Then it is well-known that the tensor product of line bundles is again a line bundle. Moreover, on the level of isomorphism classes the tensor product is associative, the dual bundle L^* is an inverse to L with respect to the tensor product, and the unit element is represented by the trivial line bundle.

We recall the following fact from differential geometry: every automorphism $\Phi \in \operatorname{Aut}(\mathscr{C}^\infty(M))$ of the complex-valued functions on M is necessarily of the form

$$\Phi(f) = \phi^* f \tag{5.2.49}$$

with some unique diffeomorphism $\phi\colon M \longrightarrow M$. Thus we canonically have

$$\operatorname{Aut}(\mathscr{C}^\infty(M)) \cong \operatorname{Diffeo}(M), \tag{5.2.50}$$

see e.g. the discussion in [58,90]. Since every pull-back commutes with complex conjugation, it follows that

$$\operatorname{Aut}(\mathscr{C}^\infty(M)) = \operatorname{Aut}^*(\mathscr{C}^\infty(M)). \tag{5.2.51}$$

Using these facts we can now formulate the following theorem:

Theorem 5.2.15 (Picard group of $\mathscr{C}^\infty(M)$) *Let M be a manifold.*

i.) $\mathsf{Pic}(\mathscr{C}^\infty(M)) \cong \operatorname{Diffeo}(M) \ltimes \mathsf{Pic}(M)$.

ii.) $\mathsf{Pic}^{\mathrm{str}}(\mathscr{C}^\infty(M)) \cong \operatorname{Diffeo}(M) \ltimes \mathsf{Pic}(M)$.

In both cases the diffeomorphisms act on line bundles via pull-back.

Proof: Thanks to Proposition 5.2.12 we have to determine $\mathsf{SPic}(\mathscr{C}^\infty(M))$ and $\mathsf{SPic}^{\mathrm{str}}(\mathscr{C}^\infty(M))$, respectively, together with the corresponding action of $\Phi = \phi^*$ needed for (5.2.43). Let $\mathscr{E}$ be a central equivalence bimodule. Since $\mathscr{C}^\infty(M)$ is unital, by Corollary 4.2.5 and Theorem 4.3.5 the right module $\mathscr{E}_{\mathscr{C}^\infty(M)}$ is a finitely generated and projective module. By the Serre-Swan Theorem 2.3.9 one finds a vector bundle $E \longrightarrow M$ with $\mathscr{E}_{\mathscr{C}^\infty(M)} \cong \Gamma^\infty(E)_{\mathscr{C}^\infty(M)}$ as right $\mathscr{C}^\infty(M)$-modules. Implementing such an isomorphism of right $\mathscr{C}^\infty(M)$-modules we can assume that $\mathscr{E} = \Gamma^\infty(E)$ as right modules. From Theorem 4.3.5 we know that $\mathscr{C}^\infty(M) \cong \operatorname{End}_{\mathscr{C}^\infty(M)}(\Gamma^\infty(E)_{\mathscr{C}^\infty(M)})$ via the left action. Since for any vector bundle it is known that $\operatorname{End}_{\mathscr{C}^\infty(M)}(\Gamma^\infty(E)_{\mathscr{C}^\infty(M)}) = \Gamma^\infty(\operatorname{End}(E))$ with the usual pointwise action of the endomorphisms, we conclude that necessarily $E = L$ is a line bundle. Moreover, since $f \cdot s = s \cdot f$ for a central bimodule, the left action of $\mathscr{C}^\infty(M)$ on $\Gamma^\infty(L)$ is the pointwise multiplication. Hence we determined the bimodule structure completely. If in addition $\mathscr{E}$ is a strong equivalence bimodule then the inner product $\langle \cdot, \cdot \rangle_{\mathscr{C}^\infty(M)}$ is given by a (positive) Hermitian fiber metric $h(\cdot, \cdot)$ which is even unique up to isometric isomorphisms. Since for every fiber metric we have $h(f \cdot s, s') = h(s, \overline{f} \cdot s')$, the fiber metric, the left action, and the complex conjugation are automatically compatible. Thus the left action gives even a *-isomorphism $\mathscr{C}^\infty(M) \cong \Gamma^\infty(\operatorname{End}(L))$ where the *-involution of $\Gamma^\infty(\operatorname{End}(L))$ is the one induced by h. From this we conclude that on the level of isomorphism classes we have injective maps

$$\mathsf{SPic}(\mathscr{C}^\infty(M)) \longrightarrow \mathsf{Pic}(M) \tag{$*$}$$

and

$$\mathsf{SPic}^{\mathrm{str}}(\mathscr{C}^\infty(M)) \longrightarrow \mathsf{Pic}(M). \tag{$**$}$$

Now let $L \longrightarrow M$ be a line bundle. To show surjectivity of $(*)$ and $(**)$, respectively, we have to show that $\Gamma^\infty(L)_{\mathscr{C}^\infty(M)}$ is a *full* module. But this is almost trivial via the Serre-Swan Theorem: we have $\Gamma^\infty(L)_{\mathscr{C}^\infty(M)} \cong e\mathscr{C}^\infty(M)^N$ for some idempotent $e = e^2 \in \mathrm{M}_n(\mathscr{C}^\infty(M)) = \mathscr{C}^\infty(M, \mathrm{M}_n(\mathbb{C}))$. This realizes L as a subbundle of the trivial vector bundle $M \times \mathbb{C}^N$ with fiber at $x \in M$ given by $L_x = \operatorname{im} e(x) \subseteq \mathbb{C}^N$. As $\dim L_x = 1$ we see $\operatorname{tr}(e(x)) = 1$ for all $x \in M$. This shows $\operatorname{tr} e = 1 \in \mathscr{C}^\infty(M)$ and hence $1 \in \mathscr{C}^\infty(M)\, e\, \mathscr{C}^\infty(M)$. By definition, this is the fullness of e which we wanted to show and thus ${}_{\mathscr{C}^\infty(M)}\Gamma^\infty(L)_{\mathscr{C}^\infty(M)}$ is an equivalence bimodule. Since without restriction we can assume e to be a projection according to Example 2.3.18, *ii.)*, and Theorem 2.3.16, the sections $\Gamma^\infty(L)$, equipped with the canonical inner product inherited from $\mathscr{C}^\infty(M)^N$, turn out to form a strong equivalence bimodule. This shows that $(*)$ as well as $(**)$ are surjective. Finally, the (algebraic) tensor product gives

$$\Gamma^\infty(L) \otimes_{\mathscr{C}^\infty(M)} \Gamma^\infty(L') \cong \Gamma^\infty(L \otimes L'),$$

again by using the Serre-Swan theorem. From this it immediately follows that $(*)$ as well as $(**)$ are group isomorphisms. It remains to identify the action of $\operatorname{Diffeo}(M)$ on $\mathsf{Pic}(M)$ under the isomorphism $(*)$ inherited from (5.2.43). Thus let $\Phi = \phi^* \in \operatorname{Aut}(\mathscr{C}^\infty(M))$ be given and let $L \longrightarrow M$ be a line bundle. Then for $s, s' \in \Gamma^\infty(L)$ and $f \in \mathscr{C}^\infty(M)$ we have

$$f \cdot_\Phi s = \Phi^{-1}(f)s = \phi_*(f)s \quad \text{and} \quad s \cdot_{\Phi^{-1}} f = s\Phi^{-1}(f) = s\phi_*(f),$$

where $\phi_*(f) = f \circ \phi^{-1}$ denotes the push-forward of f with ϕ. In case of a strong equivalence bimodule we have for the inner product

$$\langle s, s' \rangle^\Phi_{\mathscr{C}^\infty(M)} = \Phi\Big(\langle s, s' \rangle_{\mathscr{C}^\infty(M)}\Big) = \phi^*(h(s, s')),$$

where again h denotes the corresponding Hermitian fiber metric. Now consider the pull-back bundle $\phi^\# L \longrightarrow M$. Then for $s \in \Gamma^\infty(L)^\Phi_{\mathscr{C}^\infty(M)}$ we have

$$\phi^\#(s \cdot_{\Phi^{-1}} f) = \phi^\#(s\phi_*(f)) = (\phi^\# s)\phi^*\phi_*(f) = \phi^\#(s)f,$$

and thus

$$\phi^\# \colon \Gamma^\infty(L)^{\Phi^{-1}}_{\mathscr{C}^\infty(M)} \longrightarrow \Gamma^\infty(\phi^\# L) \tag{☺}$$

is a morphism of right $\mathscr{C}^\infty(M)$-modules. Since we consider central bimodules anyway, it is even a bimodule morphism. Moreover, since ϕ is a diffeomorphism, (☺) is an isomorphism. If in addition h is the Hermitian fiber metric on L then $\phi^\# h$ defined by

$$(\phi^\# h)(\phi^\# s, \phi^\# s') = \phi^*(h(s, s')) = \langle s, s' \rangle^\Phi_{\mathscr{C}^\infty(M)}$$

gives a Hermitian fiber metric on $\phi^\# L$. These computations show that $\phi^\#$ is an (isometric) isomorphism

$${}^{\Phi}_{\mathscr{C}^\infty(M)}\Gamma^\infty(L)^{\Phi^{-1}}_{\mathscr{C}^\infty(M)} \stackrel{\phi^\#}{\longrightarrow} {}_{\mathscr{C}^\infty(M)}\Gamma^\infty(\phi^\# L)_{\mathscr{C}^\infty(M)}.$$

Thus the left action of $\operatorname{Aut}(\mathscr{C}^\infty(M))$ on $\mathsf{SPic}(\mathscr{C}^\infty(M))$ used in (5.2.46) translates to the right action of $\operatorname{Diffeo}(M)$ on $\mathsf{Pic}(M)$ by pull-backs. □

Remark 5.2.16 (Static Picard group) Let M be a smooth manifold.

i.) Since $\mathsf{SPic}(M) \cong \mathsf{Pic}(M)$ is given by the geometric Picard group of line bundles this motivates the name "static" in contrast to the "dynamic" diffeomorphisms $\mathrm{Diffeo}(M)$. However, also the term *commutative* Picard group is common since one finds easily that for central bimodules ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\mathscr{E}'_{\mathscr{A}}$ and a commutative *-algebra the canonical flip

$$ {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{A}} \ni x \otimes x' \;\mapsto\; x' \otimes x \in {}_{\mathscr{A}}\mathscr{E}'_{\mathscr{A}} \otimes_{\mathscr{A}} {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \tag{5.2.52} $$

yields a bimodule isomorphism. For the inner products we find by a simple computation that

$$ \left\langle x \otimes x', y \otimes y' \right\rangle_{\mathscr{A}}^{\mathscr{E}\otimes\mathscr{E}'} = \langle x', y' \rangle_{\mathscr{A}}^{\mathscr{E}'} \langle x, y \rangle_{\mathscr{A}}^{\mathscr{E}}, \tag{5.2.53} $$

from which it immediately follows that (5.2.52) is isometric. Thus $\mathsf{SPic}(\mathscr{A})$ as well as $\mathsf{SPic}^*(\mathscr{A})$ and $\mathsf{SPic}^{\mathrm{str}}(\mathscr{A})$ are *commutative* whenever $\mathscr{A}$ is commutative.

ii.) The *-Picard group $\mathsf{Pic}^*(\mathscr{C}^\infty(M))$ can also be determined: a bimodule $\mathscr{E}$ yields an element in $\mathsf{SPic}^*(\mathscr{C}^\infty(M))$ if $\mathscr{E} \cong \Gamma^\infty(L)$ as central bimodule and if there is a *pseudo* Hermitian fiber metric h for L determining the $\mathscr{C}^\infty(M)$-valued inner products on $\Gamma^\infty(L)$. Now, if M is connected then for a pseudo Hermitian fiber metric h on a line bundle either h or $-h$ is positive. Since the tensor product of two negative Hermitian fiber metrics is a positive one, we obtain

$$ \mathsf{SPic}^*(\mathscr{C}^\infty(M)) \cong \mathsf{SPic}^{\mathrm{str}}(\mathscr{C}^\infty(M)) \times \mathbb{Z}_2 \cong \mathsf{Pic}(M) \times \mathbb{Z}_2 \tag{5.2.54} $$

for the static *-Picard group. In general, let n be the number of connected components of M. Then we can choose the signature of the fiber metric on each connected component separately. Consequently, we have

$$ \mathsf{SPic}^*(\mathscr{C}^\infty(M)) \cong \mathsf{SPic}^{\mathrm{str}}(\mathscr{C}^\infty(M)) \times (\mathbb{Z}_2)^n \cong \mathsf{Pic}(M) \times (\mathbb{Z}_2)^n. \tag{5.2.55} $$

In particular, the *-Picard group can be much larger than the naive expectation $\mathsf{Pic}^{\mathrm{str}}(\mathscr{C}^\infty(M)) \times \mathbb{Z}_2$.

iii.) The geometric Picard group $\mathsf{Pic}(M)$ can be described alternatively using the second cohomology of M. Without entering the details, we remark that the Chern class c_1 yields a group isomorphism

$$ c_1\colon\; \mathsf{Pic}(M) \longrightarrow \check{\mathrm{H}}^2(M, \mathbb{Z}) \tag{5.2.56} $$

from the geometric Picard group to the second integer Čech cohomology group of M, see e.g. the discussion in [121, Sect. III.4].

5.2.5 Kernel and Image of $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$

To conclude this section we shall use Theorem 4.2.12 to investigate the groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ for those unital *-algebras which satisfy the additional properties **(K)** and **(H)**. We start with the following observation:

Theorem 5.2.17 (Injectivity of $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$) *Restricted to the class of unital *-algebras with* **(K)** *and* **(H$^-$)**, *the canonical groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ is injective.*

PROOF: Let $\mathscr{A}$ and $\mathscr{B}$ be such *-algebras and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a strong equivalence bimodule. Then the $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{E}}$ is already fixed by the module structure $\mathscr{E}_{\mathscr{A}}$, unique up to isometric isomorphisms. For a choice within this isometric isomorphism class, the $\mathscr{B}$-valued inner product ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}}$ is fixed to be $\Theta_{\cdot,\cdot}$ by the compatibility of the inner products. This is the content of Theorem 4.2.12. It follows that the forgetting of the inner products is injective up to isometric isomorphisms. This shows that $\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A})$ is injective. ☐

Note that this argument simplifies our previous hands-on proof for the computation of the strong Picard group $\mathsf{Pic}^{\mathrm{str}}(\mathscr{C}^\infty(M))$ in Theorem 5.2.15, see Exercise 5.4.9.

The question of surjectivity is more subtle and will depend more strongly on the type of *-algebras under consideration. A slightly weaker result than surjectivity can be obtained under the following assumption. We consider a class of unital *-algebras with the following additional property [29, Sect. 7]:

Definition 5.2.18 (Property $(*)$) *A class of unital *-algebra has the property $(*)$ if for any two *-algebras $\mathscr{A}$ and $\mathscr{B}$ in this class and every projection $P = P^* = P^2 \in \mathrm{M}_n(\mathscr{A})$ one has: if $\mathscr{B}$ is isomorphic to $P\mathrm{M}_n(\mathscr{A})P$ then $\mathscr{B}$ is also *-isomorphic to $P\mathrm{M}_n(\mathscr{A})P$.*

Unlike the properties **(K)** and **(H)** this property is a feature of a certain *class* of *-algebras and not just of a single *-algebra. Nevertheless, for our two standard classes of *-algebras we have the property $(*)$:

Example 5.2.19 The class of unital C^*-algebras fulfills $(*)$. First recall that two C^*-algebras are *-isomorphic if and only if they are isomorphic, see e.g. [103, Thm. 4.1.20]. Since for $P = P^* = P^2 \in \mathrm{M}_n(\mathscr{A})$ the algebra $P\mathrm{M}_n(\mathscr{A})P$ endowed with the inherited *-involution is a C^*-algebra, the property $(*)$ follows immediately for the class of unital C^*-algebras. However, there is some caution in due. A C^*-algebra $\mathscr{A}$ can very well have another *-involution $^+$ such that $(\mathscr{A}, ^*)$ and $(\mathscr{A}, ^+)$ are *not* *-isomorphic. In this case, $(\mathscr{A}, ^+)$ is no longer a C^*-algebra, thus not contradicting the above statement. A simple geometric example is obtained for the continuous functions $\mathscr{C}(\mathbb{S}^2)$ on $\mathbb{S}^2$. Instead of using the pointwise complex conjugation making $\mathscr{C}(\mathbb{S}^2)$ a C^*-algebra one can take $f^+(x) = \overline{f(-x)}$ where $x \mapsto -x$ is the antipode map. This shows that one really has to specify the *class* of algebras first and then check whether $(*)$ is satisfied or not.

Example 5.2.20 Another class of examples are the smooth functions $\mathscr{C}^\infty(M)$ on differentiable manifolds equipped with the complex conjugation as *-involution. Note that we have to exclude other *-involutions by hand, for the same reason as for C^*-algebras. Now, if $P\mathrm{M}_n(\mathscr{C}^\infty(M))P \cong \Gamma^\infty(\mathrm{End}(E))$, with $E = \operatorname{im} P$, is isomorphic to $\mathscr{C}^\infty(M')$ then E is a line bundle and thus $P\mathrm{M}_n(\mathscr{C}^\infty(M))P \cong \mathscr{C}^\infty(M)$ are even *-isomorphic. Thanks to (5.2.50), from $\mathscr{C}^\infty(M) \cong \mathscr{C}^\infty(M')$ we conclude that M and M' are diffeomorphic and hence $\mathscr{C}^\infty(M)$ and $\mathscr{C}^\infty(M')$ are *-isomorphic, too. Thus $(*)$ is fulfilled.

Another large class will be found when discussing deformation theory in Section **??**.

For the next theorem we recall that the automorphism group $\mathrm{Aut}(\mathscr{B})$ acts on $\mathsf{Pic}(\mathscr{B}, \mathscr{A})$ from the left, provided $\mathsf{Pic}(\mathscr{B}, \mathscr{A})$ is non-empty. This follows directly from the existence of the canonical groupoid morphism $\ell\colon \mathrm{Iso} \longrightarrow \mathsf{Pic}$ according to Remark 5.2.5. Together with the canonical groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ we obtain a map

$$\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A}) \big/ \mathrm{Aut}(\mathscr{B}) \tag{5.2.57}$$

to the space of $\mathrm{Aut}(\mathscr{B})$-orbits in $\mathsf{Pic}(\mathscr{B}, \mathscr{A})$. Then we have at least surjectivity on this orbit space [29, Prop. 7.7]:

Theorem 5.2.21 (Surjectivity of $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$) *For unital *-algebras $\mathscr{A}$ and $\mathscr{B}$ within a class of *-algebras satisfying* **(K)**, **(H$^-$)**, *and* $(*)$ *the map*

$$\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A}) \big/ \mathrm{Aut}(\mathscr{B}) \tag{5.2.58}$$

is surjective (if the left hand side is empty, then the right hand side is empty as well). In particular, the orbits of Pic *and* $\mathsf{Pic}^{\mathrm{str}}$ *coincide implying that $\mathscr{A}$ and $\mathscr{B}$ are strongly Morita equivalent iff they are Morita equivalent in the ring-theoretic sense.*

Proof: Assume that we are given an equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Pic}}(\mathscr{B}, \mathscr{A})$. Then $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n$ with a full idempotent $e = e^2 \in \mathrm{M}_n(\mathscr{A})$ by Theorem 4.3.5 and $\mathscr{B} \cong e\mathrm{M}_n(\mathscr{A})e$ via the left action on $e\mathscr{A}^n$ as associative algebras. Using **(K)** and **(H)**, by Theorem 4.2.12 we can assume without restriction that $e = P$ is already a projection. This way we obtain a full and completely positive non-degenerate $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ on $\mathscr{E}_{\mathscr{A}}$ by restricting the canonical inner product of $\mathscr{A}^n$ to $P\mathscr{A}^n$. Since as associative algebras we have $\mathscr{B} \cong P\mathrm{M}_n(\mathscr{A})P \cong \mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ via the left action, the algebra $\mathscr{B}$ acts by adjointable operators on $\mathscr{E}_{\mathscr{A}}$. Thus by

$$\langle x, b \cdot y \rangle^{\mathscr{E}}_{\mathscr{A}} = \langle b^{+} \cdot x, y \rangle^{\mathscr{E}}_{\mathscr{A}}$$

a *-involution is induced on $\mathscr{B}$ such that $(\mathscr{B}, ^{+})$ is *-isomorphic to $P\mathrm{M}_n(\mathscr{A})P$. With respect to *this* *-involution, ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a strong equivalence bimodule. In general, here one reaches a dead end as this *-involution might be completely different from the original *-involution * of $\mathscr{B}$. However, thanks to the assumption **(∗)** we know that $\mathscr{B}$ is even *-isomorphic to $P\mathrm{M}_n(\mathscr{A})P$ and hence to $(\mathscr{B}, ^{+})$ since the later two are *-isomorphic via the left action. Thus let

$$\Phi \colon (\mathscr{B}, ^{+}) \longrightarrow (\mathscr{B}, ^{*})$$

be such a *-isomorphism. In particular, $\Phi \in \mathrm{Aut}(\mathscr{B})$ is an ordinary automorphism. It follows that the *-isomorphism form $\mathscr{B}$ to $\mathfrak{B}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) \cong P\mathrm{M}_n(\mathscr{A})P$ is given by

$$b \;\mapsto\; \big(x \mapsto \Phi^{-1}(b) \cdot x\big).$$

This way, the twisted bimodule ${}^{\Phi}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ equipped with $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ and the $\mathscr{B}$-valued inner product ${}_{\mathscr{B}}\langle \cdot\,, \cdot \rangle^{\mathscr{E}} = \Phi(\Theta_{\cdot,\cdot})$ becomes a strong equivalence bimodule. Since $[{}^{\Phi}\mathscr{E}] = \ell(\Phi)[\mathscr{E}]$ the classes of the bimodules $[\mathscr{E}]$ and $[{}^{\Phi}\mathscr{E}]$ lie in the same $\mathrm{Aut}(\mathscr{B})$-orbit. This finally shows that every $\mathrm{Aut}(\mathscr{B})$-orbit in $\mathsf{Pic}(\mathscr{B}, \mathscr{A})$ is reached by $\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A})$. □

Remark 5.2.22 We can also re-interpret the result as follows: for a given unital *-algebra $\mathscr{A}$ satisfying **(K)** and **(H⁻)** and another unital algebra $\mathscr{B}$ being Morita equivalent to $\mathscr{A}$ we find for every equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Pic}}(\mathscr{B}, \mathscr{A})$ a *-involution $^{+}$ for $\mathscr{B}$ such that ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ with the obvious $\mathscr{B}$-valued inner product becomes a strong equivalence bimodule.

Remark 5.2.23 (Beer's theorem) The above theorem gives also a refined version of Beer's theorem: for two unital *-algebras $\mathscr{A}$ and $\mathscr{B}$ within a class of unital *-algebras satisfying **(K)**, **(H⁻)**, and **(∗)** Morita equivalence implies strong Morita equivalence. Beer showed this statement for C^*-algebras [6], for which we can apply Theorem 5.2.21 thanks to Example 5.2.19. Note, however, that already for C^*-algebras the above theorem gives a more detailed result than just stating that the orbits of $\mathsf{Pic}^{\mathrm{str}}$ and Pic coincide.

The question about surjectivity of $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ is not yet answered completely by Theorem 5.2.21. To this end we have to investigate the relation between $\mathrm{Aut}(\mathscr{B})$ and $\mathrm{Aut}^*(\mathscr{B})$ more closely. For $\Phi \in \mathrm{Aut}(\mathscr{B})$ one defines the map

$$\Phi^* \colon \mathscr{B} \ni b \;\mapsto\; \Phi^*(b) = \Phi(b^*)^* \in \mathscr{B}, \tag{5.2.59}$$

which gives again an automorphism $\Phi^* \in \mathrm{Aut}(\mathbb{B})$. It follows that the map $\Phi \mapsto \Phi^*$ is an involutive group automorphism of $\mathrm{Aut}(\mathscr{B})$ such that $\Phi^* = \Phi$ if and only if $\Phi \in \mathrm{Aut}^*(\mathscr{B})$, see also Exercise 5.4.10.

Proposition 5.2.24 *Let $\mathscr{A}$ be a unital *-algebra satisfying* **(K)** *and* **(H⁻)**. *Moreover, let $\mathscr{B}$ be another unital *-algebra Morita equivalent to $\mathscr{A}$ via an equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. Assume that $[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]$ is in the image of $\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A})$. Then the following statements are equivalent:*

i.) *The whole* $\mathrm{Aut}(\mathscr{B})$*-orbit of* $[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]$ *is in the image.*

ii.) *For all* $\Phi \in \mathrm{Aut}(\mathscr{B})$ *there is an invertible* $u \in \mathscr{B}$ *such that* $\Phi^*\Phi^{-1} = \mathrm{Ad}(u^*u)$.

PROOF: Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be in the image then we find on ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ a full and completely positive $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ such that the left $\mathscr{B}$-action gives a *-representation, i.e. $\langle b \cdot x, y\rangle^{\mathscr{E}}_{\mathscr{A}} = \langle x, b^* \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}}$ for the given *-involution of $\mathscr{B}$. From the *-isomorphism $\mathscr{B} \cong \mathfrak{F}(\mathscr{E}_{\mathscr{A}})$ the $\mathscr{B}$-valued inner product is then determined to be $\Theta_{\cdot,\cdot}$ as usual. By the properties of $\mathscr{A}$, the inner product $\langle \cdot\,, \cdot \rangle^{\mathscr{E}}_{\mathscr{A}}$ is unique up to isometric isomorphisms. Thus consider first ${}^{\Phi}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ being in the image, too, for some $\Phi \in \mathrm{Aut}(\mathscr{B})$. By assumption there is again an $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\Phi\mathscr{E}}_{\mathscr{A}}$ being compatible with the Φ-twisted $\mathscr{B}$-module structure. Necessarily, the two inner product are isometrically isomorphic as the right $\mathscr{A}$-module structure was not changed and we have **(K)** and **(H⁻)** for $\mathscr{A}$. Thus there is an isometric isomorphism $V \in \mathfrak{B}_{\mathscr{A}}({}^{\Phi}\mathscr{E}_{\mathscr{A}}, \mathscr{E}_{\mathscr{A}})$ with

$$\langle x, y\rangle^{\Phi\mathscr{E}}_{\mathscr{A}} = \langle V(x), V(y)\rangle^{\mathscr{E}}_{\mathscr{A}} \qquad (*)$$

for all $x, y \in \mathscr{E}$. Since necessarily V is right $\mathscr{A}$-linear, there exists a (unique) invertible $v \in \mathscr{B}$ with $V(x) = v \cdot x$. Now on one hand we have for every $b \in \mathscr{B}$

$$\left\langle x, b \cdot_{\Phi} y\right\rangle^{\Phi\mathscr{E}}_{\mathscr{A}} = \left\langle V(x), V(\Phi^{-1}(b) \cdot y)\right\rangle^{\mathscr{E}}_{\mathscr{A}} = \left\langle v \cdot x, v \cdot (\Phi^{-1}(b) \cdot y)\right\rangle^{\mathscr{E}}_{\mathscr{A}} = \left\langle x, (v^*v\Phi^{-1}(b)) \cdot y\right\rangle^{\mathscr{E}}_{\mathscr{A}},$$

and on the other hand

$$\left\langle x, b{\cdot}_{\Phi}y\right\rangle^{\Phi\mathscr{E}}_{\mathscr{A}} = \left\langle b^*{\cdot}_{\Phi}x, y\right\rangle^{\Phi\mathscr{E}}_{\mathscr{A}} = \left\langle V(\Phi^{-1}(b^*){\cdot}x), V(y)\right\rangle^{\mathscr{E}}_{\mathscr{A}} = \left\langle v{\cdot}\Phi^{-1}(b^*){\cdot}x, v{\cdot}y\right\rangle^{\mathscr{E}}_{\mathscr{A}} = \left\langle x, (\Phi^{-1}(b^*)^*v^*v){\cdot}y\right\rangle^{\mathscr{E}}_{\mathscr{A}}$$

for all $x, y \in \mathscr{E}$. Thus $v^*v\Phi^{-1}(b) = \Phi^{-1}(b^*)^*v^*v$ follows from the usual non-degeneracy argument. This shows that Φ^{-1} satisfies the condition mentioned in the second statement. Since Φ was arbitrary, the second statement follows in general. Conversely, assume the second statement holds. Then for $\Phi \in \mathrm{Aut}(\mathscr{B})$ we choose the appropriate u and define the inner product $\langle \cdot\,, \cdot \rangle^{\Phi\mathscr{E}}_{\mathscr{A}}$ by $(*)$. By an analogous computation one shows that this inner product is then compatible with the Φ-twisted left $\mathscr{B}$-module structure. □

Remark 5.2.25 Since $\Phi \in \mathrm{Aut}(\mathscr{B})$ is a *-automorphism if and only if $\Phi^* = \Phi$, the property *ii.)* in Proposition 5.2.24 is always fulfilled for *-automorphisms by $u = \mathbb{1}_{\mathscr{B}}$. Thus the condition is relevant for those automorphism which are *not* *-automorphisms: for them the combination $\Phi^*\Phi^{-1}$ has to be an inner automorphism of a particular form. Whether or not this is the case typically depends strongly on the example.

Corollary 5.2.26 *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be unital* **-algebras within a class of* **-algebras satisfying* **(K)**, **(H⁻)**, *and* $(*)$.

i.) *The map* $\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A})$ *is surjective if and only if* $\mathscr{B}$ *satisfies the condition ii.) of Proposition 5.2.24.*

ii.) *Within this class of* **-algebras the condition ii.) of Proposition 5.2.24 is a strong Morita invariant.*

PROOF: The first part is clear by Theorem 5.2.21 and Proposition 5.2.24. The second part is clear as well, as the surjectivity of $\mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Pic}(\mathscr{B}, \mathscr{A})$ is a property of the groupoid orbit through $\mathscr{B}$ since we have a groupoid morphism. □

We conclude this discussion with an example of a C^*-algebra which does *not* fulfill the condition *ii.)* of Proposition 5.2.24, see also [29, Sect. 7] for more details.

Example 5.2.27 Let $\mathfrak{H}$ be a Hilbert space with countably infinite Hilbert basis $\{e_n\}_{n\in\mathbb{N}}$. Let $\mathfrak{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be the unital C^*-subalgebra consisting of operators of the form $c\,\mathrm{id}_{\mathfrak{H}} + K$ with $c \in \mathbb{C}$ and $K \in \mathfrak{K}(\mathfrak{H})$ being *compact*. Moreover, let $A = A^* \in \mathfrak{B}(\mathfrak{H})$ be determined by $Ae_{2n} = 2e_{2n}$ and $Ae_{2n+1} = e_{2n+1}$. Then $\Phi = \mathrm{Ad}(A)\big|_{\mathfrak{A}}\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ is an automorphism with $\Phi^* = \Phi^{-1}$ and hence $\Phi^*\Phi^{-1} = \mathrm{Ad}(A^{-2})$. We claim there is no invertible $B \in \mathfrak{A}$ with $\mathrm{Ad}(A^{-2}) = \mathrm{Ad}(B^*B)$. Indeed, if there would be such an operator then $A^{-2}B^*BC(A^{-2}B^*B)^{-1} = C$ for all $C \in \mathfrak{A}$. This would imply $A^{-2}B^*B = c\,\mathrm{id}_{\mathfrak{H}}$ for some $c \in \mathbb{C}$ and hence $B^*B = cA^2$ which is clearly not possible for $B \in \mathfrak{A}$ by the choice of A. Thus for this C^*-algebra $\mathfrak{A}$ the property *ii.)* as in Proposition 5.2.24 fails.

On the other hand, for a *commutative* unital *-algebra $\mathscr{A}$ it is very easy to decide whether the condition *ii.)* as in Proposition 5.2.24 holds or not: since all automorphisms are outer, this condition holds iff

$$\mathrm{Aut}^*(\mathscr{A}) = \mathrm{Aut}(\mathscr{A}). \tag{5.2.60}$$

In particular, $\mathscr{C}^\infty(M)$ always fulfills the condition *ii.)* as in Proposition 5.2.24.

5.3 Morita Invariants

In the last sections we have already seen several quantities which are preserved under Morita equivalence in its various incarnations. In this section we shall give a more systematic approach to these Morita invariants and put them into a more unified framework. Then we will also discuss some further, new invariants for all three flavours of Morita theory. It turns out that Morita equivalent algebras share a lot of properties.

5.3.1 From Groupoid Actions to Morita Invariants

The notion of "invariant" we are aiming at will rely on actions of a groupoid. We start recalling the following definition:

Definition 5.3.1 (Groupoid action) *Let $\mathfrak{G}$ be a groupoid and let $\mathfrak{C}$ be a category. A left action of $\mathfrak{G}$ on $\mathfrak{C}$ (better: on the objects of $\mathfrak{C}$) is a covariant functor*

$$\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{C}. \tag{5.3.1}$$

Analogously, a right action is defined to be a contravariant functor $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{C}$.

Remark 5.3.2 (Groupoid action) Let $\mathfrak{C}$ be a category.

i.) For a groupoid $\mathfrak{G}$ with just one object $*$, i.e. for a group, a functor $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{C}$ consists in the choice of an object $\Phi(*) \in \mathrm{Obj}(\mathfrak{C})$ together with the specification of morphisms $\Phi_g \in \mathrm{Morph}(\Phi(*), \Phi(*)) = \mathrm{End}(\Phi(*))$ such that $\Phi_e = \mathrm{id}$ and $\Phi_g \circ \Phi_h = \Phi_{g\circ h}$. Since all g are invertible, this really gives a group action in the sense that we have a group homomorphism into the automorphism group of $\Phi(*)$.

ii.) There are alternative and more sophisticated definitions of groupoid actions which can ultimately be reduced to the above definition. However, they allow to characterize some more specific features of the functor in addition, see also Exercise 5.4.11. Our choice is in some sense the most innocent one.

Example 5.3.3 (Isotropy groups) The isotropy groups of a groupoid are in some sense the most fundamental invariants. For a groupoid $\mathfrak{G}$ one constructs the following functor

$$\mathsf{Isotropy}\colon \mathfrak{G} \longrightarrow \mathsf{Group} \tag{5.3.2}$$

where $\mathsf{Isotropy}(a) = \mathfrak{G}(a)$ and for a morphism $g\colon a \longrightarrow b$ one considers the group isomorphism $\mathsf{Isotropy}(g)\colon \mathfrak{G}(a) \longrightarrow \mathfrak{G}(b)$ defined by $\mathsf{Isotropy}(g)h = ghg^{-1}$. It is an easy check that this is indeed a functor. Thus the isotropy groups are invariants of the groupoid, a fact which we have already seen in Proposition 5.1.4.

The following simple statement shows how we obtain an "invariant" from a groupoid action:

Theorem 5.3.4 (Invariants from groupoid action) *Let $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{C}$ be a left action of a groupoid $\mathfrak{G}$ on a category $\mathfrak{C}$.*

i.) For every $a \in \mathfrak{G}_0$ the object $\Phi(a) \in \mathrm{Obj}(\mathfrak{C})$ becomes a $\mathfrak{G}(a)$-space in the sense that

$$\mathfrak{G}(a) \ni g \;\mapsto\; \Phi_g \in \mathrm{End}(\Phi(a)) \tag{5.3.3}$$

is a group morphism.

ii.) If $a, b \in \mathfrak{G}_0$ are in the same orbit then $\Phi(a)$ and $\Phi(b)$ are isomorphic.

iii.) Every morphism $g\colon a \longrightarrow b$ in $\mathfrak{G}_1$ yields an isomorphism $\Phi_g\colon \Phi(a) \longrightarrow \Phi(b)$ which intertwines the $\mathfrak{G}(a)$-structure of $\Phi(a)$ into the $\mathfrak{G}(b)$-structure of $\Phi(b)$.

Proof: The proof is just a simple reformulation of the statement that Φ is a functor and the fact that every morphism in a groupoid is invertible. □

In this sense, the objects $\Phi(a)$ are constant along the orbit of a in $\mathfrak{G}$ and can thus be understood as an invariant of the groupoid. Note that every invariant carries a canonical structure of a $\mathfrak{G}(a)$-space which we always consider as being part of the invariant.

Even though the above theorem is an almost trivial statement about functoriality we shall see important applications and not so trivial examples: the Morita invariants. Here the groupoid in question is of course the Picard groupoid in one of its various forms. Thus a *Morita invariant* is, by definition, a functor $\Phi\colon \mathsf{Pic} \longrightarrow \mathfrak{C}$ from the Picard groupoid into some category and similarly for *-Morita invariants and strong Morita invariants, respectively. Analogously, an invariant of rings is defined to be a functor $\Phi\colon \mathrm{Iso} \longrightarrow \mathfrak{C}$ and a *-invariant of *-algebras is a functor $\Phi\colon \mathrm{Iso}^* \longrightarrow \mathfrak{C}$. We will have to argue that this point of view is not superfluous, but gives nice insights even for the case of Iso-invariants.

Example 5.3.5 (Picard groups) In view of Example 5.3.3 the isotropy groups of Pic, Pic^*, and $\mathsf{Pic}^{\mathrm{str}}$, respectively, i.e. the Picard groups, are the most fundamental Morita invariants.

Thanks to the groupoid morphism ℓ every Morita invariant is also an invariant of unital rings or *-algebras, respectively.

Proposition 5.3.6 *Every Morita invariant $\Phi\colon \mathsf{Pic} \longrightarrow \mathfrak{C}$ induces an invariant*

$$\Phi \circ \ell\colon \mathrm{Iso} \longrightarrow \mathfrak{C}, \tag{5.3.4}$$

where on objects we have $(\Phi \circ \ell)(\mathscr{A}) = \Phi(\mathscr{A})$. The same holds for a *-Morita invariant and a strong Morita invariant.

Proof: Clearly, as the composition of functors is a functor, this follows immediately from Theorem 5.2.4, *iv.)*, and $\ell(\mathscr{A}) = [{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]$. □

5.3.2 The Center

After the Picard groups themselves, the centers are the next simple examples of Morita invariants. We consider unital *-algebras for simplicity. Then one defines

$$\mathscr{Z}\colon \operatorname{Obj}(\mathsf{Pic}^*) \ni \mathscr{A} \mapsto \mathscr{Z}(\mathscr{A}) \in \operatorname{Obj}({}^*\text{-}\mathsf{Alg}_{\mathsf{com}}), \tag{5.3.5}$$

where ${}^*\text{-}\mathsf{Alg}_{\mathsf{com}} \subseteq {}^*\text{-}\mathsf{Alg}$ denotes the subcategory of commutative unital *-algebras. For a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ one sets

$$\mathscr{Z}\colon \mathsf{Pic}^*(\mathscr{B}, \mathscr{A}) \ni [{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}] \mapsto \mathscr{Z}([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]) = \big(h_{\mathscr{E}}\colon \mathscr{Z}(\mathscr{A}) \longrightarrow \mathscr{Z}(\mathscr{B})\big) \in \operatorname{Iso}^*(\mathscr{Z}(\mathscr{B}), \mathscr{Z}(\mathscr{A})), \tag{5.3.6}$$

where $h_{\mathscr{E}}$ is the map from Proposition 5.2.9. For the ring-theoretic version from (5.2.30) we shall use the same symbols. One obtains the following result:

Theorem 5.3.7 (Morita invariance of center) *The map* (5.3.6) *is well-defined and yields, together with* (5.3.5)*, a *-Morita invariant*

$$\mathscr{Z}\colon \mathsf{Pic}^* \longrightarrow {}^*\text{-}\mathsf{Alg}_{\mathsf{com}}. \tag{5.3.7}$$

The corresponding *-isomorphism invariant $\mathscr{Z} \circ \ell$ is given on *-isomorphisms by the restriction of the *-isomorphism to the center, i.e.

$$(\mathscr{Z} \circ \ell)(\Phi) = \Phi\Big|_{\mathscr{Z}(\mathscr{A})}. \tag{5.3.8}$$

Analogously, one obtains a Morita invariant with values in the commutative unital rings

$$\mathscr{Z}\colon \mathsf{Pic} \longrightarrow \mathsf{Ring}_{\mathsf{com}}. \tag{5.3.9}$$

PROOF: We have already shown in Proposition 5.2.9, *iv.)*, that the map $h_{\mathscr{E}}$ actually depends on $[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]$ only. Thus the well-definedness follows. Moreover, from Proposition 5.2.9, *iii.)*, it follows that $\mathscr{Z}$ is a functor. It remains to show (5.3.8). Thus let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a *-isomorphism. Then $\ell(\Phi)$ is represented by the twisted bimodule ${}_{\mathscr{B}}\mathscr{B}^{\Phi}_{\mathscr{A}}$. Now let $a \in \mathscr{Z}(\mathscr{A})$ then for all $b \in \mathscr{B}$ one has $b \cdot_{\Phi} a = b\Phi(a) = \Phi(a)b = \Phi(a) \cdot b$ since $\Phi(a) \in \mathscr{Z}(\mathscr{B})$. From this it follows that $h_{\ell(\Phi)} = \Phi$. The ring-theoretic version is shown analogously, see again Exercise 5.4.8. □

Note that the above point of view gives more than just a random isomorphism $\mathscr{Z}(\mathscr{A}) \cong \mathscr{Z}(\mathscr{B})$. Instead we have a very precise way how to pass from one center to the other, compatible with the action of the isomorphisms of the ambient algebras on each side.

5.3.3 K_0-Theory

As a next example of a Morita invariant we consider the various versions of K_0-theory. The ring-theoretic result is classical, see e.g. [4, 99]. For *-algebras, we restrict to the case of unital *-algebras. To construct a functor whose values on objects are the K_0-groups we have to consider the finitely generated and projective modules with strongly non-degenerate inner product. Let $(\mathscr{P}_{\mathscr{B}}, \langle \cdot\,, \cdot \rangle^{\mathscr{P}}_{\mathscr{B}})$ be such an inner-product right $\mathscr{B}$-module. Moreover, let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a *-equivalence bimodule. Then we consider the inner-product module $\mathscr{P}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$.

Theorem 5.3.8 (Morita invariance of K_0) *For unital *-algebras we have:*

i.) For $\mathscr{P}_{\mathscr{B}} \in \underline{\mathsf{Proj}}^*(\mathscr{B})$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ one has $\mathscr{P}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \underline{\mathsf{Proj}}^*(\mathscr{A})$. Moreover, the equivalence class $[\mathscr{P}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{Proj}^*(\mathscr{A})$ depends only on $[\mathscr{P}_{\mathscr{B}}] \in \mathsf{Proj}^*(\mathscr{B})$ and $[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{Pic}^*(\mathscr{B}, \mathscr{A})$.

ii.) *The map*

$$\mathsf{Proj}^*([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}])\colon \mathsf{Proj}^*(\mathscr{B}) \ni [\mathscr{P}_{\mathscr{B}}] \mapsto \left[\mathscr{P}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\right] \in \mathsf{Proj}^*(\mathscr{A}) \tag{5.3.10}$$

is a morphism of abelian semi-groups.

iii.) *One has*

$$\mathsf{Proj}^*([{}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}}]) = \mathrm{id}_{\mathsf{Proj}^*(\mathscr{B})} \tag{5.3.11}$$

and

$$\mathsf{Proj}^*\left(\left[{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\right]\right) = \mathsf{Proj}^*([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]) \circ \mathsf{Proj}^*([{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}]). \tag{5.3.12}$$

iv.) *The definition of* $\mathsf{Proj}^*\colon \mathsf{Pic}^* \longrightarrow \mathsf{AbSemiGroup}$ *by*

$$\mathrm{Obj}(\mathsf{Pic}^*) \ni \mathscr{A} \mapsto \mathsf{Proj}^*(\mathscr{A}) \in \mathrm{Obj}(\mathsf{AbSemiGroup}) \tag{5.3.13}$$

on objects and

$$\mathsf{Pic}^*(\mathscr{B}, \mathscr{A}) \ni [{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}] \mapsto \mathsf{Proj}^*([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]) \in \mathrm{Morph}(\mathsf{Proj}^*(\mathscr{B}), \mathsf{Proj}^*(\mathscr{A})) \tag{5.3.14}$$

on morphisms yields a right action of Pic^* *on the category of abelian semi-groups* $\mathsf{AbSemiGroup}$.

v.) *This right action induces a right action*

$$\mathsf{K}_0^*\colon \mathsf{Pic}^* \longrightarrow \mathsf{Ab}, \tag{5.3.15}$$

and hence the K_0^**-theory becomes a* **-Morita invariant.*

vi.) *Analogously, one obtains the right actions*

$$\mathsf{Proj}^{\mathrm{str}}\colon \mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{AbSemiGroup} \tag{5.3.16}$$

and

$$\mathsf{K}_0^{\mathrm{str}}\colon \mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Ab} \tag{5.3.17}$$

for the strong Morita equivalence as well as in the ring-theoretic framework

$$\mathsf{Proj}\colon \mathsf{Pic} \longrightarrow \mathsf{AbSemiGroup} \tag{5.3.18}$$

and

$$\mathsf{K}_0\colon \mathsf{Pic} \longrightarrow \mathsf{Ab}. \tag{5.3.19}$$

PROOF: For the first part we can apply Proposition 4.2.7 to show that $\mathscr{P}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ has a finite Hermitian dual basis. Thus $\mathscr{P}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is finitely generated and projective. Moreover, the $\mathscr{A}$-valued inner product $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}^{\mathscr{P}\otimes\mathscr{E}}$ is already strongly non-degenerate implying that $\mathscr{P}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = \mathscr{P}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is an object in $\underline{\mathsf{Proj}}^*(\mathscr{A})$. Together with the functoriality of $\widehat{\otimes}$ the first part follows. The second is clear as $\widehat{\otimes}$ is compatible with orthogonal direct sums. The third part is clear as well: on one hand $\mathscr{P}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \cong \mathscr{P}_{\mathscr{B}}$ for *all* strongly non-degenerate right $\mathscr{B}$-modules and $\mathscr{P}_{\mathscr{B}} \in \underline{\mathsf{Proj}}^*(\mathscr{B})$ has this property since with a finite Hermitian dual basis $e_\alpha, f_\alpha \in \mathscr{P}_{\mathscr{B}}$ we have for all $p \in \mathscr{P}_{\mathscr{B}}$

$$p \cdot \mathbb{1}_{\mathscr{B}} = \sum\nolimits_\alpha e_\alpha \cdot \langle f_\alpha, p \cdot \mathbb{1}_{\mathscr{B}} \rangle_{\mathscr{B}}^{\mathscr{P}} = \sum\nolimits_\alpha e_\alpha \cdot \left(\langle f_\alpha, p \rangle_{\mathscr{B}}^{\mathscr{P}} \mathbb{1}_{\mathscr{B}}\right) = \sum\nolimits_\alpha e_\alpha \cdot \langle f_\alpha, p \rangle_{\mathscr{B}}^{\mathscr{P}} = p.$$

On the other hand we can use the associativity of $\widehat{\otimes}$ and observe that

$$\left(\mathscr{P}_{\mathscr{C}} \mathbin{\widehat{\otimes}_{\mathscr{C}}} {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}\right) \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \cong \mathscr{P}_{\mathscr{C}} \mathbin{\widehat{\otimes}_{\mathscr{C}}} \left({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widehat{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\right) \cong \mathscr{P}_{\mathscr{C}} \mathbin{\widehat{\otimes}_{\mathscr{C}}} \left({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}\right).$$

Note that the situation is particularly simple as we do not need any quotient procedure for $\widehat{\otimes}$ here according to Proposition 4.2.7. Now (5.3.11) and (5.3.12) mean that Proj^* is a contravariant functor and thus we have a right action of Pic^*. The composition with the covariant Grothendieck functor $\mathsf{AbSemiGroup} \longrightarrow \mathsf{Ab}$ yields again a contravariant functor K_0^*. This shows the fourth and fifth part. The last is shown analogously. □

5.3.4 The Lattice of Closed Ideals

For the following invariant we can consider idempotent and non-degenerate *-algebras again. As motivation we recall the following well-known result from C^*-algebra theory:

Theorem 5.3.9 (Closed *-ideals) *Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{J} \subseteq \mathscr{A}$ a *-ideal. Then the following statements are equivalent:*

i.) The *-ideal $\mathscr{J}$ is closed.

ii.) There is a *-representation $(\mathfrak{H}, \pi)$ of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$ such that $\ker \pi = \mathscr{J}$.

Proof: For completeness, we give a sketch of the proof: it is well-known that a *-representation of a C^*-algebra on a pre-Hilbert space is continuous and can thus be extended to the Hilbert space completion by continuity. Note that the kernel will not change under the completion. Then the kernel is closed. For the reverse, it is also known that $\mathscr{A}/\mathscr{J}$ is a C^*-algebra itself if $\mathscr{J}$ is a closed *-ideal. But every C^*-algebra has a faithful *-representation, say $\tilde{\pi}\colon \mathscr{A}/\mathscr{J} \longrightarrow \mathfrak{B}(\mathfrak{H})$. If $\mathrm{pr}\colon \mathscr{A} \longrightarrow \mathscr{A}/\mathscr{J}$ is the canonical projection, then $\pi = \tilde{\pi} \circ \mathrm{pr}$ is the desired *-representation of $\mathscr{A}$. □

Moreover, for a unital C^*-algebra we can split every *-representation on a Hilbert space into a strongly non-degenerate one and the null representation on the orthogonal complement. The kernel of the original *-representation coincides with the strongly non-degenerate part of it. For a non-unital C^*-algebra there is still an orthogonal splitting into the null representation and a *-representation on a Hilbert space which is strongly non-degenerate up to completion. Again, the kernel of the original *-representation is determined by the strongly non-degenerate part. This suggest to focus on the strongly non-degenerate *-representations in general.

In the following we want also *-representations on general inner-product modules over an arbitrary coefficient *-algebra $\mathscr{D}$ and not just $\mathscr{D} = \mathsf{C}$. As usual we insist on *-algebras being non-degenerate and idempotent. Then for any *-representation ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A})$ we can pass to a strongly non-degenerate one by tensoring with the canonical bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$: by Lemma 4.3.28 the resulting bimodule is just the submodule $\mathscr{A} \cdot {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \subseteq {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}$ up to the usual isomorphism. Since we are interested in the kernels of *-representations we want to relate the kernel of the original *-representation to the kernel of this strongly non-degenerate one. In general, one can not say much, but under mild assumptions the kernels coincide:

Proposition 5.3.10 *Let $\mathscr{A}$ and $\mathscr{D}$ be *-algebras and let $({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}, \pi)$ be a *-representation on an inner-product $(\mathscr{A}, \mathscr{D})$-bimodule and let $\pi(\mathscr{A})\mathscr{H} \subseteq {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}$ be the corresponding strongly non-degenerate *-representation.*

i.) If $\mathscr{A}$ has local Hermitian units then $\ker \pi = \ker \pi\big|_{\pi(\mathscr{A})\mathscr{H}}$.

ii.) If $\mathscr{H}$ is a pre-Hilbert bimodule and $\mathscr{D}$ is admissible, then $\ker \pi = \ker \pi\big|_{\pi(\mathscr{A})\mathscr{H}}$.

Proof: In both cases it is clear that $\ker \pi \subseteq \ker \pi\big|_{\pi(\mathscr{A})\mathscr{H}}$. Thus let $a \in \mathscr{A}$ satisfy $\pi(a) \sum_i \pi(b_i)\phi_i = 0$ for all $b_i \in \mathscr{A}$ and $\phi_i \in \mathscr{H}$. If we have local Hermitian units, we find an $\mathsf{e} \in \mathscr{A}$ with $a\mathsf{e} = a$. Thus we have $0 = \pi(a)\pi(\mathsf{e})\phi = \pi(a)\phi$ for all $\phi \in \mathscr{H}$ and thus $a \in \ker \pi$. This shows the first part. For the other situation, assume that $\mathscr{D}$ is admissible and $\mathscr{H}$ is a pre-Hilbert module over $\mathscr{D}$. For $a \in \ker \pi\big|_{\pi(\mathscr{A})\mathscr{H}}$ we have in particular $\pi(a)\pi(a^*)\phi = 0$ for all $\phi \in \mathscr{H}$. Thus $0 = \langle \phi, \pi(a)\pi(a^*)\phi \rangle_{\mathscr{D}} = \langle \pi(a^*)\phi, \pi(a^*)\phi \rangle_{\mathscr{D}}$. Since $\langle \cdot\,, \cdot \rangle_{\mathscr{D}}$ is positive definite by assumption, we conclude $\pi(a^*)\phi = 0$ for all $\phi \in \mathscr{H}$. Since π is a *-representation, this implies $a \in \ker \pi$. □

In view of this proposition, the following choice for "closed" ideals seems to be very reasonable. We state the definition in two flavours: first without positivity requirement and, second, taking into account the positivity as usual.

Definition 5.3.11 ($\mathscr{D}$-Closed *-ideals) *Let $\mathscr{A}$ and $\mathscr{D}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$.*

i.) *A *-ideal $\mathscr{J} \subseteq \mathscr{A}$ is called $\mathscr{D}$-closed if there is a strongly non-degenerate *-representation ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ of $\mathscr{A}$ on an inner-product right $\mathscr{D}$-module with kernel given by $\mathscr{J}$. The set of these *-ideals is denoted by $\mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$.*

ii.) *A *-ideal $\mathscr{J} \subseteq \mathscr{A}$ is called strongly $\mathscr{D}$-closed if there is a strongly non-degenerate *-representation ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})$ of $\mathscr{A}$ on a pre-Hilbert right $\mathscr{D}$-module with kernel given by $\mathscr{J}$. The set of these *-ideals is denoted by $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$.*

Note that the name *closed* is meant only in analogy to the C^*-algebraic case, we do not have any sort of topology around to justify closedness in a topological sense. Note also that we differ slightly from the definition in [25] where the assumption about strong non-degeneracy of the *-representation was *not* included into the definition. In view of Proposition 5.3.10 this will only make a difference for *-algebras having somewhat pathological features anyway.

Remark 5.3.12 Also in the ring-theoretic situation one can define a $\mathscr{D}$-closed ideal to be the kernel of a strongly non-degenerate representation of $\mathscr{A}$ on a right $\mathscr{D}$-module. However, at least for the case where C is a field, any ideal will be $\mathscr{D}$-closed: indeed, let $\mathscr{J} \subseteq \mathscr{A}$ be a two-sided ideal in a unital algebra $\mathscr{A}$ over C. Then the algebra $\mathscr{A}/\mathscr{J}$ is faithfully represented on itself via left multiplications as we assume that $\mathscr{A}$ and hence also $\mathscr{A}/\mathscr{J}$ are unital. Viewed as a right $\mathscr{A}$-module, the kernel of this representation is just $\mathscr{J}$. Now if $\mathscr{D}$ is any other coefficient algebra, then on $\mathscr{A}/\mathscr{J} \otimes \mathscr{D}$ we have a $(\mathscr{A}, \mathscr{D})$-bimodule structure and the kernel of the left $\mathscr{A}$-module structure is still just $\mathscr{J}$, unless we have some torsion effects in the tensor product spoiling this. In particular, if C is a field, then this will not happen. From this point of view the notion of $\mathscr{D}$-closed ideals will be non-interesting. However, requiring that on the module one has in addition an algebra-valued inner product gives some new and non-trivial extra conditions, see also Example 5.3.20.

A first interpretation of the $\mathscr{D}$-closed *-ideals (in both cases) is obtained by the following proposition. Note again that the strong non-degeneracy does not provide an extra condition in many cases as described in Proposition 5.3.10.

Proposition 5.3.13 *Let $\mathscr{A}$ and $\mathscr{D}$ be *-algebras over C and let $\mathscr{J} \subseteq \mathscr{A}$ be a *-ideal.*

i.) *The quotient *-algebra $\mathscr{A}/\mathscr{J}$ has a faithful and strongly non-degenerate *-representation on an inner product right $\mathscr{D}$-module iff $\mathscr{J} \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$.*

ii.) *The quotient *-algebra $\mathscr{A}/\mathscr{J}$ has a faithful and strongly non-degenerate *-representation on a pre-Hilbert right $\mathscr{D}$-module iff $\mathscr{J} \in \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$.*

Proof: This is clear as the kernel of the quotient map $\mathscr{A} \longrightarrow \mathscr{A}/\mathscr{J}$ is precisely $\mathscr{J}$. □

Proposition 5.3.14 *Let $\mathscr{A}$ be a *-algebra over C. Moreover, let I be an index set, let $\mathscr{J}_\alpha \subseteq \mathscr{A}$ be *-ideals for $\alpha \in I$, and let $\mathscr{J} = \bigcap_{\alpha \in I} \mathscr{J}_\alpha$.*

i.) *If $\mathscr{J}_\alpha \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$ for all $\alpha \in I$ then also $\mathscr{J} \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$.*

ii.) *If $\mathscr{J}_\alpha \in \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ for all $\alpha \in I$ then also $\mathscr{J} \in \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$.*

Proof: This is clear as in general

$$\ker\left(\bigoplus_{\alpha \in I} (\mathscr{H}_\alpha, \pi_\alpha)\right) = \bigcap_{\alpha \in I} \ker(\mathscr{H}_\alpha, \pi_\alpha). \qquad \square$$

In both cases, the set of $\mathscr{D}$-closed *-ideals $\mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$ or $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ of $\mathscr{A}$ carries a rich structure, namely it forms a lattice. We recall the definition:

Definition 5.3.15 (Lattice) *A lattice is a set $\mathfrak{L}$ endowed with two compositions*

$$\wedge, \vee \colon \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L} \tag{5.3.20}$$

such that

i.) *the compositions $\wedge$ and $\vee$ are associative and commutative,*

ii.) *for all $a \in \mathfrak{L}$ one has $a \wedge a = a = a \vee a$,*

iii.) *for all $a, b \in \mathfrak{L}$ one has $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.*

A homomorphism $\phi\colon (\mathfrak{L}, \wedge, \vee) \longrightarrow (\mathfrak{L}', \wedge', \vee')$ of lattices is a map with $\phi(a \wedge b) = \phi(a) \wedge' \phi(b)$ and $\phi(a \vee b) = \phi(a) \vee' \phi(b)$ for all $a, b \in \mathfrak{L}$. The category of lattices is denoted by Lattice.

Remark 5.3.16 (Lattices)

i.) Needless to say, lattices indeed form a category with respect to the above notion of homomorphisms.

ii.) A first example of a lattice is the power set $\mathfrak{L} = 2^M$ of a set M. Indeed, for subsets $U, V \subseteq M$ one defines
$$U \wedge V = U \cap V \quad \text{and} \quad U \vee V = U \cup V. \tag{5.3.21}$$
The verification of the properties of a lattice is trivial in this case. Nevertheless, this will be a guiding example.

iii.) For $a, b \in \mathfrak{L}$ one has
$$a \wedge b = a \quad \text{iff} \quad a \vee b = b. \tag{5.3.22}$$
In this case we write $a \leq b$. It is easy to see that $\leq$ defines a half-ordering. The lattice is called to have a *maximal element* $1 \in \mathfrak{L}$ if for all $a \in \mathfrak{L}$
$$a \leq 1. \tag{5.3.23}$$
If existing, the maximal element is unique. Analogously, one defines a *minimal element* 0 by requiring $0 \leq a$ for all $a \in \mathfrak{L}$.

iv.) Conversely, if $(\mathfrak{L}, \leq)$ is a set with half-ordering with the additional property that for two elements $a, b \in \mathfrak{L}$ there always exists a supremum $\sup(a, b)$ and an infimum $\inf(a, b)$ then $\mathfrak{L}$ becomes a lattice via
$$a \wedge b = \inf(a, b) \quad \text{and} \quad a \vee b = \sup(a, b). \tag{5.3.24}$$

We shall now prove that the $\mathscr{D}$-closed *-ideals of $\mathscr{A}$ form a lattice for both versions. In order to define the lattice operations we first introduce a *closure operation* for both cases. Let $\mathscr{I} \subseteq \mathscr{A}$ be an arbitrary subset. Then the **-closure* (with respect to the coefficient algebra $\mathscr{D}$) is defined by

$$\mathscr{I}^{*\text{-cl}} = \bigcap_{\mathscr{J} \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A}) \text{ with } \mathscr{I} \subseteq \mathscr{J}} \mathscr{J}, \tag{5.3.25}$$

while the *strong closure* (with respect to the coefficient algebra $\mathscr{D}$) is defined by

$$\mathscr{I}^{\text{strcl}} = \bigcap_{\mathscr{J} \in \mathsf{Ideals}^{\text{str}}_{\mathscr{D}}(\mathscr{A}) \text{ with } \mathscr{I} \subseteq \mathscr{J}} \mathscr{J}. \tag{5.3.26}$$

By Proposition 5.3.14, both operations yield $\mathscr{D}$-closed *-ideals of the desired type, i.e. $\mathscr{I}^{*\text{-cl}} \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$ and $\mathscr{I}^{\text{strcl}} \in \mathsf{Ideals}^{\text{str}}_{\mathscr{D}}(\mathscr{A})$, respectively. Note that, by some mild abuse of notation, we do not indicate the coefficient algebra in the notation of the closure operations.

Lemma 5.3.17 *For arbitrary subsets* $\mathscr{I}, \mathscr{J} \subseteq \mathscr{A}$ *we have*

i.) $\mathscr{I} \subseteq \mathscr{I}^{\mathrm{cl}}$,

ii.) $\mathscr{I} \subseteq \mathscr{J}$ *implies* $\mathscr{I}^{\mathrm{cl}} \subseteq \mathscr{J}^{\mathrm{cl}}$,

iii.) $(\mathscr{I}^{\mathrm{cl}})^{\mathrm{cl}} = \mathscr{I}^{\mathrm{cl}}$,

where $^{\mathrm{cl}}$ *is either* $^{*\text{-cl}}$ *or* $^{\mathrm{strcl}}$.

The proof of these facts is obvious. Nevertheless, these simple properties guarantee that we obtain a lattice:

Proposition 5.3.18 *Let* $\mathscr{D}$ *be a* **-algebra. Then the* $\mathscr{D}$*-closed* **-ideals* $\mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$ *of* $\mathscr{A}$ *form a lattice via*

$$\mathscr{J} \wedge \mathscr{I} = \mathscr{J} \cap \mathscr{I} \quad \textit{and} \quad \mathscr{J} \vee \mathscr{I} = (\mathscr{J} \cup \mathscr{I})^{*\text{-cl}}, \tag{5.3.27}$$

with maximal element $\mathscr{A}$ *and a minimal element denoted by*

$$\mathscr{J}^*_{\min,\mathscr{D}}(\mathscr{A}) = \{0\}^{*\text{-cl}} = \bigcap_{\mathscr{J} \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})} \mathscr{J}. \tag{5.3.28}$$

We have

$$\mathscr{J} \leq \mathscr{I} \quad \textit{iff} \quad \mathscr{J} \subseteq \mathscr{I}. \tag{5.3.29}$$

Analogously, $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ *forms a lattice, too, with minimal element* $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{A}) = \{0\}^{\mathrm{strcl}}$.

PROOF: In fact, the above construction is much more general and relies on the fact that we have a closure operation $^{\mathrm{cl}}$ with the properties as described in Lemma 5.3.17. Though this is a folklore construction of a lattice out of a closure operation we outline the proof. First it is clear that $\wedge$ is associative and commutative. The commutativity of $\vee$ is clear, too. To show the associativity of $\vee$ we choose $\mathscr{I}, \mathscr{J}, \mathscr{K} \in \mathsf{Ideals}^*_{\mathscr{D}}(\mathscr{A})$. We have $\mathscr{K}, \mathscr{I} \cup \mathscr{J} \subseteq \mathscr{I} \cup \mathscr{J} \cup \mathscr{K}$ and hence $\mathscr{K} = \mathscr{K}^{*\text{-cl}} \subseteq (\mathscr{I} \cup \mathscr{J} \cup \mathscr{K})^{*\text{-cl}}$ as well as $\mathscr{I} \vee \mathscr{J} = (\mathscr{I} \cup \mathscr{J})^{*\text{-cl}} \subseteq (\mathscr{I} \cup \mathscr{J} \cup \mathscr{K})^{*\text{-cl}}$. Thus we have $(\mathscr{I} \vee \mathscr{J}) \vee \mathscr{K} \subseteq (\mathscr{I} \cup \mathscr{J} \cup \mathscr{K})^{*\text{-cl}}$. On the other hand, we have $\mathscr{I}, \mathscr{J}, \mathscr{K} \subseteq (\mathscr{I} \vee \mathscr{J}) \vee \mathscr{K}$ and hence $(\mathscr{I} \cup \mathscr{J} \cup \mathscr{K}) \subseteq (\mathscr{I} \vee \mathscr{J}) \vee \mathscr{K}$ implying $(\mathscr{I} \cup \mathscr{J} \cup \mathscr{K}) \subseteq (\mathscr{I} \vee \mathscr{J}) \vee \mathscr{K}$ and hence $(\mathscr{I} \cup \mathscr{J}) \cup \mathscr{K})^{*\text{-cl}} \subseteq (\mathscr{I} \vee \mathscr{J}) \vee \mathscr{K}$. This shows

$$(\mathscr{I} \vee \mathscr{J}) \vee \mathscr{K} = (\mathscr{I} \cup \mathscr{J} \cup \mathscr{K})^{*\text{-cl}}.$$

Analogously one finds $\mathscr{I} \vee (\mathscr{J} \vee \mathscr{K}) = (\mathscr{I} \cup \mathscr{J} \cup \mathscr{K})^{*\text{-cl}}$ which shows associativity. The properties $\mathscr{J} \wedge \mathscr{J} = \mathscr{J} \cap \mathscr{J} = \mathscr{J}$ and $\mathscr{J} \vee \mathscr{J} = (\mathscr{J} \cup \mathscr{J})^{*\text{-cl}} = \mathscr{J}^{*\text{-cl}} = \mathscr{J}$ are fulfilled, too. Now $\mathscr{J} = \mathscr{J} \cup (\mathscr{J} \cap \mathscr{I})$ implies $\mathscr{J} = \mathscr{J}^{*\text{-cl}} = (\mathscr{J} \cup (\mathscr{J} \cap \mathscr{I}))^{*\text{-cl}} = \mathscr{J} \vee (\mathscr{J} \wedge \mathscr{I})$. Finally, we have $\mathscr{J} \cap (\mathscr{J} \vee \mathscr{I}) \subseteq \mathscr{J}$ and since $\mathscr{J} \vee \mathscr{I}$ contains $\mathscr{J}$ we have $\mathscr{J} = \mathscr{J} \cap (\mathscr{J} \cup \mathscr{I}) \subseteq \mathscr{J} \cap (\mathscr{J} \vee \mathscr{I})$. From this we see the last requirement for a lattice. The claim (5.3.29) follows at once since $\wedge$ coincides with the set-theoretic $\cap$. Finally, $\{0\}$ is contained in every *-ideal and hence in every $\mathscr{D}$-closed *-ideal. Thus $\{0\}^{*\text{-cl}}$ is a $\mathscr{D}$-closed *-ideal, still contained in every other $\mathscr{D}$-closed *-ideal. This shows that $\{0\}^{*\text{-cl}}$ is the minimal element of the lattice $\mathsf{Ideals}_{\mathscr{D}}(\mathscr{A})$. Finally, $\mathscr{A}$ is always a $\mathscr{D}$-closed *-ideal since it is the kernel of the *-representation on the 0-module. The case of strongly $\mathscr{D}$-closed *-ideals is treated analogously. □

Remark 5.3.19 (Minimal $\mathscr{D}$-closed *-ideal) The minimal $\mathscr{D}$-closed *-ideal $\mathscr{J}^*_{\min,\mathscr{D}}(\mathscr{A})$ of a *-algebra $\mathscr{A}$ plays an important role: it encodes whether or not there is a faithful strongly non-degenerate *-representation of $\mathscr{A}$ on a inner-product module $\mathscr{H}_{\mathscr{D}}$. Analogously, the minimal strongly $\mathscr{D}$-closed *-ideal $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{A})$ encodes whether we have a faithful and strongly non-degenerate *-representation on a pre-Hilbert module over $\mathscr{D}$. In particular, the sizes of $\mathscr{J}^*_{\min,\mathscr{D}}(\mathscr{A})$ as well as $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{A})$ strongly depend on the coefficient *-algebra $\mathscr{D}$. The scalar case, i.e. $\mathscr{D} = \mathsf{C}$, was extensively discussed in Exercise 1.4.15 based on [25]. In general, we have

$$\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{A}}(\mathscr{A}) = \{0\}, \tag{5.3.30}$$

since ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ is always a faithful strongly non-degenerate *-representation on the canonical pre-Hilbert $\mathscr{A}$-module. Here one uses that $\mathscr{A}$ is non-degenerate and idempotent.

Example 5.3.20 Let $\mathscr{A} = \Lambda^\bullet \mathsf{C}^n$ be the Grassmann algebra and $\mathscr{D} = \mathsf{C}$. If $\mathscr{H}$ is a pre-Hilbert space and $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathscr{H})$ a *-representation then we have for the base vectors $e_i \in \mathsf{C}^n \subseteq \Lambda^\bullet \mathsf{C}^n$ obviously $e_i^* e_i = e_i e_i = 0$ and hence $\langle \pi(e_i)\phi, \pi(e_i)\phi \rangle = \langle \phi, \pi(e_i^* e_i)\phi \rangle = 0$. Thus necessarily $\pi(e_i) = 0$. But together with the unit element $\mathbb{1}$ the e_i generate $\Lambda^\bullet \mathsf{C}^n$. It follows that

$$\bigoplus_{k \geq 1} \Lambda^k \mathsf{C}^n \subseteq \ker \pi. \tag{5.3.31}$$

On the other hand, there do exist *-representations of the Grassmann algebra on pre-Hilbert spaces with $\pi(\mathbb{1}) = \mathrm{id}$ and hence we conclude that

$$\mathscr{J}^{\mathrm{str}}_{\mathrm{min}, \mathsf{C}}(\Lambda^\bullet \mathsf{C}^n) = \bigoplus_{k \geq 1} \Lambda^k \mathsf{C}^n. \tag{5.3.32}$$

In view of (5.3.30), this example shows that the minimal strongly $\mathscr{D}$-closed *-ideal can depend very much on the coefficient *-algebra.

Remark 5.3.21 Since every strongly non-degenerate *-representation on a pre-Hilbert module is in particular a *-representation on an inner-product module, a strongly $\mathscr{D}$-closed *-ideal is also $\mathscr{D}$-closed. This shows that

$$\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A}) \subseteq \mathsf{Ideals}^{*}_{\mathscr{D}}(\mathscr{A}). \tag{5.3.33}$$

However, the inclusion map needs not to be a lattice homomorphism as the closure operation ${}^{*\text{-cl}}$ in $\mathsf{Ideals}^{*}_{\mathscr{D}}(\mathscr{A})$ may yield strictly smaller closures than the strong closure ${}^{\mathrm{strcl}}$ in $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$. Thus we will not have compatibility with respect to the two different $\vee$-products.

The next theorem shows that the whole lattice of (strongly) $\mathscr{D}$-closed *-ideals $\mathsf{Ideals}^{*}_{\mathscr{D}}(\mathscr{A})$ and $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ is in fact a *-Morita (strong Morita) invariant, respectively. In order to prove this we define for a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ the map $\Phi_{\mathscr{E}}\colon 2^{\mathscr{A}} \longrightarrow 2^{\mathscr{B}}$ by

$$\Phi_{\mathscr{E}}(\mathscr{J}) = \left\{ b \in \mathscr{B} \;\middle|\; \langle x, b \cdot y \rangle^{\mathscr{E}}_{\mathscr{A}} \in \mathscr{J} \text{ for all } x, y \in \mathscr{E} \right\}. \tag{5.3.34}$$

This map turns out to implement the action of Pic^* and $\mathsf{Pic}^{\mathrm{str}}$, respectively, yielding the *-Morita invariance of $\mathsf{Ideals}^{*}_{\mathscr{D}}(\mathscr{A})$ and the strong Morita invariance of $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$, respectively.

Theorem 5.3.22 (Morita invariance of $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}$) *Let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be strong equivalence bimodules.*

i.) If $(\mathscr{H}, \pi) \in {}^*\text{-}\mathsf{rep}_{\mathscr{D}}(\mathscr{A})$ then

$$\Phi_{\mathscr{E}}(\ker \pi) = \ker \mathsf{R}_{\mathscr{E}}(\mathscr{H}, \pi), \tag{5.3.35}$$

where $\mathsf{R}_{\mathscr{E}}$ is the Rieffel induction with ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$.

ii.) The map $\Phi_{\mathscr{E}}\colon \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A}) \longrightarrow \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{B})$ depends on the isometric isomorphism class $[\mathscr{E}] \in \mathsf{Pic}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$ only.

iii.) For all C-submodules $\mathscr{J} \subseteq \mathscr{A}$ we have

$$\Phi_{\mathscr{F}}(\Phi_{\mathscr{E}}(\mathscr{J})) = \Phi_{\mathscr{F} \widetilde{\otimes} \mathscr{E}}(\mathscr{J}). \tag{5.3.36}$$

iv.) For all $\mathscr{J} \in \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ one has

$$\Phi_{\mathscr{A}}(\mathscr{J}) = \mathscr{J}. \tag{5.3.37}$$

v.) *The definition* $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}\colon \mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Lattice}$*, with* $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ *as before and*

$$\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]) = \big(\Phi_{\mathscr{E}}\colon \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A}) \longrightarrow \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{B})\big) \tag{5.3.38}$$

on morphisms, yields a left action of $\mathsf{Pic}^{\mathrm{str}}$ *on* $\mathsf{Lattice}$*.*

vi.) *One has* $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{A}) = \{0\}$ *if and only if* $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{B}) = \{0\}$*.*

Analogously, using **-equivalence bimodules instead, we obtain a left action*

$$\mathsf{Ideals}^{*}_{\mathscr{D}}\colon \mathsf{Pic}^{*} \longrightarrow \mathsf{Lattice}. \tag{5.3.39}$$

PROOF: Let $b \in \Phi_{\mathscr{E}}(\ker \pi)$ then $\langle x, b \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}} \in \ker \pi$ for all $x, y \in \mathscr{E}$. But then we have

$$0 = \big\langle \psi, \pi\big(\langle x, b \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}}\big)\phi\big\rangle^{\mathscr{H}}_{\mathscr{D}} = \big\langle x \otimes \psi, b \cdot (y \otimes \phi)\big\rangle^{\mathscr{E}\widehat{\otimes}\mathscr{H}}_{\mathscr{D}} = \big\langle x \otimes \psi, (\mathsf{R}_{\mathscr{E}}(\pi))(b)y \otimes \phi\big\rangle^{\mathsf{R}_{\mathscr{E}}(\mathscr{H})}_{\mathscr{D}}.$$

It follows that $(\mathsf{R}_{\mathscr{E}}(\pi))(b) = 0$ and hence b is in the kernel of the Rieffel induced representation. Conversely, let $b \in \ker \mathsf{R}_{\mathscr{E}}(\pi)$ then we have for all $\phi, \psi \in \mathscr{H}_{\mathscr{D}}$ and $x, y \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$

$$0 = \big\langle \psi, \pi\big(\langle x, b \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}}\big)\phi\big\rangle^{\mathscr{H}}_{\mathscr{D}}.$$

Hence $\langle x, b \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}} \in \ker \pi$ follows. This shows the first part. In particular, we see that $\Phi_{\mathscr{E}}$ maps strongly $\mathscr{D}$-closed *-ideals of $\mathscr{A}$ to strongly $\mathscr{D}$-closed *-ideals of $\mathscr{B}$ since the Rieffel induction with an equivalence bimodule maps strongly non-degenerate *-representations to strongly non-degenerate ones. Thus we obtain a map $\Phi_{\mathscr{E}}\colon \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A}) \longrightarrow \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{B})$. Now let $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ be an isometric isomorphism and let $b \in \Phi_{\mathscr{E}}(\mathscr{J})$. Then for all $x', y' \in \mathscr{E}'$ we have

$$\big\langle x', b \cdot y'\big\rangle^{\mathscr{E}'}_{\mathscr{A}} = \big\langle T^{-1}(x'), T^{-1}(b \cdot y')\big\rangle^{\mathscr{E}}_{\mathscr{A}} = \big\langle T^{-1}(x'), b \cdot T^{-1}(y')\big\rangle^{\mathscr{E}}_{\mathscr{A}} \in \mathscr{J},$$

showing $b \in \Phi_{\mathscr{E}'}(\mathscr{J})$. The reverse implication follows by symmetry proving the second part. For the third part we consider $c \in \mathscr{C}$ together with $x, y \in \mathscr{E}$ and $\phi, \psi \in \mathscr{F}$. Then

$$\big\langle \phi \otimes x, c \cdot (\psi \otimes y)\big\rangle^{\mathscr{F}\otimes\mathscr{E}}_{\mathscr{A}} = \big\langle x, \langle \phi, c \cdot \psi\rangle^{\mathscr{F}}_{\mathscr{B}} \cdot y\big\rangle^{\mathscr{E}}_{\mathscr{A}} \tag{$*$}$$

as usual. If $c \in \Phi_{\mathscr{F}\widetilde{\otimes}\mathscr{E}}(\mathscr{J})$ then the left hand side of $(*)$ is in $\mathscr{J}$. Thus $\langle \phi, c \cdot \psi\rangle^{\mathscr{F}}_{\mathscr{B}} \in \Phi_{\mathscr{E}}(\mathscr{J})$ for all ϕ, ψ. This shows $c \in \Phi_{\mathscr{F}}(\Phi_{\mathscr{E}}(\mathscr{J}))$. Conversely, if $c \in \Phi_{\mathscr{F}}(\Phi_{\mathscr{E}}(\mathscr{J}))$ then the right hand side of $(*)$ is in $\mathscr{J}$. Since the elementary tensors span $\mathscr{F} \mathbin{\widetilde{\otimes}} \mathscr{E}$ we conclude $c \in \Phi_{\mathscr{F}\widetilde{\otimes}\mathscr{E}}(\mathscr{J})$. This completes the third part. Now let $\mathscr{J} = \ker \pi$ with a strongly non-degenerate *-representation $(\mathscr{H}, \pi) \in {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})$. By *i.)* we have $\Phi_{\mathscr{A}}(\mathscr{J}) = \ker \mathsf{R}_{\mathscr{A}}(\mathscr{H}, \pi)$. Now $\mathscr{A} \mathbin{\widehat{\otimes}_{\mathscr{A}}} \mathscr{H}$ is unitarily equivalent to $\mathscr{H}$ again thanks to the assumption that $\mathscr{H}$ is a strongly non-degenerate *-representation of $\mathscr{A}$, see Lemma 4.3.15. This implies the fourth part. For the fifth part we already know that $\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}([{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]) = \mathrm{id}$ and

$$\mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}([{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}]) \circ \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]) = \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}\big([{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \mathbin{\widetilde{\otimes}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]\big).$$

It remains to show that $\Phi_{\mathscr{E}}$ is indeed a morphism in the category $\mathsf{Lattice}$. To this end, let $\mathscr{I}, \mathscr{J} \in \mathsf{Ideals}^{\mathrm{str}}_{\mathscr{D}}(\mathscr{A})$ be given. Then

$$\Phi_{\mathscr{E}}(\mathscr{J} \wedge \mathscr{I}) = \Phi_{\mathscr{E}}(\mathscr{J} \cap \mathscr{I}) = \Phi_{\mathscr{E}}(\mathscr{J}) \cap \Phi_{\mathscr{E}}(\mathscr{I}) = \Phi_{\mathscr{E}}(\mathscr{J}) \wedge \Phi_{\mathscr{E}}(\mathscr{I})$$

is obvious. Moreover, it is clear that

$$\Phi_{\mathscr{E}}(\mathscr{J}) \cup \Phi_{\mathscr{E}}(\mathscr{I}) = \Phi_{\mathscr{E}}(\mathscr{J} \cup \mathscr{I}) \subseteq \Phi_{\mathscr{E}}(\mathscr{J} \vee \mathscr{I}).$$

Since $\Phi_{\mathscr{E}}(\mathscr{J} \vee \mathscr{I})$ is already closed we find

$$\Phi_{\mathscr{E}}(\mathscr{J}) \vee \Phi_{\mathscr{E}}(\mathscr{I}) = \Phi_{\mathscr{E}}(\mathscr{J} \cup \mathscr{I})^{\mathrm{cl}} \subseteq \Phi_{\mathscr{E}}(\mathscr{J} \vee \mathscr{I}).$$

Applying $\Phi_{\overline{\mathscr{E}}} = \Phi_{\mathscr{E}}^{-1}$ yields

$$\Phi_{\overline{\mathscr{E}}}\Big(\Phi_{\mathscr{E}}(\mathscr{J} \cup \mathscr{I})^{\mathrm{cl}}\Big) \subseteq \Phi_{\overline{\mathscr{E}}}\Phi_{\mathscr{E}}(\mathscr{J} \vee \mathscr{I}) = \mathscr{J} \vee \mathscr{I}$$

Now $\mathscr{J}$ and $\mathscr{I}$ are contained in the left hand side and the left hand side is a strongly $\mathscr{D}$-closed *-ideal. But $\mathscr{J} \vee \mathscr{I}$ is the *smallest* strongly $\mathscr{D}$-closed *-ideal containing $\mathscr{J}$ as well as $\mathscr{I}$ and hence both sides actually coincide. Thus we conclude

$$\Phi_{\overline{\mathscr{E}}}(\Phi_{\mathscr{E}}(\mathscr{J}) \vee \Phi_{\mathscr{E}}(\mathscr{I})) = \mathscr{J} \vee \mathscr{I},$$

from which the lattice homomorphism property follows at once. This shows the fifth part. Note that from the fifth part we deduce that the minimal ideal is mapped to the minimal ideal. Hence the last part sharpens this statement: Assume $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{A}) = \{0\}$. We choose a faithful strongly non-degenerate *-representation $(\mathscr{H}, \pi) \in {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})$. If now $b \in \mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{B}) = \Phi_{\mathscr{E}}(\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{A}))$ then we have $\langle x, b \cdot y\rangle^{\mathscr{E}}_{\mathscr{A}} \in \ker\pi = \{0\}$ for all $x, y \in {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. Since ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a strong equivalence bimodule this implies $b = 0$ and thus $\mathscr{J}^{\mathrm{str}}_{\min,\mathscr{D}}(\mathscr{B}) = \{0\}$, too. The other implication follows by symmetry. ☐

5.3.5 The Representation Theories

We come now to the last Morita invariant which was essentially the main motivation to develop the tools of equivalence bimodules after all: the representation theories $^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})$. In the usual approach to (strong) Morita theory one shows that the categories of *-representations or modules, respectively, are equivalent categories. The equivalence is implemented by the Rieffel induction functors. Instead of showing this directly, which would be easy with the present tools at hand, we formulate the equivalence in a slightly more sophisticated version in order to prepare the ground for a Picard groupoid action. The following theorem is shown analogously to Proposition 3.1.15.

Theorem 5.3.23 (Morita invariance of representation theories) *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be *-algebras and let ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{B}}(\mathscr{C})$ as well as ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B})$ be inner-product modules. Moreover, let $\mathscr{D}$ be an auxiliary *-algebra.*

i.) The associativity of $\widehat{\otimes}$ yields a natural unitary isomorphism

$$\mathsf{asso}\colon \mathsf{R}_{\mathscr{F}\widehat{\otimes}\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{F}} \circ \mathsf{R}_{\mathscr{E}} \tag{5.3.40}$$

between the Rieffel induction functors $\mathsf{R}_{\mathscr{F}\widehat{\otimes}\mathscr{E}}, \mathsf{R}_{\mathscr{F}} \circ \mathsf{R}_{\mathscr{E}}\colon {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{C})$.

ii.) Restricted to $^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ the left identity yields a natural unitary isomorphism

$$\mathsf{left}\colon \mathsf{R}_{\mathscr{A}} \longrightarrow \mathrm{id}_{^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})}\,. \tag{5.3.41}$$

iii.) If $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ is an intertwiner then one has a natural transformation

$$I_T\colon \mathsf{R}_{\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{E}'} \tag{5.3.42}$$

with

$$I_T^* = I_{T^*}, \tag{5.3.43}$$

defined for ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A})$ by

$$I_T({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}\,. \tag{5.3.44}$$

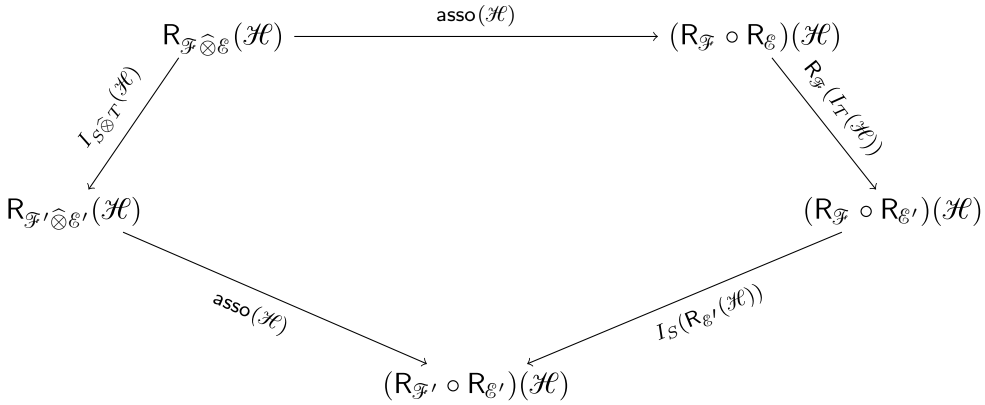
iv.) For intertwiners $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ and $T'\colon {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}''_{\mathscr{A}}$ one has

$$I_{T'}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) \circ I_T({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = I_{T' \circ T}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) \tag{5.3.45}$$

for every ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A})$ as well as

$$I_{\mathrm{id}_{\mathscr{E}}}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = \mathrm{id}_{\mathscr{E}\widehat{\otimes}\mathscr{H}}\,. \tag{5.3.46}$$

v.) For intertwiners $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ *and* $S\colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \longrightarrow {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}}$ *the diagram*



(5.3.47)

commutes for all ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A})$*.*

Proof: Let ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{mod}_{\mathscr{D}}(\mathscr{A})$ be given and define $\mathsf{asso}(\mathscr{H})$ to be the usual isometric isomorphism

$$\mathsf{asso}(\mathscr{H})\colon (\mathscr{F} \mathbin{\widehat{\otimes}} \mathscr{E}) \mathbin{\widehat{\otimes}} \mathscr{H} \longrightarrow \mathscr{F} \mathbin{\widehat{\otimes}} (\mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{H}).$$

The naturalness of the associativity asso from Lemma 4.3.14 in all three arguments implies the naturalness in the third argument. Note that $\mathsf{asso}(\mathscr{H})$ is unitary for all $\mathscr{H}$. This shows the first part. If now ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ is strongly non-degenerate then

$$\mathsf{left}(\mathscr{H})\colon \mathsf{R}_{\mathscr{A}}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) = \mathscr{A} \mathbin{\widehat{\otimes}} {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \longrightarrow {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}$$

defined by $\mathsf{left}(\mathscr{H})\colon a \otimes \phi \mapsto a \cdot \phi$ yields an isometric isomorphism which is natural. The necessary computations are completely analogous to the one for Lemma 4.3.15. Again, $\mathsf{left}(\mathscr{H})$ is unitary for all $\mathscr{H}$. That $I_T(\mathscr{H})$ is natural also follows from the functoriality of the $\widehat{\otimes}$-tensor product: for an intertwiner $U\colon {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \longrightarrow {}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}}$ we have

$$I_T({}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}}) \circ \mathsf{R}_{\mathscr{E}}(U) = (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}'}) \circ (\mathrm{id}_{\mathscr{E}} \mathbin{\widehat{\otimes}} U) = T \mathbin{\widehat{\otimes}} U = (\mathrm{id}_{\mathscr{E}'} \mathbin{\widehat{\otimes}} U) \circ (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}) = \mathsf{R}_{\mathscr{E}'}(U) \circ I_T({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}).$$

This shows that I_T is natural. Clearly, for all $\mathscr{H}$ we have

$$I_T(\mathscr{H})^* = (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}})^* = T^* \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}} = I_{T^*}(\mathscr{H}).$$

For the fourth part we take intertwiners $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ and $T'\colon {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}''_{\mathscr{A}}$. Then we have

$$I_{T'}(\mathscr{H}) \circ I_T(\mathscr{H}) = (T' \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}) \circ (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}) = (T' \circ T) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}} = I_{T' \circ T}(\mathscr{H}),$$

which shows (5.3.45). The statement (5.3.46) is obvious. Finally, let $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ and $S\colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \longrightarrow {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}}$ be intertwiners. Then we have for all $y \in \mathscr{F}$, $x \in \mathscr{E}$ and $\phi \in \mathscr{H}$

$$\begin{aligned}
&(I_S(\mathsf{R}_{\mathscr{E}'}(\mathscr{H})) \circ \mathsf{R}_{\mathscr{F}}(I_T(\mathscr{H})) \circ \mathsf{asso}(\mathscr{H}))((y \otimes x) \otimes \phi) \\
&= \Big(\Big(S \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathsf{R}_{\mathscr{E}'}(\mathscr{H})}\Big) \circ \big(\mathrm{id}_{\mathscr{F}} \mathbin{\widehat{\otimes}} I_T(\mathscr{H})\big)\Big)(y \otimes (x \otimes \phi)) \\
&= \Big(S \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathsf{R}_{\mathscr{E}'}(\mathscr{H})}\Big)(y \otimes (T(x) \otimes \phi)) \\
&= S(y) \otimes (T(x) \otimes \phi).
\end{aligned}$$

For the other direction in the diagram we obtain

$$\Big(\mathsf{asso}(\mathscr{H}) \circ \big(I_{S\widehat{\otimes}T}(\mathscr{H})\big)\Big)((y \otimes x) \otimes \phi) = \mathsf{asso}(\mathscr{H})((S(y) \otimes T(x)) \otimes \phi) = S(y) \otimes (T(x) \otimes \phi),$$

and hence both sides agree. Since it is sufficient to consider elementary tensors also the fifth part is shown. ☐

Corollary 5.3.24 (Morita invariance of representation theories) *Let $\mathscr{A}$ and $\mathscr{B}$ be idempotent and non-degenerate and let $\mathscr{D}$ be an arbitrary *-algebra over C.*

i.) If $\mathscr{A}$ and $\mathscr{B}$ are *-Morita equivalent then the Rieffel induction with a *-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ yields a *-equivalence of *-categories

$$\mathsf{R}_{\mathscr{E}} \colon {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{B}). \tag{5.3.48}$$

ii.) If $\mathscr{A}$ and $\mathscr{B}$ are strongly Morita equivalent then the Rieffel induction with a strong equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ yields a *-equivalence of *-categories

$$\mathsf{R}_{\mathscr{E}} \colon {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{B}). \tag{5.3.49}$$

Proof: Let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Pic}^*(\mathscr{B}, \mathscr{A})$ and let ${}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}} \in \mathsf{Pic}^*(\mathscr{A}, \mathscr{B})$ be its complex conjugate. By Proposition 4.3.25 the bimodule ${}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$ is an inverse to ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ via the canonical unitary isomorphism

$$\phi_{\mathrm{can}} \colon {}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}} \mathbin{\widetilde{\otimes}_{\mathscr{B}}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}.$$

By Theorem 5.3.23 we obtain the following natural transformations

$$\begin{array}{ccc}
\mathsf{R}_{\overline{\mathscr{E}}\widetilde{\otimes}\mathscr{E}} & \xrightarrow{\ \mathsf{asso}\ } & \mathsf{R}_{\overline{\mathscr{E}}} \circ \mathsf{R}_{\mathscr{E}} \\
{\scriptstyle I_{\phi_{\mathrm{can}}}}\big\downarrow & & \vdots \\
\mathsf{R}_{\mathscr{A}} & \xrightarrow[\ \mathsf{left}\ ]{} & \mathrm{id}_{{}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})}
\end{array}$$

between *-functors on ${}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$. Since ϕ_{can} is a unitary isomorphism it follows from (5.3.44), (5.3.43), and (5.3.45) that $I_{\phi_{\mathrm{can}}}$ is a natural unitary isomorphism. Since asso and left are natural unitary isomorphisms as well, it follows that $\mathsf{R}_{\overline{\mathscr{E}}} \circ \mathsf{R}_{\mathscr{E}}$ is naturally unitarily isomorphic to the identity functor on ${}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$. Exchanging the role of $\mathscr{E}$ and $\overline{\mathscr{E}}$ we see that $\mathsf{R}_{\mathscr{E}} \circ \mathsf{R}_{\overline{\mathscr{E}}}$ is naturally unitarily isomorphic to the identity on ${}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{B})$. This shows the first part. The second part follows analogously since for a strong equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ also the complex conjugation ${}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$ is a strong equivalence bimodule and all operations from Theorem 5.3.23 preserve complete positivity. □

With this corollary we have finally answered the question asked in the introduction of this chapter: the equivalence of representation theories is encoded in Morita equivalence and the functors implementing the equivalence are the Rieffel induction functors. However, Theorem 5.3.23 gives a much more precise formulation of this Morita invariant. The Equations (5.3.40) and (5.3.41) can be interpreted *almost* like a left action of the Picard groupoid on the representation theories. However, in these equations, we do not get equality of the functors but only a natural unitary isomorphism between them. Thus, if we want to interpret the construction of Theorem 5.3.23 as an *action*, we have to enlarge our notion of groupoid actions to an action of the Picard *bigroupoids*.

To this end, we first introduce the bicategory of *all* representation theories ${}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\,\cdot\,)$ and ${}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\,\cdot\,)$, respectively, where $\mathscr{D}$ is a fixed *-algebra. In fact, this will even lead to a 2-category and not just to a bicategory. Moreover, we have even a *-bicategory:

Definition 5.3.25 (The *-bicategories ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$) *Let $\mathscr{D}$ be a *-algebra over C. Then we define the *-bicategories ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$ as follows:*

i.) The class of objects of ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$, respectively, are the non-degenerate and idempotent *-algebras over C.

ii.) *For two such* **-algebras* $\mathscr{A}$ *and* $\mathscr{B}$ *one defines the class of* 1*-morphisms* $1\text{-}\mathrm{Morph}(\mathscr{B}, \mathscr{A})$ *as the class of* **-functors* $\mathsf{R}\colon {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{B})$ *and* $\mathsf{R}\colon {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{B})$*, respectively. The class of* 2*-morphisms* $2\text{-}\mathrm{Morph}(\mathsf{R}', \mathsf{R})$ *is defined to be the class of natural transformations* $\eta\colon \mathsf{R} \longrightarrow \mathsf{R}'$*. The corresponding categories are denoted by* ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}(\mathscr{B}, \mathscr{A})$ *and* ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}(\mathscr{B}, \mathscr{A})$*.*

iii.) *The composition of* 1*-morphisms* $\otimes$ *is the usual composition of functors. The identity* 1*-morphisms are the* **-functors* $\mathsf{Id}_{\mathscr{A}} = \mathrm{id}_{{}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})}$ *and* $\mathsf{Id}_{\mathscr{A}} = \mathrm{id}_{{}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})}$*, respectively. The* C*-module structure for* 2*-morphisms is the* C*-module structure on natural transformations between* **-functors according to* (4.3.57) *and* (4.3.58)*.*

iv.) *The associativity* asso*, the left identity* left*, and the right identity* right *are always the identity transformations.*

Before we actually show that this definition gives indeed a *-bicategory (even a 2-category by *iv.)*) there are some remarks in due: first we note that since the categories ${}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ as well as ${}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A})$ are *-categories over C with respect to the C-module structure of $\mathfrak{B}_{\mathscr{D}}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}, {}_{\mathscr{A}}\mathscr{H}'_{\mathscr{D}})$ and the operator adjoint as *-involution, it is meaningful to speak of *-functors. Natural transformation between those automatically consist of collections of adjointable bimodule morphisms. Even though this may look unfamiliar for the moment, we shall denote the composition of functors in the context of ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$ by $\mathsf{R} \otimes \mathsf{S}$ and not as usual by $\mathsf{R} \circ \mathsf{S}$. We shall use this notation to stay conform with the bicategory formulations.

Theorem 5.3.26 (2-Categories of *-representations) *For a* **-algebra* $\mathscr{D}$ *the definitions of* ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ *as well as of* ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$ *yield* **-bicategories (in fact even* 2*-categories).*

Proof: This relies on the more general fact that the "category" of all categories with functors as 1-morphisms and natural equivalences as 2-morphisms is actually a 2-category, see e.g. [85, Sect. II.5]. In our situation we can rely on these general results. The only thing to be check is that the composition $\otimes$ of *-functors yields again a *-functor and that the natural transformations inherit C-module structures compatible with the composition. But this is clear from the definitions. ☐

The key observation in understanding ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$ as Morita invariants is to view them as bicategories even though they are 2-categories. This allows for more and hence for more interesting morphisms. In particular, we can look for a functor of bicategories

$$\underline{\mathsf{Pic}}^* \longrightarrow {}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}} \tag{5.3.50}$$

or

$$\underline{\mathsf{Pic}}^{\mathrm{str}} \longrightarrow {}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}, \tag{5.3.51}$$

respectively, since $\underline{\mathsf{Pic}}^*$ and $\underline{\mathsf{Pic}}^{\mathrm{str}}$ are only bicategories but not 2-categories. Before doing so we have to explain the notion of functor between bicategories in some more detail. In fact, the notions are far from being uniform in the literature requiring that we should be specific here, see e.g. [7,83] for a more detailed discussion and other options. Instead of presenting the most general form we concentrate directly on the notion of a left action of a bigroupoid. Moreover, since ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and ${}^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$ are even *-bicategories, we can require the left action of the Picard bigroupoids to be *unitary* in a second step. Putting everything together, the following definition is suitable for us:

Definition 5.3.27 (Left action of a bigroupoid) *Let* $\underline{\mathfrak{G}}$ *be a bigroupoid and* $\underline{\mathfrak{B}}$ *a bicategory. A left action* Φ *of* $\underline{\mathfrak{G}}$ *on* $\underline{\mathfrak{B}}$ *consists of the following data:*

i.) *A map* $\underline{\Phi}\colon \mathfrak{G}_0 \longrightarrow \mathfrak{B}_0$*.*

ii.) For any two objects $a, b \in \mathfrak{B}_0$ a functor

$$\Phi_{ba} \colon \mathfrak{G}_1(b, a) \longrightarrow \mathfrak{B}_1(\Phi(b), \Phi(a)). \tag{5.3.52}$$

iii.) For every three objects $a, b, c \in \mathfrak{G}_0$ a natural isomorphism

$$\varphi_{cba} \colon \otimes_{\Phi(b)} \circ (\Phi_{cb} \times \Phi_{ba}) \longrightarrow \Phi_{ca} \circ \otimes_b. \tag{5.3.53}$$

iv.) For every object $a \in \mathfrak{G}_0$ a 2-isomorphism

$$\varphi_a \colon \mathsf{Id}_{\Phi(a)} \longrightarrow \Phi_{aa}(\mathsf{Id}_a). \tag{5.3.54}$$

These data are required to fulfill the following coherence conditions: For all 1-morphisms $g \in \mathfrak{G}_1(b, a)$, $h \in \mathfrak{G}_1(c, b)$, and $k \in \mathfrak{G}_1(d, c)$ the diagrams

$$\begin{array}{ccc}
 & (\Phi_{dc}(k) \otimes \Phi_{cb}(h)) \otimes \Phi_{ba}(g) & \\
\swarrow^{\mathsf{asso}_{\mathfrak{B}}} & & \searrow^{\varphi_{dcb}(k,h) \otimes \mathrm{id}} \\
\Phi_{dc}(k) \otimes (\Phi_{cb}(h) \otimes \Phi_{ba}(g)) & & \Phi_{db}(k \otimes h) \otimes \Phi_{ba}(g) \\
\downarrow{\scriptstyle \mathrm{id} \otimes \varphi_{cba}(h,g)} & & \downarrow{\scriptstyle \varphi_{dba}(k \otimes h, g)} \\
\Phi_{dc}(k) \otimes \Phi_{ca}(h \otimes g) & & \Phi_{da}((k \otimes h) \otimes g) \\
\searrow_{\varphi_{dca}(k, h \otimes g)} & & \swarrow_{\mathsf{asso}_{\mathfrak{G}}} \\
 & \Phi_{da}(k \otimes (h \otimes g)) &
\end{array} \tag{5.3.55}$$

as well as

$$\begin{array}{ccc}
\Phi_{ba}(g) \otimes \mathsf{Id}_{\Phi(a)} & \xrightarrow{\mathrm{id}_{\Phi_{ba}(g)} \otimes \varphi_a} & \Phi_{ba}(g) \otimes \Phi_{aa}(\mathsf{Id}_a) \\
\downarrow{\scriptstyle \mathsf{right}_{ba}} & & \downarrow{\scriptstyle \varphi_{baa}(g, \mathsf{Id}_a)} \\
\Phi_{ba}(g) & \xleftarrow[\Phi(\mathsf{right}_{ba})]{} & \Phi_{ba}(g \otimes \mathsf{Id}_a)
\end{array} \tag{5.3.56}$$

and

$$\begin{array}{ccc}
\mathsf{Id}_{\Phi(b)} \otimes \Phi_{ba}(g) & \xrightarrow{\varphi_b \otimes \mathrm{id}_{\Phi_{ba}(g)}} & \Phi_{bb}(\mathsf{Id}_b) \otimes \Phi_{ba}(g) \\
\downarrow{\scriptstyle \mathsf{left}_{ba}} & & \downarrow{\scriptstyle \varphi_{bba}(\mathsf{Id}_b, g)} \\
\Phi_{ba}(g) & \xleftarrow[\Phi(\mathsf{left}_{ba})]{} & \Phi_{ba}(\mathsf{Id}_b \otimes g)
\end{array} \tag{5.3.57}$$

commute.

Remark 5.3.28 (Morphisms of bicategories) Our definition of an action is a particular case of a more general notion of morphisms between bicategories. First, we can use an arbitrary bicategory $\underline{\mathfrak{G}}$ instead of a bigroupoid. Moreover, φ_{cba} can be relaxed to an arbitrary natural transformation and φ_a

can be allowed to be an arbitrary 2-morphisms instead of isomorphisms as we did in Definition 5.3.27. In this case, one ends up with Benabou's definition of a *morphism of bicategories*, see [7]. If both φ_{cba} and φ_a are isomorphisms, then Benabou speaks of a *homomorphism*. We write for a homomorphism also

$$\underline{\Phi}\colon \underline{\mathfrak{G}} \longrightarrow \underline{\mathfrak{C}}. \tag{5.3.58}$$

If only the φ_a are isomorphisms then $\underline{\Phi}$ is called a *unitary morphism*, not to be confused with our notions of unitarity referring to the *-involutions. Finally, if even $\varphi_a = \mathrm{id}$ then one has a *strict unitary morphism* which yields a *strict homomorphism* if in addition also $\varphi_{cba} = \mathrm{id}$. For us, the notion of a homomorphism turns out to be the relevant one. The notion of "bifunctor" or "weak 2-functor" is not used uniformly in the literature. Thus we shall avoid this term.

Definition 5.3.29 (Unitary left action of a bigroupoid) *Let $\underline{\mathfrak{G}}$ be a bigroupoid and let $\underline{\mathfrak{B}}$ be a *-bicategory. A unitary left action of $\underline{\mathfrak{G}}$ on $\underline{\mathfrak{B}}$ is a left action such that in addition one has the following properties:*

i.) For every three objects $a, b, c \in \mathfrak{G}_0$ the natural isomorphism

$$\varphi_{cba}\colon \otimes_{\Phi(b)} \circ (\Phi_{cb} \times \Phi_{ba}) \longrightarrow \Phi_{ca} \circ \otimes_b \tag{5.3.59}$$

is unitary.

ii.) For every object $a \in \mathfrak{G}_0$ the 2-isomorphism

$$\varphi_a\colon \mathsf{Id}_{\Phi(a)} \longrightarrow \Phi_{aa}(\mathsf{Id}_a). \tag{5.3.60}$$

is unitary.

Analogously to Theorem 5.3.4 we obtain an invariant also from a bigroupoid action in the following sense:

Proposition 5.3.30 *Let $\underline{\Phi}\colon \underline{\mathfrak{G}} \longrightarrow \underline{\mathfrak{B}}$ be a left action of a bigroupoid $\underline{\mathfrak{G}}$ on a bicategory $\underline{\mathfrak{B}}$.*

i.) If $a, b \in \mathfrak{G}_0$ are objects in the same orbit then $\underline{\Phi}(a)$ and $\underline{\Phi}(b)$ are isomorphic objects in the bicategory sense.

ii.) Every 1-morphism $g\colon a \longrightarrow b$ in $\mathfrak{G}_1$ yields an isomorphism $\Phi_{ba}(g)\colon \underline{\Phi}(a) \longrightarrow \underline{\Phi}(b)$.

iii.) The classifying groupoid $\mathfrak{G}$ of $\underline{\mathfrak{G}}$ acts from the left on the classifying category $\mathfrak{B}$ via the classifying action Φ which is defined by the functor $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{B}$ explicitly given by

$$\Phi(a) = \underline{\Phi}(a) \quad \textit{and} \quad \Phi([g]) = [\Phi_{ba}(g)]. \tag{5.3.61}$$

PROOF: For $g\colon a \longrightarrow b$ also $\Phi(g)$ is invertible up to a 2-morphism. Indeed, if $h\colon b \longrightarrow a$ is a 1-morphism and

$$\phi\colon h \otimes g \longrightarrow \mathsf{Id}_a \quad \text{and} \quad \psi\colon g \otimes h \longrightarrow \mathsf{Id}_b$$

are the corresponding 2-isomorphisms, then for $\Phi(g)$ and $\Phi(h)$ we find that

$$\Phi(g) \otimes \Phi(h) \stackrel{\varphi(g,h)}{\longrightarrow} \Phi(g \otimes h) \stackrel{\Phi(\phi)}{\longrightarrow} \Phi(\mathsf{Id}_a) \stackrel{\varphi_a^{-1}}{\longrightarrow} \mathsf{Id}_{\Phi(a)}$$

is a 2-isomorphism in $\underline{\mathfrak{B}}$ since all involved arrows are 2-isomorphisms. Analogously, we find the 2-isomorphism

$$\varphi_b^{-1} \circ \Phi(\psi) \circ \varphi(h, g)\colon \Phi(h) \otimes \Phi(g) \longrightarrow \mathsf{Id}_{\Phi(b)}.$$

Thus $\Phi(g)$ is indeed invertible and $\Phi(a)$ and $\Phi(b)$ turn out to be isomorphic objects in the bicategory sense. This shows the first and second part. Note that we need that the φ_a are all invertible: a *morphism* of bicategories would not be sufficient for this statement. For the third part, we have already seen in Proposition 5.1.6 and Theorem 4.3.22 that $\mathfrak{G}$ is a groupoid and $\mathfrak{B}$ is a category, respectively. We have to show that $\Phi([g])$ is well-defined. Thus let $\phi\colon g \longrightarrow g'$ be a 2-isomorphism. Then by the functoriality of Φ also $\Phi(\phi)\colon \Phi(g) \longrightarrow \Phi(g')$ is a 2-isomorphism. It follows that (5.3.61) is well-defined. Let now $g\colon a \longrightarrow b$ and $h\colon b \longrightarrow c$ be given. Then $\Phi(h) \otimes \Phi(g)$ is (naturally) isomorphic to $\Phi(h \otimes g)$ via φ_{cba}. Similarly, $\Phi(\mathsf{Id}_a)$ is (naturally) isomorphic to $\mathsf{Id}_{\Phi(a)}$, hence

$$[\Phi(h)] \circ [\Phi(g)] = [\Phi(h) \otimes \phi(g)] = \Phi([g \otimes h]) = \Phi([g] \circ [h]),$$

and $[\Phi(\mathsf{Id}_a)] = [\mathsf{Id}_{\Phi(a)}]$ follows. This shows that (5.3.61) is indeed functorial. □

After this general preparation we can now formulate the Morita invariants $^*\text{-}\mathsf{Mod}_{\mathscr{D}}$ and $^*\text{-}\mathsf{Rep}_{\mathscr{D}}$ from Corollary 5.3.24 in the framework of a bicategory action:

Theorem 5.3.31 (Picard bigroupoid action on representation theory) *The Rieffel induction yields a unitary left action $\underline{\Phi}$ of $\underline{\mathsf{Pic}}^*$ on $^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ and of $\underline{\mathsf{Pic}}^{\mathrm{str}}$ on $^*\text{-}\underline{\mathsf{Rep}}_{\mathscr{D}}$, respectively. More precisely, $\underline{\Phi}$ is obtained as follows:*

i.) For the objects in $\underline{\mathsf{Pic}}^*$ or $\underline{\mathsf{Pic}}^{\mathrm{str}}$, i.e. idempotent and non-degenerate *-algebras $\mathscr{A}$, one sets $\Phi(\mathscr{A}) = \mathscr{A}$.

ii.) For a 1-morphism ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ in $\underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ or in $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$, respectively, one sets

$$\Phi({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}) = \mathsf{R}_{\mathscr{E}}\colon {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A}) \tag{5.3.62}$$

or

$$\Phi({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}) = \mathsf{R}_{\mathscr{E}}\colon {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathscr{D}}(\mathscr{B}), \tag{5.3.63}$$

respectively.

iii.) For a 2-morphism $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ one uses the natural transformation

$$\Phi(T) = I_T\colon \mathsf{R}_{\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{E}'} \tag{5.3.64}$$

according to (5.3.42).

iv.) For $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ as well as for ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ in $\underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ or in $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A})$, respectively, and ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ in $\underline{\mathsf{Pic}}^*(\mathscr{C}, \mathscr{B})$ or in $\underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{C}, \mathscr{B})$, respectively, one sets

$$\varphi_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E}) = \mathsf{asso}^{-1}\colon \mathsf{R}_{\mathscr{F}} \otimes \mathsf{R}_{\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{F} \widetilde{\otimes} \mathscr{E}} \tag{5.3.65}$$

according to (5.3.40).

v.) Finally, one uses

$$\varphi_{\mathscr{A}} = \mathsf{left}^{-1}\colon \mathrm{id}_{^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})} \longrightarrow \mathsf{R}_{\mathscr{A}} \tag{5.3.66}$$

with left *as in (5.3.41).*

Proof: First of all, it is clear from Theorem 5.3.23 that $\Phi\colon \underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A}) \longrightarrow {}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}(\mathscr{B}, \mathscr{A})$ is indeed functorial: for $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ and $T'\colon {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}''_{\mathscr{A}}$ the corresponding I_T is on one hand a natural transformation between $\mathsf{R}_{\mathscr{E}}$ and $\mathsf{R}_{\mathscr{E}'}$ and thus a morphism in $^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}(\mathscr{B}, \mathscr{A})$. On the other hand we have $I_{T' \circ T} = I_{T'} \circ I_T$ as well as $I_{\mathrm{id}} = \mathrm{id}_{\mathsf{R}_{\mathscr{E}}}$. Thus *ii.)* of Definition 5.3.27 is fulfilled. Moreover, for unitary T we have $I_T^{-1} = I_{T^{-1}} = I_{T^*} = I_T^*$ showing that the natural transformation is unitary in this case. By Theorem 5.3.23, *i.)*, the associativity asso and hence also its inverse asso^{-1} is a natural unitary isomorphism, thus a unitary isomorphism in $^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}(\mathscr{C}, \mathscr{A})$ as wanted. It remains to show that this unitary isomorphism is also natural in $\mathscr{E}$ and $\mathscr{F}$. Thus let 2-morphisms $T\colon {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}\mathscr{E}'_{\mathscr{A}}$ and $S\colon {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \longrightarrow {}_{\mathscr{C}}\mathscr{F}'_{\mathscr{B}}$ be given. Then we have $(\otimes \circ (\Phi \times \Phi))(\mathscr{F}, \mathscr{E}) = \mathsf{R}_{\mathscr{F}} \otimes \mathsf{R}_{\mathscr{E}}$ as well as $(\Phi \circ \otimes)(\mathscr{F}, \mathscr{E}) =$

$\mathsf{R}_{\mathscr{F}\tilde{\otimes}\mathscr{E}}$ where we write again $\otimes$ for the composition of functors in ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ as this is the tensor product of this bicategory. Moreover, we have $(\otimes \circ (\Phi \times \Phi))(S, T) = I_S \otimes I_T \colon \mathsf{R}_{\mathscr{F}} \otimes \mathsf{R}_{\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{F}'} \otimes \mathsf{R}_{\mathscr{E}'}$ with the tensor product of natural transformations $I_S \otimes I_T$. Evaluating this on a given ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}$ we get

$$\begin{aligned}(I_S \otimes I_T)({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}) &= I_S(\mathsf{R}_{\mathscr{E}'}({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}})) \circ \mathsf{R}_{\mathscr{F}}(I_T({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}})) \\ &= I_S\big(\mathscr{E}' \mathbin{\widehat{\otimes}} {}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}\big) \circ \big(\mathrm{id}_{\mathscr{F}} \mathbin{\widehat{\otimes}} (I_T({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}))\big) \\ &= \Big(S \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{E}'\widehat{\otimes}\mathscr{H}}\Big) \circ \big(\mathrm{id}_{\mathscr{F}} \mathbin{\widehat{\otimes}} (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}})\big) \\ &= S \mathbin{\widehat{\otimes}} (T \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}})\end{aligned}$$

for every ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$. Finally, we have $(\Phi \circ \otimes)(S, T) = I_{S \tilde{\otimes} T} \colon \mathsf{R}_{\mathscr{F}\tilde{\otimes}\mathscr{E}} \longrightarrow \mathsf{R}_{\mathscr{F}'\tilde{\otimes}\mathscr{E}'}$. Now putting things together we see that the naturalness of $\varphi_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E}) = \mathsf{asso}^{-1}_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E})$ means that we have to show

$$\mathsf{asso}^{-1}_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}', \mathscr{E}') \circ (I_S \otimes I_T) = I_{S\tilde{\otimes}T} \circ \mathsf{asso}^{-1}_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E}).$$

But this is just the statement that the diagram (5.3.47) commutes. Thus $\varphi_{\mathscr{C}\mathscr{B}\mathscr{A}}$ is natural as wanted, showing that *iii.)* of Definition 5.3.27 is satisfied. Since we already know that left and thus also left^{-1} are natural unitary isomorphisms, we conclude that $\varphi_{\mathscr{A}} = \mathsf{left}^{-1}$ is a unitary 2-isomorphism in ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ as needed. It remains to show the coherence conditions between these data. To this end we first note that the associativity in ${}^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$ holds strictly. Thus the hexagon in (5.3.55) degenerates to a pentagon. To prove its commutativity we first write all involved natural transformations explicitly for some ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$. We have

$$\begin{aligned}\big(\mathrm{id}_{\mathsf{R}_{\mathscr{G}}} \otimes \varphi_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E})\big)(\mathscr{H}) &= \mathrm{id}_{\mathsf{R}_{\mathscr{F}\tilde{\otimes}\mathscr{E}}(\mathscr{H})} \circ \mathsf{R}_{\mathscr{G}}(\varphi_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E})(\mathscr{H})) \\ &= \mathsf{R}_{\mathscr{G}}\big(\mathsf{asso}^{-1}(\mathscr{F}, \mathscr{E}, \mathscr{H})\big) \\ &= \mathrm{id}_{\mathscr{G}} \mathbin{\widehat{\otimes}} \mathsf{asso}^{-1}(\mathscr{F}, \mathscr{E}, \mathscr{H}),\end{aligned}$$

and $\varphi_{\mathscr{C}'\mathscr{C}\mathscr{A}}(\mathscr{G}, \mathscr{F} \mathbin{\tilde{\otimes}} \mathscr{E})(\mathscr{H}) = \mathsf{asso}^{-1}(\mathscr{G}, \mathscr{F} \mathbin{\tilde{\otimes}} \mathscr{E}, \mathscr{H})$. Thus we get

$$\begin{aligned}&\big(\varphi_{\mathscr{C}'\mathscr{C}\mathscr{B}}(\mathscr{G}, \mathscr{F} \mathbin{\tilde{\otimes}} \mathscr{E}) \circ \big(\mathrm{id}_{\mathsf{R}_{\mathscr{G}}} \otimes \varphi_{\mathscr{C}\mathscr{B}\mathscr{A}}(\mathscr{F}, \mathscr{E})\big)\big)(\mathscr{H}) \\ &\quad = \mathsf{asso}^{-1}(\mathscr{G}, \mathscr{F} \mathbin{\tilde{\otimes}} \mathscr{E}, \mathscr{H}) \circ \big(\mathrm{id}_{\mathscr{G}} \mathbin{\widehat{\otimes}} \mathsf{asso}^{-1}(\mathscr{F}, \mathscr{E}, \mathscr{H})\big).\end{aligned} \tag{$*$}$$

On the other hand, we get first

$$\big(\varphi_{\mathscr{C}'\mathscr{C}\mathscr{B}}(\mathscr{G}, \mathscr{F}) \otimes \mathrm{id}_{\mathsf{R}_{\mathscr{E}}}\big)(\mathscr{H}) = (\varphi_{\mathscr{C}'\mathscr{C}\mathscr{B}}(\mathscr{G}, \mathscr{F}))(\mathsf{R}_{\mathscr{E}}(\mathscr{H})) = \mathsf{asso}^{-1}(\mathscr{G}, \mathscr{F}, \mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{H}),$$

and second $\big(\varphi_{\mathscr{C}'\mathscr{B}\mathscr{A}}(\mathscr{G} \mathbin{\tilde{\otimes}} \mathscr{F}, \mathscr{E})\big)(\mathscr{H}) = \mathsf{asso}^{-1}(\mathscr{G} \mathbin{\tilde{\otimes}} \mathscr{F}, \mathscr{E}, \mathscr{H})$ as well as

$$(\Phi_{\mathscr{C}'\mathscr{A}}(\mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E})))(\mathscr{H}) = I_{\mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E})}(\mathscr{H}) = \mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E}) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}.$$

Collecting this yields the result

$$\begin{aligned}&((\Phi_{\mathscr{C}'\mathscr{A}}(\mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E})))(\mathscr{H})) \circ \big(\big(\varphi_{\mathscr{C}'\mathscr{B}\mathscr{A}}(\mathscr{G} \mathbin{\tilde{\otimes}} \mathscr{F}, \mathscr{E})\big)(\mathscr{H})\big) \circ \big(\big(\varphi_{\mathscr{C}'\mathscr{C}\mathscr{B}}(\mathscr{G}, \mathscr{F}) \otimes \mathrm{id}_{\mathsf{R}_{\mathscr{E}}}\big)(\mathscr{H})\big) \\ &\quad = \big(\mathsf{asso}(\mathscr{G}, \mathscr{F}, \mathscr{E}) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}.\big) \circ \big(\mathsf{asso}^{-1}(\mathscr{G} \mathbin{\tilde{\otimes}} \mathscr{F}, \mathscr{E}, \mathscr{H})\big) \circ \big(\mathsf{asso}^{-1}(\mathscr{G}, \mathscr{F}, \mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{H})\big).\end{aligned} \tag{$**$}$$

The fact that $(*)$ and $(**)$ coincide is just the associativity coherence of asso as in Proposition 4.3.8, *i.)*. Indeed, for the associativity coherence it was never important to use the fact $\mathscr{H} \cdot \mathscr{D} = \mathscr{H}$, thus the equality of $(*)$ and $(**)$ holds also without this feature of $\mathscr{H}$. Next we consider the diagram (5.3.56). We have

$$\Phi_{\mathscr{B}\mathscr{A}}(\mathscr{E}) \otimes \mathsf{Id}_{\Phi(\mathscr{A})} = \mathsf{R}_{\mathscr{E}} \otimes \mathsf{Id}_{\mathscr{A}} = \mathsf{R}_{\mathscr{E}} \circ \mathrm{id}_{{}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})} = \mathsf{R}_{\mathscr{E}},$$

since the identity functor is a strict unit element in the 2-category $^*\text{-}\underline{\mathsf{Mod}}_{\mathscr{D}}$. Hence the diagram (5.3.56) degenerates to a triangle. We consider ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ and compute

$$\big(\mathrm{id}_{\mathsf{R}_{\mathscr{E}}} \otimes \varphi_{\mathscr{A}}\big)(\mathscr{H}) = \mathrm{id}(\Phi(\mathsf{Id}_{\mathscr{A}})) \circ \mathsf{R}_{\mathscr{E}}(\varphi_{\mathscr{A}}(\mathscr{H})) = \mathsf{R}_{\mathscr{E}}(\mathsf{left}^{-1}(\mathscr{H})) = \mathrm{id}_{\mathscr{E}} \mathbin{\widehat{\otimes}} \mathsf{left}^{-1}(\mathscr{H}),$$

where $\mathsf{left}^{-1}(\mathscr{H})\colon \mathscr{H} \longrightarrow \mathscr{A} \mathbin{\widehat{\otimes}} \mathscr{H}$ is the inverse to the unitary isomorphism $\mathsf{left}(\mathscr{H})\colon \mathscr{A} \mathbin{\widehat{\otimes}} \mathscr{H} \longrightarrow \mathscr{H}$. Moreover, we have $\varphi_{\mathscr{E}\mathscr{B}\mathscr{A}}(\mathscr{E}, \mathsf{Id}_{\mathscr{A}})(\mathscr{H}) = \mathsf{asso}^{-1}(\mathscr{E}, \mathscr{A}, \mathscr{H})$ and $(\Phi(\mathsf{right}(\mathscr{E})))(\mathscr{H}) = \mathsf{right}(\mathscr{E}) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}$ where $\mathsf{right}(\mathscr{E})\colon \mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{A} \longrightarrow \mathscr{E}$ is the canonical isomorphism. Combining these things we get

$$\begin{aligned}
&(\Phi(\mathsf{right}(\mathscr{E}))(\mathscr{H})) \circ (\varphi_{\mathscr{B}\mathscr{A}\mathscr{A}}(\mathscr{E}, \mathscr{A})(\mathscr{H})) \circ \big((\mathrm{id}_{\Phi(\mathscr{E})} \otimes \varphi_{\mathscr{A}})(\mathscr{H})\big) \\
&\quad = \big(\mathsf{right}(\mathscr{E}) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}\big) \circ \mathsf{asso}^{-1}(\mathscr{E}, \mathscr{A}, \mathscr{H}) \circ \big(\mathrm{id}_{\mathscr{E}} \mathbin{\widehat{\otimes}} \mathsf{left}^{-1}(\mathscr{H})\big) \\
&\quad = \mathrm{id}_{\mathscr{E} \widehat{\otimes} \mathscr{H}},
\end{aligned}$$

according to the identity coherence for the three bimodules $\mathscr{E}$, $\mathscr{A}$, and $\mathscr{H}$ as in Proposition 4.3.8, *ii.)*. Again, for the identity coherence the strong non-degeneracy $\mathscr{H} \cdot \mathscr{D} = \mathscr{H}$ was not needed. Finally, we have

$$\Phi(\mathsf{left}(\mathscr{E}))(\mathscr{H}) = I_{\mathsf{left}(\mathscr{E})}(\mathscr{H}) = \mathsf{left}(\mathscr{E}) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}$$

with the canonical unitary isomorphism $\mathsf{left}(\mathscr{E})\colon \mathscr{B} \mathbin{\widehat{\otimes}} \mathscr{E} \longrightarrow \mathscr{E}$. Moreover, we have $\varphi_{\mathscr{B}\mathscr{B}\mathscr{A}}(\mathscr{B}, \mathscr{E})(\mathscr{H}) = \mathsf{asso}^{-1}(\mathscr{B}, \mathscr{E}, \mathscr{H})$ and

$$\big(\varphi_{\mathscr{B}} \otimes \mathrm{id}_{\mathsf{R}_{\mathscr{E}}}\big)(\mathscr{H}) = \varphi_{\mathscr{B}}(\mathsf{R}_{\mathscr{E}}(\mathscr{H})) = \mathsf{left}^{-1}(\mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{H}).$$

With this, we consider the remaining diagram (5.3.57) which also degenerates to a triangle. We have

$$\begin{aligned}
&(\Phi(\mathsf{left}(\mathscr{E}))(\mathscr{H})) \circ (\varphi_{\mathscr{B}\mathscr{B}\mathscr{A}}(\mathscr{B}, \mathscr{E})(\mathscr{H})) \circ \big((\varphi_{\mathscr{B}} \otimes \mathrm{id}_{\mathsf{R}_{\mathscr{E}}})(\mathscr{H})\big) \\
&\quad = (\mathsf{left}(\mathscr{E}) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathscr{H}}) \circ (\mathsf{asso}^{-1}(\mathscr{B}, \mathscr{E}, \mathscr{H})) \circ \mathsf{left}^{-1}(\mathscr{E} \mathbin{\widehat{\otimes}} \mathscr{H}) \\
&\quad = \mathrm{id}_{\mathscr{E} \widehat{\otimes} \mathscr{H}},
\end{aligned}$$

as a simple computation shows. In fact, since we have a bicategory, the last step also follows from the "all diagrams commute" theorem for bicategories, as we only have morphisms being data of the bicategory itself. Thus the last coherence diagram is shown and we indeed have a unitary action of the *-Picard bigroupoid. Note that the strong version follows analogously. ☐

Remark 5.3.32 Using this theorem we obtain Corollary 5.3.24 immediately from our general considerations on bigroupoid actions in Proposition 5.3.30. Thus the above theorem can be viewed as the deeper reason for the equivalence of the representation theories for Morita equivalent *-algebras in both flavours. As usual for our point of view, we have the equivalence not just "by accident" but coming with a more systematic structure yielding in particular a systematic way to actually implement the natural isomorphisms between the Rieffel induction functors.

5.4 Exercises

Exercise 5.4.1 (Isomorphisms yield a groupoid) Prove Proposition 5.1.2.

Exercise 5.4.2 (Isotropy groups of a groupoid) Prove Proposition 5.1.4.

Exercise 5.4.3 (Groupoid morphisms) Let $\mathfrak{G}$ and $\mathfrak{H}$ be two groupoids. A *groupoid morphism* $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{H}$ is a covariant functor.

i.) Show that groupoids with groupoid morphisms form a category $\mathsf{Groupoid}$.

ii.) Show that a groupoid morphism Φ induces group morphisms $\Phi\colon \mathfrak{G}(a) \longrightarrow \mathfrak{H}(\Phi(a))$ between the isotropy groups for all objects a of $\mathfrak{G}$.

iii.) Show that the kernels as well as the images of the induced group morphisms $\Phi\colon \mathfrak{G}(a) \longrightarrow \mathfrak{H}(\Phi(a))$ are isomorphic along the orbit of a.

iv.) Let a and b be objects of $\mathfrak{G}$ with $\operatorname{Morph}(b,a) \neq \emptyset$. Show that for $g, h \in \operatorname{Morph}(b,a)$ with $\Phi(g) = \Phi(h)$ there exists a unique element $u \in \ker \Phi_a \subseteq \mathfrak{G}(a)$ such that $h = g \circ u$. In this sense, the kernels of the group morphisms $\Phi\colon \mathfrak{G}(a) \longrightarrow \mathfrak{H}(\Phi(a))$ encode the (non-) injectivity of the groupoid morphism $\Phi\colon \mathfrak{G} \longrightarrow \mathfrak{H}$ completely, even though there is no naive groupoid definition of a kernel directly.

Exercise 5.4.4 (Inversion functors in $\underline{\mathsf{Pic}}^*$ and $\underline{\mathsf{Pic}}^{\mathrm{str}}$) Consider the complex conjugation of bimodules in the bigroupoids $\underline{\mathsf{Pic}}^*$ and $\underline{\mathsf{Pic}}^{\mathrm{str}}$ as inversion.

i.) Show that for two idempotent and non-degenerate *-algebras $\mathscr{A}$ and $\mathscr{B}$ the inversion gives functors

$$\mathrm{inv}\colon \underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^*(\mathscr{A}, \mathscr{B}) \tag{5.4.1}$$

as well as

$$\mathrm{inv}\colon \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{B}, \mathscr{A}) \longrightarrow \underline{\mathsf{Pic}}^{\mathrm{str}}(\mathscr{A}, \mathscr{B}), \tag{5.4.2}$$

provided there are equivalence bimodules between $\mathscr{A}$ and $\mathscr{B}$ at all.

ii.) Consider now the inversion of a tensor product $_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \widetilde{\otimes}_{\mathscr{B}} \mathscr{E}_{\mathscr{A}}$ and compare it to the tensor product of the inversions $_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$ and $_{\mathscr{B}}\overline{\mathscr{F}}_{\mathscr{C}}$: here one has a unitary isomorphism. Show that this isomorphism is natural.

iii.) Show that the inversion of the identity bimodule $_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ is unitarily isomorphic to $_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ with a natural isomorphism, too.

iv.) Investigate the relations between the above natural isomorphisms and formulate corresponding coherence properties.

Exercise 5.4.5 (Inner automorphisms) Let $\mathscr{A}$ be a unital ring.

i.) Check directly that the inner automorphisms $\operatorname{InnAut}(\mathscr{A})$ form a normal subgroup of all automorphisms $\operatorname{Aut}(\mathscr{A})$.

ii.) Denote by $\operatorname{Gl}(\mathscr{A})$ the invertible elements in $\mathscr{A}$ and show that they form a group under multiplication. Define

$$\mathrm{Ad}\colon \mathrm{GL}(\mathscr{A}) \ni g \mapsto \mathrm{Ad}_g \in \operatorname{InnAut}(\mathscr{A}) \tag{5.4.3}$$

by $\mathrm{Ad}_g(a) = gag^{-1}$ for all $a \in \mathscr{A}$. Show that this defines a group homomorphism.

iii.) Determine the kernel of Ad.

Assume now in addition that $\mathscr{A}$ is a unital *-algebra. Proceed analogously for $\operatorname{InnAut}^*(\mathscr{A})$ instead of $\operatorname{InnAut}(\mathscr{A})$ and the group of unitary elements $\mathrm{U}(\mathscr{A})$ of $\mathscr{A}$ instead of the invertible ones.

Exercise 5.4.6 (The center acts on equivalence bimodules) Let $\mathscr{A}$ and $\mathscr{B}$ be idempotent and non-degenerate *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a *-equivalence bimodule.

i.) Show that the center $\mathscr{Z}(\mathscr{A})$ is a *-subalgebra of $\mathscr{A}$, containing $\mathbb{1}$ whenever $\mathscr{A}$ is unital.

ii.) Show that the endomorphisms $\operatorname{End}(_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$ of $_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ in the sense of the category $\underline{\mathsf{Pic}}^*(\mathscr{B}, \mathscr{A})$ form a unital *-algebra.

iii.) Let $a \in \mathscr{Z}(\mathscr{A})$ be a central element of $\mathscr{A}$. Show that the right multiplication with a is an endomorphism of $_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. Show that this gives a *-homomorphism

$$\mathscr{Z}(\mathscr{A}) \longrightarrow \operatorname{End}(_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}), \tag{5.4.4}$$

such that the image is in the center of $\operatorname{End}(_{\mathscr{B}}\mathscr{E}_{\mathscr{A}})$.

Exercise 5.4.7 (The strong Picard group of $\mathrm{M}_n(\mathsf{C})$) Assume that R is a real closed field (with its canonical ordering) such that $\mathsf{C} = \mathsf{R}(\mathrm{i})$ is an algebraically closed field.

i.) Discuss whether C satisfies **(K)** and **(H)**.

ii.) Compute the strong Picard groups $\mathsf{Pic}^{\mathrm{str}}(\mathsf{C})$ and $\mathsf{Pic}^{\mathrm{str}}(M_n(\mathsf{C}))$ for all $n \in \mathbb{N}$.

Hint: What are the finitely generated projective modules over C? Use Exercise 2.4.21 and Exercise 2.4.22.

iii.) Show that all *-automorphisms of $\mathrm{M}_n(\mathsf{C})$ are inner.

Exercise 5.4.8 (Ring-theoretic Picard group and the center) Formulate and prove the ring-theoretic versions of Proposition 5.2.9, Proposition 5.2.12, and Theorem 5.3.7.

Exercise 5.4.9 (The strong Picard group of $\mathscr{C}^\infty(M)$) Use Theorem 5.2.17 and Corollary 5.2.26 to simplify the computation of the strong Picard group of $\mathscr{C}^\infty(M)$ as in Theorem 5.2.15.

Exercise 5.4.10 (The involution of $\mathrm{Aut}(\mathscr{A})$) Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Consider the map $^*\colon \mathrm{Aut}(\mathscr{A}) \longrightarrow \mathrm{Aut}(\mathscr{A})$ as in (5.2.59).

i.) Show that * is an involutive automorphism of the group $\mathrm{Aut}(\mathscr{A})$.

ii.) Let $\Phi \in \mathrm{Aut}(\mathscr{A})$. Show that $\Phi^* = \Phi$ iff $\Phi \in \mathrm{Aut}^*(\mathscr{A})$.

iii.) Let $g \in \mathrm{Gl}(\mathscr{A})$ be a invertible element in $\mathscr{A}$ and compute $(\mathrm{Ad}_g)^*$.

iv.) Show that * induces a well-defined involutive group automorphism of $\mathrm{OutAut}(\mathscr{A})$.

Exercise 5.4.11 (Geometric groupoid action) Let $\mathfrak{G}$ be a (small) groupoid and M a set. An alternative way to define a groupoid action of $\mathfrak{G}$ on M is as follows: one has two maps, the *anchor*

$$\varrho\colon M \longrightarrow \mathfrak{G}_0 \tag{5.4.5}$$

and the multiplication map

$$\triangleright\colon \mathfrak{G}_1 \times_{\mathfrak{G}_0} M \longrightarrow M \tag{5.4.6}$$

written as $(g, p) \mapsto g \triangleright p$. Here $\times_{\mathfrak{G}_0}$ denotes the usual fiber product

$$\mathfrak{G}_1 \times_{\mathfrak{G}_0} M = \big\{(g, p) \bigm| \mathrm{source}(g) = \varrho(p)\big\} \subseteq \mathfrak{G}_1 \times M \tag{5.4.7}$$

with respect to the source map and the anchor. Then these two maps constitute a *geometric groupoid action* if $\varrho(g \triangleright p) = \mathrm{target}(g)$ and $\mathrm{id}_a \triangleright p = p$ for $\varrho(p) = a$ as well as

$$h \triangleright (g \triangleright p) = (hg) \triangleright p, \tag{5.4.8}$$

whenever the compositions are defined. Show that this can be interpreted as a groupoid action $\Phi\colon \mathfrak{G} \longrightarrow \mathsf{Set}$ in the sense of Definition 5.3.1.

Hint: Define $\Phi(a) = \varrho^{-1}(a) \subseteq M$ and let $\Phi(g)\colon \Phi(a) \longrightarrow \Phi(b)$ be the obvious map where $g\colon a \longrightarrow b$ is an arrow in $\mathfrak{G}$.

The advantage of the definition of a geometric groupoid action compared to Definition 5.3.1 is that one can encode useful information in the two maps ϱ and $\triangleright$ which are hard to encode in the functor Φ directly. As example one can consider instead of a bare set a topological space or a manifold M and require continuity or smoothness properties of ϱ and $\triangleright$.

Exercise 5.4.12 (Lattices) We collect some basic properties of lattices in this exercise:

i.) Let $\mathfrak{L}$ be a lattice and let $a, b \in \mathfrak{L}$. Show that $a \wedge b = b$ holds iff $a \vee b = b$ holds. Prove that $a \leq b$ if $a \wedge b = b$ defines a partial ordering on $\mathfrak{L}$.

ii.) Show that for a lattice $\mathfrak{L}$ the partial ordering $\leq$ has for all $a, b \in \mathfrak{L}$ an infimum and a supremum given by (5.3.24).

iii.) Show that in a lattice $\mathfrak{L}$ a maximal (minimal) element is necessarily unique, if it exists at all.

iv.) Now suppose that $(\mathfrak{L}, \leq)$ is a partially ordered set such that for any two elements one has an infimum and a supremum. Show that in this case (5.3.24) defines the structure of a lattice.

Exercise 5.4.13 (The closed *-ideals of $\mathscr{C}^\infty(M)$) Consider the *-algebra $\mathscr{C}^\infty(M)$ for a smooth manifold M. Show that the scalar closed *-ideals of $\mathscr{C}^\infty(M)$ are precisely the vanishing ideals

$$\mathscr{J}_A = \left\{ f \in \mathscr{C}^\infty(M) \;\middle|\; f\big|_A = 0 \right\} \tag{5.4.9}$$

of closed subsets $A \subseteq M$.

Hint: Use the characterization of positive functionals of $\mathscr{C}^\infty(M)$ from Exercise 1.4.17 as well as Exercise 1.4.15, see also [25, Sect. 6].

Exercise 5.4.14 (The minimal ideal: general case) This exercise is a continuation of the investigations of the minimal ideal started in Exercise 1.4.15. Let $\mathscr{A}$ be an idempotent and non-degenerate *-algebra over C and let $\mathscr{D}$ be an admissible coefficient *-algebra.

i.) Show that dividing by the minimal strongly $\mathscr{D}$-closed *-ideal $\mathscr{A} \rightsquigarrow \mathscr{A} \big/ \mathscr{J}^{\mathrm{str}}_{\mathrm{min},\mathscr{D}}(\mathscr{A})$ is functorial.

ii.) Prove that the minimal strongly $\mathscr{D}$-closed *-ideal of the quotient $\mathscr{A} \big/ \mathscr{J}^{\mathrm{str}}_{\mathrm{min},\mathscr{D}}(\mathscr{A})$ is trivial.

iii.) Show that the representation theories of $\mathscr{A}$ and $\mathscr{A} \big/ \mathscr{J}^{\mathrm{str}}_{\mathrm{min},\mathscr{D}}(\mathscr{A})$ on pre-Hilbert right $\mathscr{D}$-modules are equivalent categories.

iv.) Show that the *-algebras $\mathscr{A}$ and $\mathscr{A} \big/ \mathscr{J}^{\mathrm{str}}_{\mathrm{min},\mathscr{D}}(\mathscr{A})$ are strongly Morita equivalent iff $\mathscr{J}^{\mathrm{str}}_{\mathrm{min},\mathscr{D}}(\mathscr{A}) = \{0\}$.

Exercise 5.4.15 (Representation theory of $\Lambda(\mathsf{C}^n)$) Consider the Grassmann algebra $\mathscr{A} = \Lambda(\mathsf{C}^n)$.

i.) Show that the scalar *-representation theories *-Rep of C and $\mathscr{A}$ on pre-Hilbert spaces are equivalent by explicitly constructing functors which establish the equivalence.

ii.) Show that the Grassmann algebra is not strongly Morita equivalent to C.

Hint: Here one can argue in many ways. Try to use one of the Morita invariants or argue with Exercise 5.4.14.

Exercise 5.4.16 (Unitary intertwiner for Rieffel induction) Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$ be *-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Let ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}} \in {}^*\text{-}\mathsf{Mod}_{\mathscr{D}}(\mathscr{A})$ be a strongly non-degenerate *-representation of $\mathscr{A}$ with the coefficient *-algebra $\mathscr{D}$. Moreover, let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a *-equivalence bimodule. Theorem 5.3.31 provides now a natural and unitary intertwiner from $(\mathsf{R}_{\overline{\mathscr{E}}} \circ \mathsf{R}_{\mathscr{E}})({}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}})$ to ${}_{\mathscr{A}}\mathscr{H}_{\mathscr{D}}$. Find an explicit formula for this intertwiner in terms of the algebra-valued inner products on ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and ${}_{\mathscr{A}}\overline{\mathscr{E}}_{\mathscr{B}}$ and verify the claimed properties according to Theorem 5.3.31 explicitly.

Chapter 6

Deformations of Algebras, States, and Modules

After developing the representation theory of *-algebras in detail, we shall now use the general results to investigate *-algebras which are obtained as deformations of other *-algebras. First we remind that for an ordered ring R also the formal power series $\mathsf{R}[\![\lambda]\!]$ are ordered in a natural way. Thus we do not leave the general framework of *-algebras over ordered rings if we study formal deformations. In fact, this was one of the main motivations to study general ordered rings instead of just the real numbers $\mathbb{R}$. This way, one is able to gain new examples with entirely new features by deformation. Including such *-algebras into the discussion will also help to understand *-algebras over $\mathbb{C}$ from a slightly different angle, broadening the point of view considerably.

The main theme of this and the following chapter is that existence and classification theorems in deformation theory are usually very hard and difficult. However, the inverse process, i.e. taking a *classical limit*, can be defined and constructed in a rather simple way. In this chapter we will therefore discuss the classical limit of various mathematical structures: algebras, states, and modules. For all of them we obtain classical limit constructions in a rather straightforward way, posing the typically much more subtle question of *quantization* as the way to go the other direction.

6.1 Deformations of *-Algebras

In this introductory section we provide the necessary background in deformation theory of associative algebras and *-algebras. This will not be a comprehensive discussion, more detailed expositions can be found e.g. in the textbooks [47, 116] as well as in the seminal papers of Gerstenhaber [52–56] who coined the basic notions of deformation theory of associative algebras. The field itself is by now vast and fast developing. We will indicate more recent literature when needed.

The basic idea of formal deformation theory is that we have an algebraic structure like e.g. an algebra multiplication μ_0 which we would like to understand and study. One (traditional) approach is to relate μ_0 to other structures of the same type by trying to find a classification of these structures and determine the isomorphism class of μ_0 within this classification scheme. While in simple situations this might succeed, in more realistic scenarios of interest, a full classification might be out of reach. Now what is still possible is to investigate which structures μ of the same sort are still *close to* the given structure μ_0: one wants to understand the classification only of the nearby structures. This suggest that one has a notion of *nearby* in order to make sense out of such ideas on a mathematically sound basis. As soon as one has algebraic structures combined with analytic features this can be done in various ways. However, Gerstenhaber's original idea is to consider a *formal* neighbourhood of the structure μ_0 thereby avoiding the usage of any a priori topological concepts. One thus considers structures μ of the form

$$\mu = \mu_0 + \lambda\mu_1 + \lambda^2\mu_2 + \cdots \tag{6.1.1}$$

as a formal power series expansion called *formal deformations* of μ_0. Here we have to assume that the structures we are investigating can be placed inside some linear space. It then turns out that studying whether μ is *equivalent* to the original structure μ_0 tells much about μ_0. If there are non-equivalent such deformations one wants to understand how many equivalence classes are possible.

Ultimately, there are many occasions where the algebraic structures allow for some more serious analytic context. Hence a study of the convergence of the above formal series is a primary goal *after* the algebraic questions have been settled. This will of course be of yet another level of complication: we will not touch these questions related to convergence here any more.

While the above interpretation of deformations and rigidity is perfectly adequate within mathematics there are applications far beyond. Soon after Gerstenhaber's general theory of deforming associative algebras it became clear that the transition from classical physics to quantum physics can be viewed as a deformation process with the formal parameter λ playing the role of Planck's constant $\hbar$. This is the basis for the seminal work of Bayen et al. [5] establishing the notion of star products and deformation quantization, see also e.g. [44,60,109] for recent reviews as well as the textbooks [47,116] for more details. We will occasionally come back to this main class of examples and gain motivation from this point of view.

6.1.1 The Ring of Formal Power Series as new Scalars

We have already met the ring of formal power series $\mathsf{R}[\![\lambda]\!]$ with coefficients in a given ring R in Section 1.1.1. Here we will recall some of its basic properties we need in the sequel.

Thus let R be an associative and commutative ring and consider a module V over it. Then the formal power series $V[\![\lambda]\!]$ with coefficients in V become a module over $\mathsf{R}[\![\lambda]\!]$ by the usual multiplication of formal series, i.e. by the Cauchy product formula

$$a \cdot v = \left(\sum_{r=0}^{\infty} \lambda^r a_r\right)\left(\sum_{r=0}^{\infty} \lambda^r v_r\right) = \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} a_s \cdot v_{r-s} \tag{6.1.2}$$

for $a \in \mathsf{R}[\![\lambda]\!]$ and $v \in V[\![\lambda]\!]$. Note that the ring inclusion $\mathsf{R} \subseteq \mathsf{R}[\![\lambda]\!]$ yields the canonical R-module structure of $V[\![\lambda]\!]$.

Next, suppose that V and W are both modules over R and $\phi_r \colon V \longrightarrow W$ are R-linear maps for $r \in \mathbb{N}_0$. Then we obtain an $\mathsf{R}[\![\lambda]\!]$-linear map

$$\phi = \sum_{r=0}^{\infty} \lambda^r \phi_r \colon V[\![\lambda]\!] \longrightarrow W[\![\lambda]\!] \tag{6.1.3}$$

by setting

$$\phi(v) = \left(\sum_{r=0}^{\infty} \lambda^r \phi_r\right)\left(\sum_{s=0}^{\infty} \lambda^s v_s\right) = \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} \phi_s(v_{r-s}) \tag{6.1.4}$$

for $v \in V[\![\lambda]\!]$. We leave it as a little exercise to verify that this is indeed $\mathsf{R}[\![\lambda]\!]$-linear. In particular, we can extend a given R-linear map $\phi \colon V \longrightarrow W$ by this to an $\mathsf{R}[\![\lambda]\!]$-linear map $\phi \colon V[\![\lambda]\!] \longrightarrow W[\![\lambda]\!]$. In the following we will always assume such an extension without indicating this in our notation. A similar construction can be done with multilinear maps. It is now a simple verification that the composition of such extensions ϕ and $\psi = \sum_{r=0}^{\infty} \lambda^r \psi_r \colon W[\![\lambda]\!] \longrightarrow U[\![\lambda]\!]$ yields analogous formulas to the Cauchy product (6.1.2).

As a first observation we note that between such modules *all* $\mathsf{R}[\![\lambda]\!]$-linear maps are of the form (6.1.3):

Proposition 6.1.1 *Let $V_1, \ldots, V_n$ and W be R-modules. Then for an $\mathsf{R}[\![\lambda]\!]$-multilinear map*

$$\Phi\colon V_1[\![\lambda]\!] \times \cdots \times V_n[\![\lambda]\!] \longrightarrow W[\![\lambda]\!] \tag{6.1.5}$$

there exists unique R-multilinear maps $\phi_r\colon V_1 \times \cdots \times V_n \longrightarrow W$ such that

$$\Phi = \sum_{r=0}^{\infty} \lambda^r \phi_r. \tag{6.1.6}$$

Here we use the canonical extension of the ϕ_r to $\mathsf{R}[\![\lambda]\!]$-multilinear maps as above. The proof is discussed in Exercise 6.4.1.

Of course, there are other modules over $\mathsf{R}[\![\lambda]\!]$ which are not of the form $V[\![\lambda]\!]$ with an R-module V, see Exercise 6.4.4 for a more intrinsic characterization. Nevertheless, most of the time we will be interested in such more particular modules. Here we have a canonical projection onto the zeroth order

$$\mathrm{cl}\colon V[\![\lambda]\!] \ni v = \sum_{r=0}^{\infty} \lambda^r v_r \;\mapsto\; v_0 \in V, \tag{6.1.7}$$

which we will call the *classical limit* map. Conversely, we can consider V as a subset of $V[\![\lambda]\!]$ by including it in order zero. Note that this will only be an R-submodule but not an $\mathsf{R}[\![\lambda]\!]$-submodule.

For an element $v = \sum_{r=0}^{\infty} \lambda^r v_r \in V[\![\lambda]\!]$ we define its *order* by

$$o(v) = \min\{k \mid v_k \neq 0\} \tag{6.1.8}$$

with the convention $o(0) = \infty$. Then the λ-*adic valuation* is defined by

$$\varphi(v) = 2^{-o(v)} \tag{6.1.9}$$

with the convention $2^{-\infty} = 0$ for the zero element $0 \in V[\![\lambda]\!]$. Using this valuation we can define the λ-*adic metric* by

$$d(v, w) = \varphi(v - w) = 2^{-o(v-w)} \tag{6.1.10}$$

for $v, w \in V[\![\lambda]\!]$. This turns out to be a metric with the following nice properties:

Proposition 6.1.2 *Let V, W, $V_1, \ldots, V_n$ be modules over R.*

i.) The order satisfies

$$o(v) = o(-v), \quad o(v) = \infty \text{ iff } v = 0, \quad \text{and} \quad o(v + w) \geq \min(o(v), o(w)) \tag{6.1.11}$$

for all $v, w \in V[\![\lambda]\!]$.

ii.) The λ-adic metric is an ultrametric, i.e. a metric satisfying the strong triangle inequality

$$d(v, w) \leq \max(d(v, u), d(u, w)) \tag{6.1.12}$$

for all $v, w, u \in V[\![\lambda]\!]$.

iii.) With respect to the λ-adic metric, $V[\![\lambda]\!]$ is a complete metric space and the polynomials $V[\lambda] \subseteq V[\![\lambda]\!]$ are dense. In fact, we have the convergence

$$\lim_{N \longrightarrow \infty} \sum_{r=0}^{N} \lambda^r v_r = \sum_{r=0}^{\infty} \lambda^r v_r \tag{6.1.13}$$

for all $v \in V[\![\lambda]\!]$. The induced topology on $V \subseteq V[\![\lambda]\!]$ is discrete.

iv.) The ring $\mathsf{R}[\![\lambda]\!]$ *is a topological ring and the module structure of* $V[\![\lambda]\!]$ *is continuous. More precisely, one has*

$$o(av) \geq o(a) + o(v) \quad \text{and} \quad \varphi(av) \leq \varphi(a)\varphi(v) \tag{6.1.14}$$

for all elements $a \in \mathsf{R}[\![\lambda]\!]$ *and* $v \in V[\![\lambda]\!]$.

v.) All $\mathsf{R}[\![\lambda]\!]$*-multilinear maps*

$$\Phi\colon V_1[\![\lambda]\!] \times \cdots \times V_n[\![\lambda]\!] \longrightarrow W[\![\lambda]\!] \tag{6.1.15}$$

are continuous with respect to the λ*-adic metric topologies.*

We refer to Exercise 6.4.3 for more details on the rather straightforward proof.

6.1.2 Basic Definitions and Examples

We are now able to make the heuristic considerations in the introduction of this section more precise. One defines a formal associative deformation of an associative algebra as follows [53]:

Definition 6.1.3 (Associative formal deformation) *Let* $\mathscr{A}$ *be an associative algebra over a ring* R *with multiplication* $\mu_0\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$. *Then an associative formal deformation of* $\mathscr{A}$ *is an associative* $\mathsf{R}[\![\lambda]\!]$*-bilinear multiplication* μ *for* $\mathscr{A}[\![\lambda]\!]$ *such that*

$$\operatorname{cl}(\mu) = \mu_0. \tag{6.1.16}$$

Let us unwind this definition. By Proposition 6.1.1 we know that any $\mathsf{R}[\![\lambda]\!]$-bilinear map $\mu\colon \mathscr{A}[\![\lambda]\!] \times \mathscr{A}[\![\lambda]\!] \longrightarrow \mathscr{A}[\![\lambda]\!]$ is actually of the form

$$\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r \tag{6.1.17}$$

with some uniquely determined R-bilinear maps $\mu_r\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$. Then the requirement (6.1.16) simply means that the zeroth order of μ is the original, given multiplication of $\mathscr{A}$, thereby justifying our sloppy notation. The associativity is a quadratic condition on μ meaning $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ for all $a, b, c \in \mathscr{A}[\![\lambda]\!]$. Inserting (6.1.17) we can evaluate this order by order and obtain the equivalent conditions

$$\sum_{s=0}^{r} \mu_s(\mu_{r-s}(a, b), c) = \sum_{s=0}^{r} \mu_s(a, \mu_{r-s}(b, c)), \tag{6.1.18}$$

which have to hold for all $r \in \mathbb{N}_0$ and $a, b, c \in \mathscr{A}$. If these equations only hold for $r = 0, \ldots, n$ then μ is called an associative deformation *up to order* n. Note that it is sufficient to check this for elements $a, b, c \in \mathscr{A} \subseteq \mathscr{A}[\![\lambda]\!]$ without higher orders of λ.

We will typically denote the new multiplication by a product symbol $\star$ and thus write

$$a \star b = \mu(a, b) = ab + \sum_{r=1}^{\infty} \lambda^r \mu_r(a, b), \tag{6.1.19}$$

where we simply write $ab = \mu_0(a, b)$ for the original multiplication of $\mathscr{A}$. If $\mathscr{A}$ is unital with unit element $\mathbb{1}$ we typically require that the same element $\mathbb{1} \in \mathscr{A}[\![\lambda]\!]$ serves as unit for $\star$ as well. This means that

$$\mathbb{1} \star a = a = a \star \mathbb{1} \tag{6.1.20}$$

for all $a \in \mathscr{A}[\![\lambda]\!]$ or, equivalently, $\mu_r(\mathbb{1}, \cdot\,) = 0 = \mu_r(\,\cdot\,, \mathbb{1})$ for all $r \geq 1$.

There is of course always one particular deformation, namely the *trivial deformation* where we simply have $\mu = \mu_0$ extended $\mathsf{R}[\![\lambda]\!]$-linearly to $\mathscr{A}[\![\lambda]\!]$.

Suppose that $\star$ is a formal associative deformation. Consider then a formal power series $T = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ of linear maps $T_r \in \mathrm{End}_{\mathsf{R}}(\mathscr{A})$ viewed as $\mathsf{R}[\![\lambda]\!]$-linear endomorphism of $\mathscr{A}[\![\lambda]\!]$. Then the definition

$$a \star' b = T(T^{-1}(a) \star T^{-1}(b)) \tag{6.1.21}$$

yields a new associative product $\star'$ for $\mathscr{A}[\![\lambda]\!]$. Note that in general a formal series is invertible iff the zeroth order is invertible. Thus in our case, T^{-1} is well-defined. We call $\star$ and $\star'$ *equivalent* if there exists such a T. Clearly, one obtains an equivalence relation. The set of equivalence classes is then the space of formal deformations of $\mathscr{A}$:

Definition 6.1.4 ($\mathrm{Equiv}(\star, \star')$ **and** $\mathrm{Def}(\mathscr{A})$**)** *Let $\mathscr{A}$ be an associative algebra over* R.

i.) An equivalence transformation T from an associative deformation $\star$ to another associative deformation $\star'$ is an algebra isomorphism starting with id *in the zeroth order. In the unital case we additionally require $T\mathbb{1} = \mathbb{1}$.*

ii.) The set of all equivalence transformations from $\star$ to $\star'$ is denoted by

$$\mathrm{Equiv}(\star', \star) = \Big\{ T \in \mathrm{Iso}\left((\mathscr{A}[\![\lambda]\!], \star), (\mathscr{A}[\![\lambda]\!], \star')\right) \;\Big|\; \mathrm{cl}(T) = \mathrm{id} \Big\}. \tag{6.1.22}$$

iii.) Two associative deformations $\star$ and $\star'$ of $\mathscr{A}$ are called equivalent if there exists an equivalence transformation from $\star$ to $\star'$.

iv.) The set of equivalence classes $[\star]$ of associative deformations, denoted by

$$\mathrm{Def}(\mathscr{A}) = \big\{ [\star] \;\big|\; \star \text{ is an associative deformation of } \mathscr{A} \big\}, \tag{6.1.23}$$

is called the deformation theory of $\mathscr{A}$.

Remark 6.1.5 One way to interpret these definitions is that Equiv becomes a sub-groupoid of the isomorphism groupoid Iso and Def encodes the orbits of this smaller groupoid.

As usual for groupoids, it is of great importance to understand the self-equivalences $\mathrm{Equiv}(\star) = \mathrm{Equiv}(\star, \star)$, i.e. those automorphisms of a deformation $\star$ which are in addition equivalence transformations. Here the following proposition gives a quite satisfying answer in the case where $\mathbb{Q} \subseteq \mathsf{R}$. For us this will always be a reasonable assumption.

Proposition 6.1.6 *Let $\mathscr{A}$ be an associative algebra over a ring $\mathsf{R} \supseteq \mathbb{Q}$ and let $\star$ be a formal associative deformation of $\mathscr{A}$. Let $T = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ be a formal series of R-linear endomorphisms of $\mathscr{A}$.*

i.) There is a unique logarithm

$$D = \sum_{r=0}^{\infty} \lambda^r D_r \colon \mathscr{A}[\![\lambda]\!] \longrightarrow \mathscr{A}[\![\lambda]\!] \tag{6.1.24}$$

of T, i.e. $T = \exp(\lambda D)$.

ii.) The map T is an automorphism of $\star$ iff D is a derivation of $\star$.

Proof: We follow [27, Lem. 5]. The first part is clear since we can define D uniquely by the Taylor series of the logarithm

$$\lambda D = \log(T) = \log\left(\mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r\right) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s} \left(\sum_{r=1}^{\infty} \lambda^r T_r\right)^s,$$

which is a well-defined formal power series. In fact, it is easy to see that it converges in the λ-adic topology. Note that it is important to have $\mathbb{Q} \subseteq \mathsf{R}$ and $T_0 = \mathrm{id}$. We denote the possible failure of D being a derivation by

$$E(a, b) = D(a \star b) - D(a) \star b - a \star D(b),$$

where $a, b \in \mathscr{A}[\![\lambda]\!]$. By induction we find constants $c_{krst} \in \mathbb{Q}$ such that

$$D^k(a \star b) = \sum_{\ell=0}^{k} \binom{k}{\ell} D^\ell(a) \star D^{k-\ell}(b) + \sum_{r,s,t=0}^{k-1} c_{krst} D^r(E(D^s(a), D^t(b))).$$

The precise combinatorics is not relevant for the following. We can now compute the possible failure of T being an automorphism and get

$$\begin{aligned} T(a \star b) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} D^k(a \star b) \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} D^\ell(a) \star D^{k-\ell}(b) + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{r,s,t=0}^{k-1} c_{krst} D^r(E(D^s(a), D^t(b))) \\ &= T(a) \star T(b) + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{r,s,t=0}^{k-1} c_{krst} D^r(E(D^s(a), D^t(b))). \end{aligned}$$

Hence T is an automorphism iff the second contribution vanishes. Now the second term will not contribute to order zero and the first order in λ is just $E(a, b)$. Hence T is an automorphism iff

$$E(a, b) = -\sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \sum_{r,s,t=0}^{k-1} c_{krst} D^r(E(D^s(a), D^t(b))). \tag{$*$}$$

We can now solve this equation by recursion: more precisely, we know that E is an $\mathsf{R}[\![\lambda]\!]$-bilinear map and hence

$$E \in \mathrm{Hom}_{\mathsf{R}[\![\lambda]\!]}(\mathscr{A}[\![\lambda]\!], \mathscr{A}[\![\lambda]\!]; \mathscr{A}[\![\lambda]\!]) = \mathrm{Hom}_{\mathsf{R}}(\mathscr{A}, \mathscr{A}; \mathscr{A})[\![\lambda]\!],$$

where we have used the identification from Proposition 6.1.1. Now the space on the right hand side is a complete metric space for the λ-adic metric. Moreover, E can be seen as a solution of a fixed point equation $E = \phi(E)$ with a map ϕ determined by the right hand side of $(*)$. Since the right hand side of $(*)$ starts with at least one more power of λ, the map ϕ is easily seen to be contracting for the λ-adic metric: by Banach's fixed point theorem we have a unique fixed point, which is of course $E = 0$ since ϕ is linear, see also Exercise 6.4.5. Hence T is an automorphism iff $E = 0$ iff D is a derivation. □

Remark 6.1.7 This proposition makes sense out of the heuristic argument that the exponential of a derivation is an automorphism: in general we can not exponentiate maps easily. This requires always an analytic argument. However, in the present case, the λ-adic topology can take care of that in a rather trivial way.

In many cases the original algebra $\mathscr{A}$ is not only associative but also commutative. Of particular interest are then the *non-commutative* deformations $\star$, but also commutative deformations are considered where $\mu_r(a, b) = \mu_r(b, a)$ holds. The antisymmetric part of the first order of a deformation of a commutative algebra is necessarily a Poisson bracket:

Proposition 6.1.8 *Let $(\mathscr{A}, \mu_0)$ be an associative and commutative algebra over* R. *Suppose that $\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r$ is an associative deformation of μ_0. Then*

$$\{a, b\} = \mu_1(a, b) - \mu_1(b, a) \tag{6.1.25}$$

defines a Poisson bracket for $\mathscr{A}$.

PROOF: The antisymmetry and R-bilinearity is clear. For the Leibniz rule one considers the Leibniz rule for the $\star$-commutator $[a, b \star c]_\star = [a, b]_\star \star c + b \star [a, c]_\star$ in lowest non-trivial order. The Jacobi identity then follows from the Jacobi identity of the $\star$-commutator by considering the second order of $[a, [b, c]_\star]_\star = [[a, b]_\star, c]_\star + [b, [a, c]_\star]_\star$. Note that it is crucial that μ_0 is commutative. ☐

Remark 6.1.9 (Quantization) This observation leads immediately to the question, whether or not a *given* Poisson bracket $\{\cdot\,, \cdot\}$ on a commutative algebra $\mathscr{A}$ actually occurs as the anti-symmetric part of the first order of an associative deformation of $\mathscr{A}$. In general, this is a highly non-trivial question, depending on the details of the situation. Remarkably, for the algebra $\mathscr{C}^\infty(M)$ of smooth functions on a smooth manifold M, Kontsevich was able to answer the question in the positive: every Poisson bracket can be *quantized* into an associative product $\star$, see [77]. This is the basic task in deformation quantization. Before Kontsevich's ground-breaking results the existence and classfication of star products for more particular Poisson manifolds, most notably symplectic manifolds, where found, see e.g. [43, 49, 50, 94], and classified, see [8, 9, 42, 61, 91, 119].

Remark 6.1.10 In case of a commutative algebra $\mathscr{A}$ it is easy to see that equivalent deformations lead to the *same* Poisson bracket, see Exercise 6.4.6. In this case, we denote the equivalence classes of deformations $\star$ of $\mathscr{A}$ for a fixed Poisson structure $\{\cdot\,, \cdot\}$ by

$$\mathrm{Def}(\mathscr{A}, \{\cdot\,, \cdot\}) = \big\{[\star] \bigm| \star \text{ induces the Poisson bracket } \{\cdot\,, \cdot\}\big\}. \tag{6.1.26}$$

Note however, that the first order part of an equivalent deformation $\star'$ can very well differ from the original deformation $\star$: the equivalence transformation may change the symmetric part.

Remark 6.1.11 There is of course no need to stop with associative algebras: one can equally well speak about formal deformations of other types of algebras like e.g. Lie algebras. Then the corresponding condition would be an order-by-order evaluation of the Jacobi identity instead of the associativity (6.1.18). But we also can deform e.g. Poisson brackets on an associative commutative algebra as Poisson brackets.

We conclude this section with one typical and yet extremely important example of an associative deformation. The construction has been used in the literature in many contexts at many places. The first appearance is due to Gerstenhaber [55, Thm. 8], see also Exercise 6.4.8 for some slight variations on this theme:

Proposition 6.1.12 *Let $(\mathscr{A}, \mu_0)$ be an associative algebra over R where we assume $\mathbb{Q} \subseteq \mathsf{R}$. Let $D_1, \ldots, D_n$, $E_1, \ldots, E_n$ be pairwise commuting derivations of $\mathscr{A}$. Then*

$$a \star b = \mu_0 \circ \mathrm{e}^{\lambda P}(a \otimes b) \quad \textit{with} \quad P = \sum_{k=1}^{n} D_n \otimes E_n \tag{6.1.27}$$

defines an associative deformation $\star$ of μ_0 where $a, b \in \mathscr{A}[\![\lambda]\!]$.

Proof: For convenience we sketch the proof. We identify the bilinear maps with linear ones on the corresponding tensor products. Consider then the auxiliary maps

$$P_{12} = P \otimes \mathrm{id}, \quad P_{13} = \sum_{k=1}^{n} D_n \otimes \mathrm{id} \otimes E_n, \quad \text{and} \quad P_{23} = \mathrm{id} \otimes P,$$

viewed as endomorphisms of $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}$. Since the maps D_k and E_k are assumed to be derivations of μ_0 we get the Leibniz rules

$$P \circ (\mathrm{id} \otimes \mu_0) = (\mathrm{id} \otimes \mu_0) \circ (P_{12} + P_{13}) \quad \text{and} \quad P \circ (\mu_0 \otimes \mathrm{id}) = (\mu_0 \otimes \mathrm{id}) \circ (P_{13} + P_{23}).$$

Moreover, since all the derivations commute pairwise, the three maps P_{12}, P_{13}, and P_{23} mutually commute, too. These two facts allow to compute

$$a \star (b \star c) = \mu_0 \circ (\mathrm{id} \otimes \mu_0) \circ \mathrm{e}^{\lambda(P_{12}+P_{13}+P_{23})}(a \otimes b \otimes c),$$

and analogously for $(a \star b) \star c$. Since μ_0 is associative, the claim follows. □

6.1.3 Hochschild Cohomology I

Following Gerstenhaber [52–56] we outline now briefly how the deformation problem of an associative algebra can be formulated using the Hochschild cohomology of the algebra. The following results will essentially not be needed and are thus only included for completeness. We will thus be rather brief and omit many of the details and proofs. More complete discussions can be found e.g. in the textbooks [47] or [116, Sect. 6.2]. The role of Hochschild cohomology beyond deformation theory is also discussed in e.g. the monographs [84], [59, Sect. 8.4], [36, Chap. IX], [118, Chap. 9], or [66, Sect. 6.11].

First we recall the definition of the *Hochschild complex* of $\mathscr{A}$:

Definition 6.1.13 (Hochschild complex I) *Let $\mathscr{A}$ be an associative algebra over* R.

i.) The Hochschild complex $\mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$ *of $\mathscr{A}$ is*

$$\mathrm{HC}^n(\mathscr{A}, \mathscr{A}) = \mathrm{Hom}_{\mathsf{R}}\big(\underbrace{\mathscr{A}, \ldots, \mathscr{A}}_{n \text{ times}}; \mathscr{A}\big) \tag{6.1.28}$$

with n-linear maps from n copies of $\mathscr{A}$ with values in $\mathscr{A}$ again. For $n = 0$ we set $\mathrm{HC}^0(\mathscr{A}, \mathscr{A}) = \mathscr{A}$.

ii.) The Hochschild differential $\delta\colon \mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A}) \longrightarrow \mathrm{HC}^{\bullet+1}(\mathscr{A}, \mathscr{A})$ *is defined by*

$$\begin{aligned}(\delta\phi)&(a_1, \ldots, a_{k+1}) \\ &= a_1\phi(a_2, \ldots, a_{k+1}) + \sum_{r=1}^{k}(-1)^r \phi(a_1, \ldots, a_r a_{r+1}, \ldots, a_{k+1}) + (-1)^{k+1}\phi(a_1, \ldots, a_k)a_{k+1},\end{aligned} \tag{6.1.29}$$

where $a_1, \ldots, a_{k+1} \in \mathscr{A}$ and $\phi \in \mathrm{HC}^k(\mathscr{A}, \mathscr{A})$.

iii.) The Hochschild cohomology of $\mathscr{A}$ is defined by

$$\mathrm{HH}^\bullet(\mathscr{A}, \mathscr{A}) = \bigoplus_{k=0}^{\infty} \mathrm{HH}^k(\mathscr{A}, \mathscr{A}) \quad \textit{with} \quad \mathrm{HH}^k(\mathscr{A}, \mathscr{A}) = \frac{\ker\left(\delta\big|_{\mathrm{HC}^k(\mathscr{A}, \mathscr{A})}\right)}{\delta(\mathrm{HC}^{k-1}(\mathscr{A}, \mathscr{A}))}. \tag{6.1.30}$$

Of course, we first have to show that $\delta^2 = 0$ in order to have a well-defined cohomology theory. We postpone this for a moment and investigate the lowest Hochschild cohomologies directly. For $a \in \mathrm{HC}^0(\mathscr{A}, \mathscr{A}) = \mathscr{A}$ we have

$$(\delta a)(b) = ba - ab = -\operatorname{ad}(a)b \tag{6.1.31}$$

for $b \in \mathscr{A}$. Hence $\delta a = 0$ iff a is *central*. Since there are no δ-exact terms in this degree we obtain

$$\mathrm{HH}^0(\mathscr{A}, \mathscr{A}) = \mathscr{Z}(\mathscr{A}). \tag{6.1.32}$$

For $k = 1$ we have

$$(\delta D)(a, b) = aD(b) - D(ab) + D(a)b \tag{6.1.33}$$

for $D \in \mathrm{HC}^1(\mathscr{A}, \mathscr{A}) = \mathrm{End}_{\mathsf{R}}(\mathscr{A})$ and $a, b \in \mathscr{A}$. Thus we have $\delta D = 0$ iff D is a *derivation*. With (6.1.31) it follows that the first Hochschild cohomology is given by the quotient

$$\mathrm{HH}^1(\mathscr{A}, \mathscr{A}) = \frac{\mathrm{Der}(\mathscr{A})}{\mathrm{InnDer}(\mathscr{A})} = \mathrm{OutDer}(\mathscr{A}), \tag{6.1.34}$$

i.e. the *outer derivations* of $\mathscr{A}$.

Instead of directly proving that δ is indeed a differential, i.e. $\delta^2 = 0$, we follow Gerstenhaber [52] and introduce some more structure on the Hochschild complex. The key observation is now that we have a super Lie algebra structure on $\mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$ if we shift the degree by one:

Definition 6.1.14 (Gerstenhaber bracket) *Let $\mathscr{A}$ be a module over a ring R. Moreover, let $\phi \in \mathrm{HC}^{k+1}(\mathscr{A}, \mathscr{A})$ and $\psi \in \mathrm{HC}^{\ell+1}(\mathscr{A}, \mathscr{A})$.*

i.) One defines the degree of ϕ to be $\deg(\phi) = k$.

ii.) Let $i = 0, \ldots, k$. Then the insertion of ψ into ϕ after the i-th position is the map $\phi \circ_i \psi \in \mathrm{HC}^{k+\ell+1}(\mathscr{A}, \mathscr{A})$ defined by

$$(\phi \circ_i \psi)(a_1, \ldots, a_{k+\ell+1}) = \phi(a_1, \ldots, a_i, \psi(a_{i+1}, \ldots, a_{i+\ell+1}), a_{i+\ell+2}, \ldots, a_{k+\ell+1}), \tag{6.1.35}$$

where $a_1, \ldots, a_{k+\ell+1} \in \mathscr{A}$.

iii.) The Gerstenhaber product $\phi \circ \psi \in \mathrm{HC}^{k+\ell+1}(\mathscr{A}, \mathscr{A})$ of ϕ and ψ is defined by

$$\phi \circ \psi = \sum_{i=0}^{\deg(\phi)} (-1)^{i \deg(\psi)} \phi \circ_i \psi. \tag{6.1.36}$$

iv.) The Gerstenhaber bracket $[\phi, \psi] \in \mathrm{HC}^{k+\ell+1}(\mathscr{A}, \mathscr{A})$ of ϕ and ψ is defined by

$$[\phi, \psi] = \phi \circ \psi - (-1)^{\deg(\phi)\deg(\psi)} \psi \circ \phi. \tag{6.1.37}$$

It is clear that the Gerstenhaber product and the Gerstenhaber bracket are graded with respect to the shifted degree deg but not with respect to the original grading of $\mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$ by the number of arguments. It turns out that the Gerstenhaber bracket is a graded Lie bracket even though the Gerstenhaber product is *not* associative:

Proposition 6.1.15 *Let $\mathscr{A}$ be a module over R.*

i.) The Gerstenhaber product is not associative but satisfies the identity

$$(\phi \circ \psi) \circ \chi - \phi \circ (\psi \circ \chi) = (-1)^{\deg(\psi)\deg(\chi)} ((\phi \circ \chi) \circ \psi - \phi \circ (\chi \circ \psi)) \tag{6.1.38}$$

for homogeneous elements $\phi, \psi, \chi \in \mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$.

ii.) The Gerstenhaber bracket is a graded Lie bracket with respect to deg*, i.e. we have graded antisymmetry and the graded Jacobi identity*

$$[\phi, [\psi, \chi]] = [[\phi, \psi], \chi] + (-1)^{\deg(\phi)\deg(\psi)} [\psi, [\phi, \chi]] \tag{6.1.39}$$

for homogeneous elements $\phi, \psi, \chi \in \mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$.

PROOF: We omit the proof which can be found at many places in the literature. The original and quite direct approach of Gerstenhaber can be found in [52] see also [116, Sect. 6.2] and Exercise 6.4.9. Now one has more elaborate techniques to minimize the actual computations needed to show the identity (6.1.38), see e.g. the operadic approaches in [84]. □

The relevance of the Gerstenhaber bracket comes now from the observation that a bilinear map $\mu\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$, which we view as element in $\mathrm{HC}^2(\mathscr{A}, \mathscr{A})$ of degree $\deg(\mu) = 1$, is an *associative* multiplication iff $\mu \circ \mu = 0$. Indeed, an explicit evaluation of (6.1.36) shows that

$$(\mu \circ \mu)(a, b, c) = \sum_{i=0}^{1} (-1)^i (\mu \circ_i \mu)(a, b, c) = \mu(\mu(a, b), c) - \mu(a, \mu(b, c)) \tag{6.1.40}$$

for $a, b, c \in \mathscr{A}$. If we assume $\frac{1}{2} \in \mathsf{R}$ then this is equivalent to the condition

$$[\mu, \mu] = 0, \tag{6.1.41}$$

since $\deg(\mu) = 1$ and thus the graded commutator becomes $[\mu, \mu] = \mu \circ \mu + \mu \circ \mu = 2\mu \circ \mu$.

Now suppose that $\mathscr{A}$ is not just an R-module but even an associative algebra with multiplication μ. Then the Hochschild differential δ can be written as follows:

Lemma 6.1.16 *Let $\mathscr{A}$ be an associative algebra over R.*

i.) One has

$$\delta\phi = (-1)^{\deg(\phi)} [\mu, \phi] = -[\phi, \mu] \tag{6.1.42}$$

for all $\phi \in \mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$.

ii.) One has $\delta^2 = 0$.

PROOF: The first statement is just a direct evaluation. Then the second is clear from the graded Jacobi identity and $[\mu, \mu] = 0$. □

Remark 6.1.17 This gives a more conceptual proof of $\delta^2 = 0$. We have chosen the traditional definition for δ. From the point of view of Lemma 6.1.16 it would perhaps be more natural to define δ as $[\mu, \cdot\,]$ causing some additional signs in (6.1.29). Of course, the cohomologies will not change at all. Moreover, the Hochschild differential satisfies now a Leibniz rule with respect to the Gerstenhaber bracket since it is even an inner derivation. Hence the bracket is well-defined also in the Hochschild cohomology $\mathrm{HH}^\bullet(\mathscr{A}, \mathscr{A})$ which therefore becomes a graded Lie algebra as well.

We will now use the Hochschild cohomology to describe obstructions for associative deformations of a given product μ_0 on $\mathscr{A}$. For simplicity we assume $\frac{1}{2} \in \mathsf{R}$ from now on. Then the fundamental idea is the following: if μ_0 is associative and $\mu_1, \mu_2, \ldots \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})$ are given then

$$\mu = \mu_0 + \lambda\mu_1 + \lambda^2\mu_2 + \cdots \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})[\![\lambda]\!] \tag{6.1.43}$$

is a formal associative deformation iff $[\mu, \mu] = 0$. Expanding this in powers of λ we get the associativity conditions

$$0 = \sum_{r=0}^{k} [\mu_r, \mu_{k-r}] = [\mu_0, \mu_k] + \sum_{r=1}^{k-1} [\mu_r, \mu_{k-r}] + [\mu_k, \mu_0] = -2\delta\mu_k + \sum_{r=1}^{k-1} [\mu_r, \mu_{k-r}] \tag{6.1.44}$$

in each order $k \geq 1$. Note that for $\mu_i, \mu_j \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})$ the Gerstenhaber bracket is symmetric $[\mu_i, \mu_j] = [\mu_j, \mu_i]$.

Heading for a recursive construction this means that for already found $\mu_1, \ldots, \mu_{k-1}$ we want to find μ_k such that

$$\delta\mu_k = \frac{1}{2}\sum_{r=1}^{k-1}[\mu_r, \mu_{k-r}]. \tag{6.1.45}$$

The obvious necessary condition to find a solution of this equation is that the right hand side is δ-closed. This is always the case since for any μ, whether associative or not, we have $[\mu, [\mu, \mu]] = 0$ by the graded Jacobi identity. Now if μ was associative up to order λ^{k-1} then the lowest non-trivial term in this cubic equation is

$$0 = \sum_{r=0}^{k}[\mu_0, [\mu_r, \mu_{k-r}]] = [\mu_0, [\mu_0, \mu_k]] + \sum_{r=1}^{k-1}[\mu_0, [\mu_r, \mu_{k-r}]] + [\mu_0, [\mu_k, \mu_0]]. \tag{6.1.46}$$

Since $[\mu_k, \mu_0] = [\mu_0, \mu_k]$ and $[\mu_0, [\mu_0, \cdot\,]] = 0$ by the associativity of μ_0 we end up with the middle part being zero. But this is precisely

$$\delta\sum_{r=1}^{k-1}[\mu_r, \mu_{k-r}] = 0. \tag{6.1.47}$$

Hence the necessary condition for (6.1.45) is fulfilled and we are left with a cohomological problem:

Proposition 6.1.18 *Let $\mathscr{A}$ be an associative algebra over R with product μ_0. Moreover, assume that $\mu^{(k-1)} = \mu_0 + \lambda\mu_1 + \cdots + \lambda^{k-1}\mu_{k-1}$ with $\mu_1, \ldots, \mu_{k-1} \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})$ is an associative deformation of μ_0 up to order $k-1$ where $k \in \mathbb{N}$.*

i.) The error term for associativity in order k is δ-closed, i.e.

$$\delta\sum_{r=1}^{k-1}[\mu_r, \mu_{k-r}] = 0. \tag{6.1.48}$$

ii.) There exists a $\mu_k \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})$ such that $\mu^{(k)} = \mu^{(k-1)} + \lambda^k\mu_k$ is associative up to order k iff the error term is exact, i.e.

$$\delta\mu_k = \frac{1}{2}\sum_{r=1}^{k-1}[\mu_r, \mu_{k-r}]. \tag{6.1.49}$$

Thus the obstruction for associative deformations of μ_0 is controlled by the *third Hochschild cohomology* $\mathrm{HH}^3(\mathscr{A}, \mathscr{A})$. Now in typical examples the Hochschild cohomology is known to be nontrivial: thus we do have to expect obstructions. From this point of view the above analysis is rather disappointing since it only tells us that there might be problems in continuing a deformation up to the next order. The recursive approach will of course not say anything about absolute obstructions since it might or might not happen that the error term is exact.

In a second step we want to show how the question of equivalence of formal deformations can be formulated as a cohomological problem. To this end, suppose that $\mu = \mu_0 + \lambda\mu_1 + \cdots$ and $\tilde{\mu} = \mu_0 + \lambda\tilde{\mu}_1 + \cdots$ are two associative deformations of the same classical limit μ_0. If μ and $\tilde{\mu}$ are equivalent up to order $k-1$ then there is an equivalence transformation $T = \mathrm{id} + \lambda T_1 + \cdots$ such that the new multiplication $\mu'(a, b) = T^{-1}\mu(Ta, Tb)$ coincides with $\tilde{\mu}$ up to order $k-1$ and is equivalent, by construction, to μ up to all orders. Thus we can assume that $\tilde{\mu}$ is already of this form: we have $\tilde{\mu}_r = \mu_r$ for all $r = 1, \ldots, k-1$. Then the question is whether one can find an equivalence T between μ and $\tilde{\mu}$ up to one order higher. The natural Ansatz would be to consider a $T = \mathrm{id} + \lambda^k T_k + \cdots$

to accomplish this. We have $T^{-1} = \mathrm{id} - \lambda^k T_k + \cdots$ by the usual geometric series. Evaluating the equivalence condition in k-th order gives

$$\tilde{\mu}_k(a, b) = -T_k(\mu_0(a, b)) + \mu_k(a, b) + \mu_0(T_k a, b) + \mu_0(a, T_k b) \tag{6.1.50}$$

for $a, b \in \mathscr{A}$. Again, we can rephrase this as

$$\delta T_k = \tilde{\mu}_k - \mu_k. \tag{6.1.51}$$

With a similar argument as above one shows that the difference $\tilde{\mu}_k - \mu_k$ is necessarily closed whenever μ and $\tilde{\mu}$ are both associative deformations which coincide up to order $k - 1$. Hence we again have a cohomological problem:

Proposition 6.1.19 *Let μ and $\tilde{\mu}$ be associative deformations of an associative algebra $\mathscr{A}$ over* R *which coincide up to order $k - 1$.*

i.) The difference $\tilde{\mu}_k - \mu_k$ is δ-closed.

ii.) There exists an equivalence transformation between μ and $\tilde{\mu}$ up to order k of the form $T = \mathrm{id} + \lambda^k T_k + \cdots$ iff the Hochschild cohomology class of $\tilde{\mu}_k - \mu_k$ is trivial.

Thus the *second* Hochschild cohomology $\mathrm{HH}^2(\mathscr{A}, \mathscr{A})$ is the source of obstructions to equivalence of deformations. However, the above proposition will not directly give a classification of inequivalent deformations as there might be other equivalence transformations T up to order k which are not of the specified form: lower order terms can very well contribute to establish an equivalence which can not be achieved by transformations of the form $T = \mathrm{id} + \lambda^k T_k + \cdots$. However, for the start point $k = 1$ the above Proposition gives a complete classification of the possible inequivalent infinitesimal deformations: they are parameterized by the second Hochschild cohomology.

Again, in most relevant examples the second Hochschild cohomology is nontrivial and hence there is the chance to get some non-equivalent deformations. Note that if $\mathrm{HH}^2(\mathscr{A}, \mathscr{A}) = \{0\}$ is trivial then any two deformations would be equivalent and thus equivalent to the undeformed multiplication μ_0. Such an algebra would be absolutely *rigid*.

6.1.4 Hermitian Deformations

In a next step we specialize deformation theory to *-algebras. Thus let R be an ordered ring with $\mathsf{C} = \mathsf{R}(\mathrm{i})$ as usual. Then the first important observation is that $\mathsf{R}[\![\lambda]\!]$ is still ordered, see Example 1.1.3, *ii.)*. Thus we stay within the correct framework of algebras over ordered rings when we pass to formal deformations. Of course we have canonically $(\mathsf{R}[\![\lambda]\!])(\mathrm{i}) = \mathsf{C}[\![\lambda]\!]$.

Now if $\mathscr{A}$ is a *-algebra over R then we can deform two algebraic structures, the multiplication and the *-involution. If we denote the *-involution by $I_0 \colon \mathscr{A} \longrightarrow \mathscr{A}$ then a deformation of I_0 would be a $\mathsf{C}[\![\lambda]\!]$-antilinear map

$$I = I_0 + \lambda I_1 + \cdots = \sum_{r=0}^{\infty} \lambda^r I_r \tag{6.1.52}$$

with corresponding antilinear maps $I_r \colon \mathscr{A} \longrightarrow \mathscr{A}$. This leads to the following definition [24]:

Definition 6.1.20 (Hermitian deformation) *Let $(\mathscr{A}, \mu_0, I_0)$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with an ordered ring* R.

i.) A *-algebra deformation $(\mathscr{A}[\![\lambda]\!], \mu, I)$ of $(\mathscr{A}, \mu_0, I_0)$ is a *-algebra over $\mathsf{C}[\![\lambda]\!]$ such that $\mathrm{cl}(\mu) = \mu_0$ and $\mathrm{cl}(I) = I_0$.

ii.) A Hermitian deformation of $\mathscr{A}$ is a *-algebra deformation where in addition $I = I_0$, i.e. the *-involution stays undeformed.

Note that a minor adaption of Proposition 6.1.1 shows that $\mathsf{C}[\![\lambda]\!]$-antilinear maps $I\colon \mathscr{A}[\![\lambda]\!] \longrightarrow \mathscr{A}[\![\lambda]\!]$ are necessarily of the form (6.1.52) with C-antilinear maps I_r. Thus we can always speak of the zeroth order of I.

If the reference to $\star$ is clear from the context, we abbreviate the deformed *-algebra also by

$$\boldsymbol{\mathscr{A}} = \big(\mathscr{A}[\![\lambda]\!], \star\big). \tag{6.1.53}$$

Moreover, by some slight abuse of language we will speak of $\star$ as the Hermitian deformation of $\mathscr{A}$.

Remark 6.1.21 Another way to phrase the definition is to say that the classical limit map

$$\mathrm{cl}\colon (\mathscr{A}[\![\lambda]\!], \mu, I) \longrightarrow (\mathscr{A}, \mu_0, I_0) \tag{6.1.54}$$

is a *-homomorphism of *-algebras along the ring morphism $\mathrm{cl}\colon \mathsf{C}[\![\lambda]\!] \longrightarrow \mathsf{C}$: the *-homomorphism property of cl means

$$\mathrm{cl}(za + wb) = z_0 a_0 + w_0 b_0 = \mathrm{cl}(z)\mathrm{cl}(a) + \mathrm{cl}(w)\mathrm{cl}(b) \tag{6.1.55}$$

$$\mathrm{cl}(\mu(a, b)) = \mu_0(a_0, b_0) = \mathrm{cl}(\mu)(\mathrm{cl}(a), \mathrm{cl}(b)) \tag{6.1.56}$$

$$\mathrm{cl}(I(a)) = I_0(a_0) = \mathrm{cl}(I)(\mathrm{cl}(a)) \tag{6.1.57}$$

for all $a, b \in \mathscr{A}[\![\lambda]\!]$ and $z, w \in \mathsf{C}[\![\lambda]\!]$. We will make constantly use of these facts without further mentioning. In particular, the classical limit of Hermitian, normal, unitary, or isometric elements, respectively, are again Hermitian, normal, unitary, or isometric, respectively. Also the classical limit of projections are projections again.

Remark 6.1.22 Hermitian deformations are sometimes also called *symmetric* deformations in the context of star products [5]. There relevance comes e.g. from quantization theory: here we want to keep the fact that certain elements (the Hermitian ones) of an observable algebra are observables in the physical sense: both in the classical, i.e. undeformed case, and in the quantum, i.e. deformed case. In such approaches to quantization one of the main advantages is that we have the same underlying space, namely $\mathscr{A}[\![\lambda]\!]$, for the observable algebra: thus we can directly and trivially identify the physical meaning of elements in $\mathscr{A}[\![\lambda]\!]$ by their classical interpretation. It is only the product which changes when passing from classical to quantum. Of course, such an interpretation is only possible for a Hermitian deformation, a *-algebra deformation would not be sufficient.

Remark 6.1.23 (Hochschild cohomology of *-algebras) It is fairly straightforward to extend the *-involution of $\mathscr{A}$ to the Hochschild complex of $\mathscr{A}$. This way, one can extend the previous cohomological discussion of the deformation problem also to Hermitian deformations, see e.g. [24] and Exercise 6.4.13. However, we shall not take this point of view here.

In the following we will mainly be interested in Hermitian deformations. This allows to write again $a \mapsto a^*$ for the involution and abandon the more clumsy notation I_0. Since from the definition of $\mathsf{C}[\![\lambda]\!]$ it is clear that

$$\overline{\lambda} = \lambda, \tag{6.1.58}$$

an associative deformation μ is Hermitian iff

$$\mu_r(a, b)^* = \mu_r(b^*, a^*) \tag{6.1.59}$$

holds for all $a, b \in \mathscr{A}$ and $r \in \mathbb{N}$.

The following technical lemma shows that not only the classical limit of certain types of elements is preserved, but we can go the other direction and deform squares and unitaries [23, Lem. 2.1, Cor. 2.2]:

Lemma 6.1.24 *Let $\star$ be a Hermitian deformation of a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and assume $\frac{1}{2} \in \mathsf{R}$.*

i.) Let $b_0 \in \mathscr{A}$ be invertible and let $a = \sum_{r=0}^{\infty} \lambda^r a_r = a^* \in \mathscr{A}[\![\lambda]\!]$ with $a_0 = b_0^* b_0$ be given. Then there exist elements $b_1, b_2, \ldots \in \mathscr{A}$ such that

$$a = b^* \star b, \tag{6.1.60}$$

where $b = \sum_{r=0}^{\infty} \lambda^r b_r$.

ii.) Let $u_0 \in \mathscr{A}$ be unitary. Then there exists a unitary $u \in \boldsymbol{\mathscr{A}}$ with $\mathrm{cl}(u) = u_0$.

PROOF: We construct b recursively. Suppose that $b_0, \ldots, b_{k-1} \in \mathscr{A}$ are found in such a way that $b^{(k-1)} = b_0 + \cdots + \lambda^{k-1} b_{k-1}$ satisfies $a - (b^{(k-1)})^* \star b^{(k-1)} = \lambda^k c_k + \cdots$. Since $a = a^*$ is Hermitian also $c_k = c_k^*$ is Hermitian. We want to find b_k such that the corresponding $b^{(k)} = b^{(k-1)} + \lambda^k b_k$ has a square $(b^{(k)})^* \star b^{(k)}$ which coincides with a up to one order higher, i.e. up to order λ^{k+1}. Collecting the terms in order λ^k gives the necessary and sufficient condition $b_k^* b_0 + b_0^* b_k = c_k$. This equation can now be solved by $b_k = \frac{1}{2}(c_k b_0^{-1})^*$. Induction gives then the first part. The second is clear as we can apply the first part to $\mathbb{1} = u_0^* u_0$. □

Note, however, that in general neither b nor u are uniquely determined: if $v = \sum_{r=0}^{\infty} \lambda^r v_r$ is unitary with respect to $\star$ then $v \star b$ also solves $a = (v \star b)^* \star (v \star b)$. In particular, there will be many unitary elements v starting with $v_0 = \mathbb{1}$. This also gives a large freedom in the second case, see also Exercise 6.4.14.

Having Hermitian deformations $\star$ and $\star'$ of $\mathscr{A}$ we can refine our notion of equivalence to *-equivalence: two Hermitian deformations $\star$ and $\star'$ are called **-equivalent* if there is an equivalence transformation $T = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ with (6.1.21) such that in addition

$$T(a^*) = T(a)^* \tag{6.1.61}$$

holds for all $a \in \mathscr{A}[\![\lambda]\!]$. Equivalently, this means $T_r(a^*) = T_r(a)^*$ for all $a \in \mathscr{A}$ and $r \in \mathbb{N}$. The set of all *-equivalence transformations is then denoted by $\mathrm{Equiv}^*(\star', \star)$. Again, we obtain an equivalence relation. The corresponding set of equivalence classes

$$\mathrm{Def}^*(\mathscr{A}) = \left\{ [\star] \;\middle|\; \star \text{ is a Hermitian deformation of } \mathscr{A} \right\} \tag{6.1.62}$$

is called the *Hermitian deformation theory* of the *-algebra $\mathscr{A}$. Forgetting about the *-involution we get a well-defined map

$$\mathrm{Def}^*(\mathscr{A}) \longrightarrow \mathrm{Def}(\mathscr{A}) \tag{6.1.63}$$

for every *-algebra. Among Hermitian deformations the orbits of the two groupoids Equiv and Equiv* coincide [27, Cor. 4]:

Proposition 6.1.25 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathbb{Q} \subseteq \mathsf{R}$. Then two Hermitian deformations $\star$ and $\star'$ of $\mathscr{A}$ are equivalent iff they are *-equivalent. Hence the map* (6.1.63) *becomes an inclusion*

$$\mathrm{Def}^*(\mathscr{A}) \subseteq \mathrm{Def}(\mathscr{A}). \tag{6.1.64}$$

PROOF: Every *-equivalence is an equivalence. Thus let S be an equivalence from $\star$ to $\star'$, i.e. $S(a \star b) = S(a) \star' S(b)$. Define $a^\dagger = S^{-1}(S(a)^*)$ which yields a *-involution for $\star$ which coincides with the original *-involution in zeroth order. Thus there is a $\mathsf{C}[\![\lambda]\!]$-linear map $T = \mathrm{id} + \lambda T_1 + \cdots$ with $a^\dagger = T(a^*)$. Since both * and † are anti-automorphisms, T is an automorphism of $\star$. Hence Proposition 6.1.6 shows that $T^{1/2}$ can be defined and is still an automorphism of $\star$. By a straightforward computation one verifies that $ST^{1/2}$ is the *-equivalence from $\star$ to $\star'$ we are looking for. □

For C^*-algebras one has a canonical decomposition of an automorphism into a *-automorphism and an exponential of an anti-Hermitian derivation. This kind of polar decomposition uses very much the strong analytic structure of C^*-algebras, see e.g. [103, Thm. 4.1.19] or [93, Thm. 7.1]. Moreover, in the commutative case every automorphism is a *-automorphism directly: there are no derivations of the algebra of continuous functions on a compact Hausdorff space. This feature is shared by other classes of *-algebras like e.g. the smooth functions on a manifold, see again (5.2.51). Surprisingly, this feature is again rigid under Hermitian deformation in the following sense [29, Prop. 8.8]:

Proposition 6.1.26 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathbb{Q} \subseteq \mathsf{R}$ such that every automorphism of $\mathscr{A}$ is a *-automorphism. Let $\star$ be a Hermitian deformation of $\mathscr{A}$. Then every automorphism $\Phi \in \mathrm{Aut}(\boldsymbol{\mathscr{A}})$ has a unique factorization*

$$\Phi = \mathrm{e}^{\mathrm{i}\lambda D} \circ \Psi \tag{6.1.65}$$

with $D \in {}^*\text{-}\mathrm{Der}(\boldsymbol{\mathscr{A}})$ and $\Psi \in \mathrm{Aut}^*(\boldsymbol{\mathscr{A}})$.

PROOF: We know that $\Phi = \Phi_0 + \lambda\Phi_1 + \cdots$ from Proposition 6.1.1. Since Φ is an automorphism of $\star$, the zeroth order Φ_0 is invertible and an automorphism of the undeformed product, i.e. $\Phi_0 \in \mathrm{Aut}(\mathscr{A})$. Hence we can write $\Phi = T \circ \Phi_0$ with T starting with id in zeroth order. Define now $a \star' b = \Phi_0(\Phi_0^{-1}(a) \star \Phi_0^{-1}(b))$ which gives again a Hermitian deformation since Φ_0 is even a *-automorphism of the undeformed product by assumption. It follows that T is an equivalence from $\star$ to $\star'$. By Proposition 6.1.25 we find even a *-equivalence $\tilde{T}$ from $\star'$ to $\star$. Thus $\Psi^{(1)} = \Phi_0 \circ \tilde{T}$ is a *-automorphism of $\star$. Hence we can factorize the automorphism Φ into $\Psi^{(1)}$ and some automorphism starting with the identity. By Proposition 6.1.6 this composition has a logarithm and hence we find a derivation $D^{(1)}$ of $\star$ with $\Phi = \mathrm{e}^{\mathrm{i}\lambda D^{(1)}} \circ \Psi^{(1)}$. The derivation $D^{(1)}$ can be decomposed into *-derivations $D^{(1)} = D_1^{(1)} + \mathrm{i}D_2^{(1)}$. By the Baker-Campbell-Hausdorff formula we find a derivation $D^{(2)}$ with $\mathrm{e}^{\mathrm{i}\lambda D^{(1)}} \circ \mathrm{e}^{\lambda D_2^{(1)}} = \mathrm{e}^{\mathrm{i}\lambda D^{(2)}}$ in such a way that the imaginary part of $D^{(2)}$ is of one order higher than the one in $D_2^{(1)}$. By induction, we can split off successively *-automorphisms $\mathrm{e}^{\lambda D_2^{(k)}}$ such that their infinite product gives the factorization (6.1.65). □

The basic example of a deformation by commuting derivations can also be turned into a Hermitian deformation by specifying the derivations as follows:

Example 6.1.27 Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ where we assume $\mathbb{Q} \subseteq \mathsf{R}$. Moreover, let $D_1, \ldots, D_n$ be pairwise commuting Hermitian derivations. Then the associative deformation

$$a \star b = \mu_0 \circ \mathrm{e}^{\mathrm{i}\lambda P_\pi}(a \otimes b), \tag{6.1.66}$$

where

$$P_\pi = \sum_{r,s=1}^{n} \pi^{rs} D_r \otimes D_s \tag{6.1.67}$$

with coefficients $\pi^{rs} \in \mathsf{C}$ is a Hermitian deformation if the matrix $\pi \in \mathrm{M}_n(\mathsf{C})$ is an anti-Hermitian matrix. This is clear since we have

$$(\tau \circ ({}^* \otimes {}^*))(P_\pi(a \otimes b)) = \sum_{r,s=1}^{n} \overline{\pi^{rs}} (D_s b)^* \otimes (D_r a)^* = \left(\sum_{r,s=1}^{n} \overline{\pi^{rs}} D_s \otimes D_r \right) (b^* \otimes a^*) = P_{\pi^*}(a^* \otimes b^*)$$

with the canonical flip $\tau(a \otimes b) = b \otimes a$ as usual. Together with the sign from the complex conjugation of i in the exponent, the claim follows at once. This way, we get a large class of Hermitian deformations, most notably examples from deformation quantization.

Example 6.1.28 A slight variation of the previous example is obtained as follows. Suppose that $D_1, \ldots, D_n$ are pairwise commuting derivations such that also D_r commutes with D_s^* for all $r, s = 1, \ldots, n$. Then the deformation

$$a \star b = \mu_0 \circ \mathrm{e}^{2\lambda P_g}(a \otimes b) \quad \text{with} \quad P_g = \sum_{r,s=1}^{n} g^{rs} D_r \otimes D_s^* \tag{6.1.68}$$

is a Hermitian deformation whenever the matrix $g \in \mathrm{M}_n(\mathsf{C})$ is Hermitian. Again, the proof relies on the observation that

$$\tau \circ ({}^* \otimes {}^*) \circ P_g = P_{g^*} \circ \tau \circ ({}^* \otimes {}^*). \tag{6.1.69}$$

One can alternatively show that this deformation is of the form (6.1.66) by decomposing $D_r = D_r^{(1)} + \mathrm{i} D_r^{(2)}$ into real and imaginary parts which are then Hermitian derivations.

We conclude this section with the observation that Hermitian deformations behave well with respect to the property **(K)**:

Proposition 6.1.29 (Rigidity of (K)) *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ which satisfies property* **(K)**. *Then any Hermitian deformation $\star$ of $\mathscr{A}$ still satisfies* **(K)**.

PROOF: This is obvious since first $\mathrm{M}_n(\mathscr{A})[[\lambda]] = \mathrm{M}_n(\mathscr{A}[[\lambda]])$ and hence $\star$ yields a Hermitian deformation of $\mathrm{M}_n(\mathscr{A})$ as well. The invertibility of $\mathbb{1} + A^* \star A$ for $A \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ is then decided in zeroth order where we can use **(K)** for the undeformed algebra. ☐

6.1.5 Deformation of Projections

Let $\star$ be an associative deformation of an associative algebra $\mathscr{A}$. We know that an idempotent element $e \in \mathscr{A}[[\lambda]]$ with respect to $\star$ has a classical limit $e_0 = \mathrm{cl}(e)$ which is idempotent for the undeformed product, see also Remark 6.1.21. The following proposition shows that we can also deform classically idempotent elements into idempotents, see [51, Eq. (6.1.4)] as well as [46, 57, 100], and determine the image of the projections [23]:

Proposition 6.1.30 *Let $\star$ be an associative deformation of an associative algebra $\mathscr{A}$ over a ring* R.

i.) Let $n, m \in \mathbb{N}$ and suppose $e \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ and $f \in \mathrm{M}_m(\boldsymbol{\mathscr{A}})$ are idempotent with $\mathrm{cl}(e) = e_0$ and $\mathrm{cl}(f) = f_0$. Then the map

$$I \colon \mathrm{M}_{n\times m}(\mathscr{A})[[\lambda]] \ni A \ \mapsto \ e \star (e_0 A f_0) \star f \in \mathrm{M}_{n\times m}(\mathscr{A})[[\lambda]] \tag{6.1.70}$$

is of the form

$$I = \sum_{r=0}^{\infty} \lambda^r I_r \tag{6.1.71}$$

with C-linear maps $I_r \colon \mathrm{M}_{n\times m}(\mathscr{A}) \longrightarrow \mathrm{M}_{n\times m}(\mathscr{A})$ such that I_0 is the projection $I_0(A) = e_0 A f_0$.

ii.) The map I restricts to a $\mathsf{C}[[\lambda]]$-linear isomorphism

$$I \colon (e_0 \mathrm{M}_{n\times m}(\mathscr{A}) f_0)[[\lambda]] \longrightarrow e \star \mathrm{M}_{n\times m}(\boldsymbol{\mathscr{A}}) \star f. \tag{6.1.72}$$

of $\mathsf{C}[[\lambda]]$-modules.

iii.) The map I induces a formal associative deformation of the subalgebra $e_0 \mathrm{M}_n(\mathscr{A}) e_0$ of $\mathrm{M}_n(\mathscr{A})$.

iv.) If $e_0 \in \mathscr{A}$ *is idempotent then there exist* $e_1, e_2, \ldots \in \mathscr{A}$ *such that* $e = \sum_{r=0}^{\infty} \lambda^r e_r \in \mathscr{A}[\![\lambda]\!]$ *is idempotent with respect to* $\star$*. If* $\mathscr{A}$ *is unital and* $\mathbb{Q} \subseteq \mathsf{R}$ *the element*

$$e = \frac{1}{2} + \left(e_0 - \frac{1}{2}\right) \star \frac{1}{\sqrt[\star]{1 + 4(e_0 \star e_0 - e_0)}} \tag{6.1.73}$$

satisfies $\operatorname{cl}(e) = e_0$ *and* $e \star e = e$.

v.) If in addition R *is an ordered ring,* $\mathscr{A}$ *is a* **-algebra over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$*, and* $\star$ *is a Hermitian deformation, the deformation* (6.1.73) *of a projection* $e_0 = e_0^2 = e_0^*$ *gives a projection in* $\boldsymbol{\mathscr{A}}$*. For a projection* $P = P^* = P \star P \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ *the subalgebra* $P \star \mathrm{M}_n(\boldsymbol{\mathscr{A}}) \star P$ *is a* **-subalgebra and induces a Hermitian deformation of* $P_0 \mathrm{M}_n(\mathscr{A}) P_0$.

Proof: Since I is $\mathsf{R}[\![\lambda]\!]$-linear, it is a formal series of R-linear maps. The classical limit of I is now given by $\operatorname{cl}(I(A_0)) = \operatorname{cl}(e \star (e_0 A_0 f_0) \star f_0) = \operatorname{cl}(e)(e_0 A_0 f_0)\operatorname{cl}(f) = e_0^2 A_0 f_0^2 = e_0 A_0 f_0$ since e_0 and f_0 are idempotent. This shows the first part. For the second part we note that I restricted to $e_0 \mathrm{M}_{n\times m}(\mathscr{A}) f_0$ is injective already in zeroth order since I_0 is the identity on this submodule. But then the $\mathsf{R}[\![\lambda]\!]$-linear extension is still injective. Thus let $B \in e \star \mathrm{M}_{n\times m}(\boldsymbol{\mathscr{A}}) \star f$ be given. Then $e \star B \star f = B$ implies $e_0 B_0 f_0 = B_0$. Hence $B_0 \in e_0 \mathrm{M}_{n\times m}(\mathscr{A}) f_0$ and thus $I(B_0) - B \in e \star \mathrm{M}_{n\times m}(\boldsymbol{\mathscr{A}}) \star f$ vanishes in zeroth order. Repeating this inductively gives the surjectivity of I. The third part is then a consequence since for an idempotent $e \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ the subset $e \star \mathrm{M}_n(\mathscr{A}) \star e$ is always a subalgebra. Thus we can pull-back the product by I to $(e_0 \mathrm{M}_n(\mathscr{A}) e_0)[\![\lambda]\!]$. Since I is the identity on this subspace in zeroth order, we get a deformation of the original product of $e_0 \mathrm{M}_n(\mathscr{A}) e_0$. For the fourth part let $e_0^2 = e_0$ be given. Following [100] we assume that we have $e_0, \ldots, e_k \in \mathrm{M}_n(\mathscr{A})$ such that $e^{(k)} = e_0 + \cdots + \lambda^k e_k$ satisfies $e^{(k)} \star e^{(k)} = e^{(k)} + \lambda^{k+1} b_{k+1} + \cdots$. Clearly, for $k = 0$ this is trivially fulfilled. Since $e^{(k)}$ commutes with $e^{(k)} \star e^{(k)} - e^{(k)}$ with respect to $\star$, we conclude that e_0 commutes with b_{k+1} with respect to the original product. Hence setting

$$e_{k+1} = -e_0 b_{k+1} - b_{k+1} e_0 + b_{k+1}$$

yields the desired correction term such that $e^{(k+1)} = e^{(k)} + \lambda^{k+1} e_{k+1}$ is an idempotent up to terms of order λ^{k+2}. By induction we find an idempotent e with respect to $\star$ deforming e_0. This recursion simplifies in the case $\mathbb{Q} \subseteq \mathsf{R}$ since then the formal Taylor series of the $\star$-square root is well-defined. With e according to (6.1.73), the verification $e \star e = e$ is then a simple computation, see also Exercise 6.4.21. In case of a *-algebra and $e_0^* = e_0$ we also get $e^* = e$ as one can see directly from the formula. Note that the recursive construction will also yield a projection directly. Then the last statement is clear. □

The importance of the explicit formula can hardly be overestimated. We see e.g. that the original idempotent e_0 and the $\star$-idempotent e will $\star$-commute

$$e \star e_0 = e_0 \star e, \tag{6.1.74}$$

since e is approximated in the λ-adic topology by polynomials in e_0. We will come back to the deformation of idempotents when we study K_0-theory.

For later use we formulate the following special case of the above construction:

Corollary 6.1.31 *Let* $\star$ *be an associative deformation of an associative algebra* $\mathscr{A}$ *over* R*. Suppose* $e \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ *is an idempotent with* $\operatorname{cl}(e) = e_0$*. Then*

$$I\colon \mathscr{A}^n[\![\lambda]\!] \ni x \ \mapsto \ e \star (e_0 x) \in \mathscr{A}^n[\![\lambda]\!] \tag{6.1.75}$$

is of the form

$$I = \sum_{r=0}^{\infty} \lambda^r I_r \tag{6.1.76}$$

with R*-linear maps* $I_r \colon \mathscr{A}^n \longrightarrow \mathscr{A}^n$ *such that* I_0 *is the projection* $I_0(A) = e_0 A$*. In particular, the restriction*

$$I \colon e_0 \mathscr{A}^n[\![\lambda]\!] \longrightarrow e \star \mathscr{A}^n[\![\lambda]\!] \tag{6.1.77}$$

is an isomorphism.

6.2 Deformation of States

After deforming the *-algebra structure we are now interested in deformations of positive functionals and of states. In general, this might not be possible. However, we will find many classes of examples of Hermitian deformations which behave well also with respect to aspects of positivity, among which we find all Hermitian star products from deformation quantization. For such algebras we are able to study the behaviour of the important property **(H)**, which turns out to be rigid. As a first application we will relate the GNS representations of deformed positive functionals to the original GNS representation.

6.2.1 Completely Positive Deformations

In general, Hermitian deformations already capture most of the interesting structure of the deformation theory of *-algebras. However, the aspect of positivity is not yet fully implemented. We first note that the classical limit of a positive functional is again positive:

Proposition 6.2.1 *Let* $\star$ *be a Hermitian deformation of a* **-algebra* $\mathscr{A}$ *over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$*. If* $\omega \colon \boldsymbol{\mathscr{A}} \longrightarrow \mathsf{C}[\![\lambda]\!]$ *is positive with respect to* $\star$*, i.e. if* $\omega(a^* \star a) \geq 0$ *for all* $a \in \boldsymbol{\mathscr{A}}$*, then* $\omega_0 = \mathrm{cl}(\omega)$ *is positive for the undeformed algebra.*

PROOF: We know $\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r$ with C-linear functionals $\omega_r \colon \mathscr{A} \longrightarrow \mathsf{C}$. If we write $\star = \mu_0 + \lambda\mu_1 + \cdots$ and consider $a \in \mathscr{A}$ then

$$\omega(a^* \star a) = \omega_0(a^* a) + \lambda\big(\omega_0(\mu_1(a^*, a)) + \omega_1(a^* a)\big) + \cdots .$$

Hence $\omega(a^* \star a) \geq 0$ implies $\omega_0(a^* a) \geq 0$ by the very definition of the ring ordering of $\mathsf{R}[\![\lambda]\!]$. ☐

The difficulty with the above observation is that the converse is not necessarily true: if $\omega_0(a^* a) = 0$ then the question of positivity $\omega_0(a^* \star a)$ is decided in the first order r where $\omega_0(\mu_r(a^*, a))$ is non-zero. For this combination one typically can not say much. In fact, it may fail to be positive. This becomes most visible in the following example from deformation quantization [18, Sect. 2]:

Example 6.2.2 (Weyl product and positivity) Consider the *-algebra $\mathscr{C}^\infty(\mathbb{R}^2)$ of smooth functions on $\mathbb{R}^2$ with canonical coordinates (q, p). We deform this by the *Weyl product*

$$f \star_{\mathrm{Weyl}} g = \mu_0 \circ \mathrm{e}^{\frac{\mathrm{i}\lambda}{2} P}(f \otimes g) \quad \text{with} \quad P = \frac{\partial}{\partial q} \otimes \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}. \tag{6.2.1}$$

Clearly, this is a Hermitian deformation as it is a particular case of Example 6.1.27. Now, classically, the δ-functional $\delta \colon \mathscr{C}^\infty(\mathbb{R}^2) \ni f \mapsto f(0) \in \mathbb{C}$ is positive since

$$\delta(\overline{f} f) = \overline{f(0)} f(0) \geq 0. \tag{6.2.2}$$

Remarkably, this will fail for $\star_{\mathrm{Weyl}}$. Taking e.g. the Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ of the harmonic oscillator one has $\overline{H} = H$ and

$$\delta(\overline{H} \star_{\mathrm{Weyl}} H) = -\frac{\lambda^2}{4} < 0. \tag{6.2.3}$$

Since this is strictly negative, δ is not a positive functional for $\star_{\mathrm{Weyl}}$ anymore.

Thus we can not expect that positive functionals of the undeformed *-algebra stay positive for a Hermitian deformation. However, in the situation of Proposition 6.2.1 we considered positive linear functionals ω of the deformed algebra which might not just consist of a classical contribution ω_0 alone: the higher orders $\omega_1, \omega_2, \ldots$ can take care of the possible failure of ω_0 being positive directly. This motivates now the following definitions [24, 30]:

Definition 6.2.3 (Completely positive deformation) *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let $\star$ be a Hermitian deformation of $\mathscr{A}$.*

i.) The deformation $\star$ is called positive if for every positive linear functional $\omega_0\colon \mathscr{A} \longrightarrow \mathsf{C}$ one finds higher orders $\omega_1, \omega_2, \ldots$ such that

$$\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r \colon \boldsymbol{\mathscr{A}} \longrightarrow \mathsf{C}[\![\lambda]\!] \tag{6.2.4}$$

is positive with respect to $\star$.

ii.) The deformation $\star$ is completely positive, if the induced deformations $\star$ for $\mathrm{M}_n(\mathscr{A})$ are positive for all $n \in \mathbb{N}$.

iii.) The deformation $\star$ is called strongly positive if every classically positive linear functional ω_0 of $\mathscr{A}$ is positive with respect to $\star$.

The above example shows that the Weyl product is not a strongly positive deformation. To decide whether the Weyl product is actually a positive deformation is a much more subtle task: we have to find correction terms $\omega_1, \omega_2, \ldots$ for a given classically positive ω_0. Since we are dealing now with *inequalities* there seems to be no powerful cohomological description of the problem as for the *equalities* needed for associativity etc. We come back to the question of the existence of such deformations in Section 6.2.3.

Remark 6.2.4 (Every classical state is a classical limit) One way to phrase this in the context of deformation quantization is that one wants every classical state to appear as the classical limit of a (possibly highly non-unique) quantum state. From a physical point of view this is of course highly desirable as anything else would contradict the idea that quantum theory is the more fundamental one: if there are classical states not accessible by quantum physics then the classical theory can hardly be called a limiting theory of the quantum theory. Thus we anticipate already here that reasonable deformations of *-algebras should be positive.

Remark 6.2.5 (Non-uniqueness of deformations) If we can deform a positive functional ω_0 then the possible deformations will typically *not* be unique: given ω with $\mathrm{cl}(\omega) = \omega_0$ and any other positive functional $\mu\colon \boldsymbol{\mathscr{A}} \longrightarrow \mathsf{C}[\![\lambda]\!]$ the convex combination $\omega + \lambda\mu$ is still positive and has the same classical limit. If ω_0 was a state, $\omega_0(\mathbb{1}) = 1$, then we can normalize $\omega + \lambda\mu$ again to become a state as well since $\omega(\mathbb{1}) + \lambda\mu(\mathbb{1})$ starts with 1 in zeroth order. Hence this number in $\mathsf{C}[\![\lambda]\!]$ is invertible again. Due to this effect it seems hopeless to classify deformations of ω_0 without further restrictions.

Remark 6.2.6 If $\mathscr{A}$ has sufficiently many positive linear functionals then every positive deformation $\star$ of $\mathscr{A}$ yields a *-algebra $\boldsymbol{\mathscr{A}}$ which still has sufficiently many positive linear functionals. In particular, the nice consequences from e.g. Corollary 1.2.10 or Proposition 2.1.18 apply to any such deformation.

Remark 6.2.7 (*-Equivalence of positive deformations) Let $\star$ and $\star'$ be *-equivalent Hermitian deformations of a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ via a *-equivalence transformation $S = \mathrm{id} + \lambda S_1 + \cdots$. Then $\star$ is (completely) positive iff $\star'$ is (completely) positive. Indeed, given a positive linear functional $\omega = \omega_0 + \lambda\omega_1 + \cdots$ of $(\mathscr{A}[\![\lambda]\!], \star)$ we obtain a positive linear functional

$$\omega' = \omega \circ S = \omega_0 + \lambda(\omega_0 \circ S_1 + \omega_1) + \cdots \tag{6.2.5}$$

with respect to $\star'$ which has the same classical limit ω_0. This way we find a deformation of ω_0 with respect to $\star'$ if we have one with respect to $\star$. Since S is invertible, the situation is obviously symmetric and works for matrices $\mathrm{M}_n(\mathscr{A})$ as well. Hence we can consider the *-equivalence classes of completely positive deformations

$$\mathrm{Def}^{\mathrm{str}}(\mathscr{A}) = \big\{[\star] \in \mathrm{Def}^*(\mathscr{A}) \;\big|\; \star \text{ is completely positive}\big\} \subseteq \mathrm{Def}^*(\mathscr{A}) \tag{6.2.6}$$

of the *-algebra $\mathscr{A}$. Note, however, that the notion of strongly positive deformation is typically *not* invariant under *-equivalences. This is e.g. the case in deformation quantization where the Weyl star product is not strongly positive but positive according to Theorem 6.2.19 while a different star product, the Wick star product, will turn out to be strongly positive.

In a next step we provide some first little tools: it will be sufficient to check the positivity of ω on elements $a_0 \in \mathscr{A}$ without higher orders in λ, see [15, Lem. A.5].

Proposition 6.2.8 *Let $\star$ be a Hermitian deformation of a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Then a $\mathsf{C}[\![\lambda]\!]$-linear functional $\omega\colon \mathscr{A}[\![\lambda]\!] \longrightarrow \mathsf{C}[\![\lambda]\!]$ is positive with respect to $\star$ iff*

$$\omega(a_0 \star a_0) \geq 0 \tag{6.2.7}$$

for all $a_0 \in \mathscr{A}$.

Proof: One direction is obvious. Thus suppose $\omega(a_0^* \star a_0) \geq 0$ for all $a_0 \in \mathscr{A}$. We prove a weaker version of the Cauchy-Schwarz inequality now only valid for elements in $\mathscr{A}$ instead of all elements of $\mathscr{A}[\![\lambda]\!]$. For $z, w \in \mathsf{C}$ the assumption (6.2.7) implies

$$0 \leq \omega((za_0 + wb_0)^* \star (za_0 + wb_0)) = \overline{z}z\omega(a_0^* \star a_0) + \overline{z}w\omega(a_0^* \star b_0) + z\overline{w}\omega(b_0^* \star a_0) + \overline{w}w\omega(b_0^* \star b_0)$$

for all $a_0, b_0 \in \mathscr{A}$ as an inequality in $\mathsf{R}[\![\lambda]\!]$. By specifying the values of z and w in C in a clever way we conclude as in the proof of the Cauchy-Schwarz inequality in Lemma 1.1.7 that

$$\omega(a_0^* \star b_0)\overline{\omega(a_0^* \star b_0)} \leq \omega(a_0^* \star a_0)\omega(b_0^* \star b_0) \quad \text{and} \quad \omega(a_0^* \star b_0) = \overline{\omega(b_0^* \star a_0)} \tag{$*$}$$

for all $a_0, b_0 \in \mathscr{A}$ as an inequality in $\mathsf{R}[\![\lambda]\!]$. Now we apply this to a_0 and a_1 which yields with $\alpha = \omega(a_0^* \star a_0)$, $\beta = \omega(a_0^* \star a_1)$, and $\gamma = \omega(a_1^* \star a_1)$ the inequality $\overline{\beta}\beta \leq \alpha\gamma$ in $\mathbb{R}[\![\lambda]\!]$. But with this inequality one has

$$\omega((a_0 + za_1)^* \star (a_0 + za_1)) = \alpha + \beta z + \overline{\beta}\overline{z} + \gamma\overline{z}z \geq 0$$

for all $z \in \mathsf{C}[\![\lambda]\!]$ and not only for $z \in \mathsf{C}$. In particular, for $z = \lambda$ we obtain the positivity

$$\omega((a_0 + \lambda a_1)^* \star (a_0 + \lambda a_1)) \geq 0.$$

Thus we have extended the original positivity property (6.2.7) to the case where we have also a first order in λ. Repeating this argument inductively, we get the positivity

$$\omega\left(\left(\sum_{r=0}^{N} \lambda^r a_r\right)^* \left(\sum_{r=0}^{N} \lambda^r a_r\right)\right) \geq 0$$

for all $N \in \mathbb{N}$. Now we use that any $\mathsf{C}[\![\lambda]\!]$-linear functional is continuous with respect to the λ-adic topology according to Proposition 6.1.2, *v.)*, and the fact that the λ-adic topology is compatible with the ordering in the sense that the subset of non-negative elements in $\mathsf{R}[\![\lambda]\!]$ is closed. Then the convergence from Proposition 6.1.25, *iii.)*, finally shows $\omega(a^* \star a) \geq 0$ for all $a = \sum_{r=0}^{\infty} \lambda^r a_r \in \mathscr{A}[\![\lambda]\!]$.□

One of the important consequences of a (completely) positive deformation is that the classical limits of positive algebra elements are positive. Without control over the quantum states this would not be possible [29, Lemma 8.1]:

Proposition 6.2.9 *Let $\star$ be a positive deformation of a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. If $a \in \boldsymbol{\mathscr{A}}$ is positive with respect to $\star$ then $\mathrm{cl}(a) = a_0 \in \mathscr{A}$ is positive with respect to the undeformed product.*

PROOF: Let $a \in \boldsymbol{\mathscr{A}}$ be positive. This means that $\omega(a) \geq 0$ for all positive $\mathsf{C}[\![\lambda]\!]$-linear functionals $\omega\colon \boldsymbol{\mathscr{A}} \longrightarrow \mathsf{C}[\![\lambda]\!]$. Since for a given classically positive C-linear functional $\omega_0\colon \mathscr{A} \longrightarrow \mathsf{C}$ we find a deformation ω into a positive functional with respect to $\star$, we have $\omega(a) \geq 0$ for this ω and thus

$$0 \leq \mathrm{cl}(\omega(a)) = \omega_0(a_0)$$

is still positive in C. This shows $a_0 \in \mathscr{A}^+$. □

Remark 6.2.10 If we would know that every positive element $a \in \boldsymbol{\mathscr{A}}^+$ is actually algebraically positive, i.e. $a = \sum_{i=1}^n \beta_i b_i^* \star b_i$ for some $b_i \in \boldsymbol{\mathscr{A}}$ and $0 < \beta_i \in \mathsf{R}[\![\lambda]\!]$, then the classical limit of a is again algebraically positive since

$$a_0 = \mathrm{cl}(a) = \sum_{i=1}^n \mathrm{cl}(\beta_i)\mathrm{cl}(b_i)^*\mathrm{cl}(b_i) \in \mathscr{A}^{++}. \tag{6.2.8}$$

Thus the above criteria becomes interesting for the case where we have a strict inclusion of algebraically positive elements inside the positive elements, a situation which we know to be relevant in many examples, see e.g. Exercise 1.4.18.

Remark 6.2.11 One can also consider the converse problem: given a positive element $a_0 \in \mathscr{A}^+$, can one find higher orders $a_1, a_2, \ldots \in \mathscr{A}$ such that $a = \sum_{r=0}^\infty \lambda^r a_r$ is positive with respect to the deformed product? Even for a positive deformation this is not completely obvious: there could be a positive functional ω_0 of $\mathscr{A}$ allowing for a positive deformation $\omega = \omega_0 + \lambda\omega_1 + \cdots$ of $\boldsymbol{\mathscr{A}}$ with respect to $\star$ such that $\omega_0(a_0) = 0$ but $\omega(a_0) < 0$ by contributions in higher orders. Thus a deformation might be necessary and it is not clear how one can show existence in general.

We conclude this section now with the following first non-trivial observation concerning completely positive deformations: the important property **(H)** is preserved under completely positive deformations [29, Prop. 8.2]:

Proposition 6.2.12 (Rigidity of (H)) *Let $\star$ be a completely positive deformation of a unital *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\frac{1}{2} \in \mathsf{R}$. If $\mathscr{A}$ satisfies* **(H)** *or* **(H$^-$)** *then $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ also satisfies* **(H)** *or* **(H$^-$)***, respective.*

PROOF: We consider the property **(H)** first. Thus let $\boldsymbol{H} = H + \cdots \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})^+$ be an invertible positive element and let $\boldsymbol{P}_\alpha \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ be projections forming an orthogonal partition of unity with $\boldsymbol{H} \star \boldsymbol{P}_\alpha = \boldsymbol{P}_\alpha \star \boldsymbol{H}$. Taking classical limits, we see that the elements $P_\alpha = \mathrm{cl}(\boldsymbol{P}_\alpha)$ are projections again, forming an orthogonal partition of unity for the undeformed algebra. Moreover, $[H, P_\alpha] = 0$ holds. Since we assume that $\star$ is a completely positive deformation, $H \in \mathrm{M}_n(\mathscr{A})$ is positive by Proposition 6.2.9. Since $\boldsymbol{H}$ is invertible, the zeroth order is invertible for the undeformed product, too. Thus we find a unitary $U \in \mathrm{M}_n(\mathscr{A})$ with $H = U^*U$ and $[U, P_\alpha] = 0$ for all α by property **(H)** for the undeformed algebra $\mathscr{A}$ applied to H and the P_α. Now we consider $P_\alpha \mathrm{M}_n(\mathscr{A}) P_\alpha$ as unital *-algebra with unit P_α as we did before in Proposition 4.2.10, *i.)*. Then $U_\alpha = P_\alpha U P_\alpha \in P_\alpha \mathrm{M}_n(\mathscr{A}) P_\alpha$ is an invertible element in this *-algebra for all α with inverse $U_\alpha^{-1} = P_\alpha U^{-1} P_\alpha$, since

$$P_\alpha U P_\alpha P_\alpha U^{-1} P_\alpha = P_\alpha U U^{-1} P_\alpha = P_\alpha$$

according to $[U, P_\alpha] = 0$. Moreover, for the α-th component $H_\alpha = P_\alpha H P_\alpha$ of H we get

$$H_\alpha = P_\alpha H P_\alpha = P_\alpha U^* P_\alpha P_\alpha U P_\alpha = U_\alpha^* U_\alpha. \tag{$*$}$$

Clearly, this H_α is the classical limit of $\boldsymbol{P}_\alpha \star \boldsymbol{H} \star \boldsymbol{P}_\alpha$. Now by Proposition 6.1.30, *v.)*, the *-algebra $\boldsymbol{P}_\alpha \star \mathrm{M}_n(\boldsymbol{\mathscr{A}}) \star \boldsymbol{P}_\alpha$ induces a Hermitian deformation $\star_\alpha$ for $P_\alpha \mathrm{M}_n(\mathscr{A}) P_\alpha$. From $(*)$ and Lemma 6.1.24, *i.)*, transferred back from $(P_\alpha \mathrm{M}_n(\mathscr{A}) P_0)[\![\lambda]\!], \star_\alpha)$ to the *-subalgebra $\boldsymbol{P}_\alpha \star \mathrm{M}_n(\boldsymbol{\mathscr{A}}) \star \boldsymbol{P}_\alpha$ of $\mathrm{M}_n(\boldsymbol{\mathscr{A}})$, we see that we find an invertible $\boldsymbol{U}_\alpha \in \boldsymbol{P}_\alpha \star \mathrm{M}_n(\boldsymbol{\mathscr{A}}) \star \boldsymbol{P}_\alpha$ with

$$\boldsymbol{P}_\alpha \star \boldsymbol{H} \star \boldsymbol{P}_\alpha = \boldsymbol{P}_\alpha \star \boldsymbol{U}_\alpha^* \star \boldsymbol{P}_\alpha \star \boldsymbol{P}_\alpha \star \boldsymbol{U}_\alpha \star \boldsymbol{P}_\alpha. \tag{$**$}$$

Then we can consider the block-diagonal $\boldsymbol{U} = \sum_\alpha \boldsymbol{U}_\alpha$ which clearly $\star$-commutes with each $\boldsymbol{P}_\alpha$ as these projection form a partition of unity. Putting this together with $(**)$ shows $\boldsymbol{H} = \boldsymbol{U}^* \star \boldsymbol{U}$ as wanted. The case $(\mathbf{H}^-)$ is a particular case with the partition of unity given by a single projection $\boldsymbol{P}$ together with the complementary projection $\mathbb{1} - \boldsymbol{P}$. ☐

As we have seen in the discussion of the groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ the property $(\mathbf{H})$ (essentially $(\mathbf{H}^-)$ was sufficient) provides the additional information which simplifies the situation drastically. With the above rigidity result and the corresponding statement from Proposition 6.1.29 for the property $(\mathbf{K})$, we can transfer all these results on Morita theory to completely positive deformations. A first application is the following corollary to Theorem 5.2.17:

Corollary 6.2.13 *For the class of completely positive deformations of unital *-algebras satisfying* $(\mathbf{K})$ *and* $(\mathbf{H}^-)$*, the groupoid morphism* $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ *is injective.*

Of course, it remains to find interesting examples of completely positive deformations first.

6.2.2 GNS Representations of Deformed *-Algebras

Before we investigate the general representation theory of deformed *-algebras we consider a particular case here: the GNS representations arising from positive functionals. Thus we consider a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ together with a Hermitian deformation $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$. It might be advantageous to consider even a (completely) positive deformation in order to have many positive functionals. But for the following it is sufficient to assume the existence of one positive functional $\omega = \omega_0 + \lambda\omega_1 + \cdots$ of $\boldsymbol{\mathscr{A}}$ with classical limit $\mathrm{cl}(\omega) = \omega_0$ which is then a positive functional of $\mathscr{A}$.

To relate the GNS representations of ω and ω_0 we first need to relate the GNS pre-Hilbert spaces. Here the situation is less clear as for the deformed algebras: it might happen that the pre-Hilbert space $\boldsymbol{\mathfrak{H}}_\omega$ is *not* of the form $\mathfrak{H}[\![\lambda]\!]$ for some C-module $\mathfrak{H}$. Thus a naive classical limit map by projecting onto the order-zero part is not available. The next try would be to consider the *quotient* $\mathfrak{H} = \boldsymbol{\mathfrak{H}}/\lambda\boldsymbol{\mathfrak{H}}$ as classical limit with the quotient map as classical limit map. Again, there is one problem, namely that there is no a priori inner product on this quotient.

This motivates to consider a slightly more general quotient procedure as classical limit in this case [26, Lem. 8.2 and Lem. 8.3]:

Proposition 6.2.14 *Let* $\mathsf{C} = \mathsf{R}(\mathrm{i})$ *with* R *an ordered ring.*

i.) Let $\boldsymbol{\mathfrak{H}}$ *be a pre-Hilbert space over* $\mathsf{C}[\![\lambda]\!]$*. Then*

$$\boldsymbol{\mathfrak{H}}_{\mathrm{Null}} = \big\{\phi \in \boldsymbol{\mathfrak{H}} \;\big|\; \mathrm{cl}(\langle\phi, \phi\rangle) = 0\big\} = \big\{\phi \in \boldsymbol{\mathfrak{H}} \;\big|\; \mathrm{cl}(\langle\psi, \phi\rangle) = 0 \textit{ for all } \psi \in \boldsymbol{\mathfrak{H}}\big\} \tag{6.2.9}$$

is a $\mathsf{C}[\![\lambda]\!]$*-submodule of* $\boldsymbol{\mathfrak{H}}$ *with*

$$\lambda\boldsymbol{\mathfrak{H}} \subseteq \boldsymbol{\mathfrak{H}}_{\mathrm{Null}} \tag{6.2.10}$$

ii.) *The quotient* $\mathrm{cl}(\mathfrak{H}) = \mathfrak{H}/\mathfrak{H}_{\mathrm{Null}}$ *becomes a pre-Hilbert space over* C *via the inner product*

$$\big\langle \mathrm{cl}(\phi), \mathrm{cl}(\psi)\big\rangle_{\mathrm{cl}(\mathfrak{H})} = \mathrm{cl}\big(\langle \phi, \psi\rangle_{\mathfrak{H}}\big), \tag{6.2.11}$$

where $\mathrm{cl}(\phi), \mathrm{cl}(\psi) \in \mathrm{cl}(\mathfrak{H})$ *denote the equivalence classes of* $\phi, \psi \in \mathfrak{H}$.

iii.) *For pre-Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$ *over* $\mathsf{C}[\![\lambda]\!]$ *and for* $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ *one has*

$$A((\mathfrak{H}_1)_{\mathrm{Null}}) \subseteq (\mathfrak{H}_2)_{\mathrm{Null}}, \tag{6.2.12}$$

and $\mathrm{cl}(A)\colon \mathrm{cl}(\mathfrak{H}_1) \ni \mathrm{cl}(\phi) \mapsto \mathrm{cl}(A\phi) \in \mathrm{cl}(\mathfrak{H}_2)$ *is a well-defined adjointable operator.*

iv.) *The classical limit map*

$$\mathrm{cl}\colon \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2) \ni A \ \mapsto\ \mathrm{cl}(A) \in \mathfrak{B}(\mathrm{cl}(\mathfrak{H}_1), \mathrm{cl}(\mathfrak{H}_2)) \tag{6.2.13}$$

satisfies

$$\mathrm{cl}(zA + wA') = \mathrm{cl}(z)\mathrm{cl}(A) + \mathrm{cl}(w)\mathrm{cl}(A'), \tag{6.2.14}$$

$$\mathrm{cl}(BA) = \mathrm{cl}(B)\mathrm{cl}(A), \tag{6.2.15}$$

and

$$\mathrm{cl}(A^*) = \mathrm{cl}(A)^* \tag{6.2.16}$$

for all $A, A' \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ *and* $B \in \mathfrak{B}(\mathfrak{H}_2, \mathfrak{H}_3)$ *as well as* $z, w \in \mathsf{C}[\![\lambda]\!]$.

PROOF: This is an elementary verification: first we show that the two sets in (6.2.9) coincide. The inclusion $\subseteq$ is clear, thus let $\phi \in \mathfrak{H}$ satisfy $\mathrm{cl}(\langle \phi, \phi\rangle) = 0$. In general, we know $\langle \phi, \psi\rangle\overline{\langle \phi, \psi\rangle} \leq \langle \phi, \phi\rangle\langle \psi, \psi\rangle$ by the Cauchy-Schwarz inequality for $\mathfrak{H}$. Since the ring morphism $\mathrm{cl}\colon \mathsf{R}[\![\lambda]\!] \longrightarrow \mathsf{R}$ is compatible with the ordering we conclude

$$\mathrm{cl}(\langle \phi, \psi\rangle)\overline{\mathrm{cl}(\langle \phi, \psi\rangle)} \leq \mathrm{cl}(\langle \phi, \phi\rangle)\mathrm{cl}(\langle \psi, \psi\rangle),$$

from which the remaining inclusion $\supseteq$ follows at once. But then it is clear that $\mathfrak{H}_{\mathrm{Null}}$ is a $\mathsf{C}[\![\lambda]\!]$-submodule containing $\lambda\mathfrak{H}$. Thus the quotient $\mathrm{cl}(\mathfrak{H})$ is a $\mathsf{C}[\![\lambda]\!]$-module with the property that $z\mathrm{cl}(\phi) = 0$ whenever $\mathrm{cl}(z) = 0$. Hence essentially only the scalars in C act non-trivially. The definition of the inner product (6.2.11) is then easily verified to yield a positive definite C-valued inner product on $\mathrm{cl}(\mathcal{H})$ since we divided by the degeneracy space automatically. Next, suppose $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $\phi \in (\mathfrak{H}_1)_{\mathrm{Null}}$. Then for $\psi \in \mathfrak{H}_2$ we have

$$\mathrm{cl}(\langle A\phi, \psi\rangle_{\mathfrak{H}_2}) = \mathrm{cl}(\langle \phi, A^*\psi\rangle_{\mathfrak{H}_1}) = 0$$

by the characterization of $(\mathfrak{H}_1)_{\mathrm{Null}}$ according to (6.2.9). Thus $A\phi \in (\mathfrak{H}_2)_{\mathrm{Null}}$ follows. Hence the operator A is well-defined on the quotients. The last part can then be verified on representatives where it is obvious. □

Corollary 6.2.15 *Let* R *be an ordered ring and* $\mathsf{C} = \mathsf{R}(\mathrm{i})$. *Then the classical limit yields a functor*

$$\mathrm{cl}\colon \mathsf{PreHilbert}(\mathsf{C}[\![\lambda]\!]) \longrightarrow \mathsf{PreHilbert}(\mathsf{C}) \tag{6.2.17}$$

from the category of pre-Hilbert spaces over $\mathsf{C}[\![\lambda]\!]$ *to the category of pre-Hilbert spaces over* C.

Example 6.2.16 Let $\mathcal{H}$ be a pre-Hilbert space over C. Then on $\mathfrak{H} = \mathcal{H}[\![\lambda]\!]$ we extend the inner product of $\mathcal{H}$ is the usual λ-bilinear way. In this case we have

$$\mathfrak{H}_{\mathrm{Null}} = \lambda\mathfrak{H}, \tag{6.2.18}$$

and the classical limit reproduces $\mathcal{H}$. Moreover, it is easy to see that

$$\mathfrak{B}(\mathcal{H}[\![\lambda]\!]) = \mathfrak{B}(\mathcal{H})[\![\lambda]\!], \tag{6.2.19}$$

so that the classical limit map cl for adjointable operators also becomes just the projection onto the zeroth order. However, we can also rescale the inner product on $\boldsymbol{\mathcal{H}}$ by λ. Then $\lambda\langle\,\cdot\,,\,\cdot\,\rangle$ would still be a positive definite inner product on $\boldsymbol{\mathcal{H}}$, but now we have $\boldsymbol{\mathcal{H}}_{\mathrm{Null}} = \boldsymbol{\mathcal{H}}$. Thus the classical limit is trivial in this case. This shows that the classical limit procedure is more subtle than just "setting $\lambda = 0$".

After this preparation we can now consider the classical limit of a GNS representation: thus let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be a Hermitian deformation of $\mathscr{A}$ with a positive linear functional $\omega\colon \boldsymbol{\mathscr{A}} \longrightarrow \mathsf{C}[\![\lambda]\!]$. Then we have the classical limit $\omega_0 = \mathrm{cl}(\omega)\colon \mathscr{A} \longrightarrow \mathsf{C}$ which is again positive. In general, the relation between the two Gel'fand ideals $\mathscr{J}_\omega \subseteq \boldsymbol{\mathscr{A}}$ and $\mathscr{J}_{\omega_0} \subseteq \mathscr{A}$ can be quite complicated. In particular, there is typically *no* isomorphism between $\mathscr{J}_\omega$ and $\mathscr{J}_{\omega_0}[\![\lambda]\!]$. However, the relation between the GNS pre-Hilbert spaces and, building on that, the relation between the GNS representations is very simple [113, Thm. 1]:

Proposition 6.2.17 *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be a Hermitian deformation of a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Let $\omega\colon \boldsymbol{\mathscr{A}} \longrightarrow \mathsf{C}[\![\lambda]\!]$ be a positive linear functional with classical limit $\mathrm{cl}(\omega) = \omega_0$. Then the classical limit of the GNS representation π_ω of $\boldsymbol{\mathscr{A}}$ is canonically unitarily equivalent to the GNS representation of the classical limit ω_0 by the unitary intertwiner*

$$U\colon \mathrm{cl}(\mathcal{H}_\omega) \ni \mathrm{cl}(\psi_a) \;\mapsto\; \psi_{\mathrm{cl}(a)} \in \mathcal{H}_{\omega_0}, \tag{6.2.20}$$

where $a \in \mathscr{A}[\![\lambda]\!]$.

Proof: Let $\psi_a \in (\mathcal{H}_\omega)_{\mathrm{Null}}$ which means $0 = \mathrm{cl}(\langle\psi_a, \psi_a\rangle) = \mathrm{cl}(\omega(a^* \star a)) = \omega_0(a_0^* a_0)$. Hence we have $\psi_a \in (\mathcal{H}_\omega)_{\mathrm{Null}}$ iff $\mathrm{cl}(a) = a_0 \in \mathscr{J}_{\omega_0}$. This shows that U, defined as above, is well-defined as we divide by $\mathscr{J}_{\omega_0}$ on the right hand side to get $\mathcal{H}_{\omega_0}$. Moreover, U is isometric and hence injective since

$$\langle U\mathrm{cl}(\psi_a), U\mathrm{cl}(\psi_b)\rangle_{\omega_0} = \omega_0(\psi_{\mathrm{cl}(a)}, \psi_{\mathrm{cl}(b)}) = \omega_0(a_0^* b_0) = \mathrm{cl}(\omega(a^* \star b)) = \mathrm{cl}(\langle\psi_a, \psi_b\rangle_\omega).$$

Finally, U is clearly surjective. The intertwiner property follows from

$$U\mathrm{cl}(\pi_\omega)(a_0)\mathrm{cl}(\psi_b) = U\mathrm{cl}(\pi_\omega(a_0)\psi_b) = U\mathrm{cl}(\psi_{a_0\star b}) = \psi_{\mathrm{cl}(a_0\star b)} = \psi_{a_0 b_0} = \pi_{\omega_0}(a_0)\psi_{b_0} = \pi_{\omega_0}(a_0)U\mathrm{cl}(\psi_b),$$

where $a_0 \in \mathscr{A}$ and $b \in \mathscr{A}[\![\lambda]\!]$. □

6.2.3 The Case of Star Products

Up to now we have not yet many examples of completely positive deformations. The fundamental observation is that Gerstenhaber's construction with commuting derivations can lead to completely positive deformations. We consider again the situation from Example 6.1.28, see also [48, Thm. 6.7] for a yet slightly more general situation:

Proposition 6.2.18 *Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathbb{Q} \subseteq \mathsf{R}$. Let $D_1, \ldots, D_N$ be pairwise commuting derivations such that D_i^* commutes with D_j for all $i, j = 1, \ldots, N$. Let $g \in \mathrm{M}_N(\mathsf{C})$ be a positive matrix. Then the deformation $\star$ from Example 6.1.28 is completely positive and strongly positive.*

Proof: Since the deformation of $\mathrm{M}_n(\mathscr{A})$ works with the same formula and since the derivations are extended to matrices in the canonical way, we only have to take care of the case $n = 1$. Thus let $\omega\colon \mathscr{A} \longrightarrow \mathsf{C}$ be a positive linear functional of the undeformed algebra and $a \in \mathscr{A}$. Then

$$\begin{aligned}\omega(a^* \star a) &= \omega\left(\sum_{m=0}^{\infty} \frac{(2\lambda)^m}{m!} \sum_{k_1,\ell_1,\ldots,k_m,\ell_m=1}^{N} g^{k_1\ell_1}\cdots g^{k_m\ell_m} D_{k_1}\cdots D_{k_m} a^* D^*_{\ell_1}\cdots D^*_{\ell_m} a\right)\\ &= \sum_{m=0}^{\infty} \frac{(2\lambda)^m}{m!} \omega\left(\sum_{k_1,\ell_1,\ldots,k_m,\ell_m=1}^{N} g^{k_1\ell_1}\cdots g^{k_m\ell_m} (D^*_{k_1}\cdots D^*_{k_m} a)^* D^*_{\ell_1}\cdots D^*_{\ell_m} a\right)\\ &\geq 0,\end{aligned}$$

since each term in the series is a positive algebra element of the undeformed algebra by Proposition 1.1.17 and since ω is positive. By Proposition 6.2.8 it suffices to check the positivity for elements $a \in \mathscr{A}$. ☐

Based on this construction we can now find a large class of completely positive deformations: the star products from deformation quantization. Here we have the following result [24, 30]:

Theorem 6.2.19 (Star products are completely positive) *Let M be a smooth manifold. Then every Hermitian star product $\star$ is a completely positive deformation. Hence*

$$\mathrm{Def}^{\mathrm{str}}(\mathscr{C}^\infty(M)) = \mathrm{Def}^*(\mathscr{C}^\infty(M)). \tag{6.2.21}$$

Proof: We will only sketch the proof as it would need to much preparation to get the full details. First, we cover M by charts $\{(U_\alpha, x_\alpha)\}_{\alpha\in I}$ with U being diffeomorphic to an open ball. We choose a quadratic partition of unity $\{\chi_\alpha\}_{\alpha\in I}$ subordinate to this atlas. Now let $\omega_0\colon \mathscr{C}^\infty(M) \longrightarrow \mathbb{C}$ be a positive linear functional. Then ω_0 is the integration with respect to a compactly supported Borel measure by Exercise 1.4.17. For all $f \in \mathscr{C}^\infty(M)$ we then have

$$\omega_0(f) = \sum_{\alpha\in I} \omega_0(\overline{\chi}_\alpha f \chi_\alpha),$$

where in the summation only those finitely many α contribute where the support of ω_0 meets the supports of the functions χ_α. Each of the functionals $\omega_\alpha(f) = \omega_0(\overline{\chi}_\alpha f\chi_\alpha)$ has now compact support inside the corresponding U_α. Hence they extend to (still positive) linear functionals $\omega_\alpha\colon \mathscr{C}^\infty(U_\alpha) \longrightarrow \mathbb{C}$. Thus we only have to show that each ω_α can be deformed into a positive linear functional for $\mathscr{C}^\infty(U_\alpha)[\![\lambda]\!]$ with respect to the restricted star product $\star_\alpha = \star\big|_{\mathscr{C}^\infty(U_\alpha)[\![\lambda]\!]}$. Note that we can restrict star products to open subsets as they consist of bidifferential operators.

From here we distinguish two cases: if the star product deforms a symplectic Poisson structure then it is known that any two (Hermitian) star products on an open subset with vanishing second de Rham cohomology are equivalent (and hence *-equivalent by Proposition 6.1.25). Now on such an open subset we have the *Wick star product* which is precisely of the form as in Proposition 6.2.18 and hence strongly positive, see also Exercise 6.4.16. If $S = \mathrm{id} + \sum_{r=1}^\infty \lambda^r S_r$ is a *-equivalence between the local Wick star product and the restriction $\star_\alpha$ then $\omega_\alpha \circ S_\alpha$ is positive for $\star_\alpha$. Summing up all these locally finitely many functionals yields the deformation of ω_0 we are looking for, see [24].

The other case is when $\star$ deforms an arbitrary Poisson structure. Here one shows that locally one can embed the deformed algebra as a *-subalgebra of a Wick star product algebra in twice the dimension. The construction of this embedding is non-trivial, see [30], but given that, we can use the positivity of the Wick star product to obtain the deformation of ω_0.

In both cases the complete positivity gives no additional difficulty as all arguments still hold for matrix-valued functions. ☐

Remark 6.2.20 There are several other classes of examples of completely positive deformations. In particular, in [48] it was shown that certain Drinfel'd twists give rise to universal deformation formulas similar to the one in Proposition 6.2.18 which turn out to be completely positive.

Since Hermitian star products are completely positive deformations, we have many features guaranteed by the general theory. In particular, Hermitian star products satisfy the properties **(K)** and **(H)** since the classical counterparts $\mathscr{C}^\infty(M)$ have these features according Example 2.3.18, *ii.)*, and Example 2.3.22, *ii.)*. While these properties are already sufficient to guarantee a reasonably well-behaved Morita theory, star products have even stronger properties like e.g. the following:

Proposition 6.2.21 *Let $\star$ be a Hermitian star product on a manifold M. Moreover, let $H \in \mathrm{M}_n(\mathscr{C}^\infty(M)[[\lambda]])$ be invertible and positive with respect to $\star$.*

i.) *The zeroth order $H_0 = \mathrm{cl}(H)$ is an invertible positive matrix-valued function on M.*

ii.) *There exists a unique Hermitian logarithm $\mathrm{Log}(H) = \log(H_0) + \cdots \in \mathrm{M}_n(\mathscr{C}^\infty(M)[[\lambda]])$ of H with respect to $\star$, i.e. we have*

$$\mathrm{Exp}(\mathrm{Log}(H)) = H, \tag{6.2.22}$$

where Exp is the $\star$-exponential as in Exercise 6.4.15. For another function $A \in \mathrm{M}_n(\mathscr{C}^\infty(M)[[\lambda]])$ we have $[H, A]_\star = 0$ iff $[\mathrm{Log}(H), A]_\star = 0$.

iii.) *There exists a unique positive invertible square root $\sqrt[\star]{H} \in \mathrm{M}_n(\mathscr{C}^\infty(M)[[\lambda]])^+$ with respect to $\star$, explicitly given by*

$$\sqrt[\star]{H} = \mathrm{Exp}(\tfrac{1}{2}\,\mathrm{Log}(H)). \tag{6.2.23}$$

For $A \in \mathrm{M}_n(\mathscr{C}^\infty(M)[[\lambda]])$ one has $[H, A]_\star = 0$ iff $[\sqrt[\star]{H}, A]_\star = 0$.

PROOF: Since $\star$ is a completely positive deformation we know that $H_0 = \mathrm{cl}(H)$ is an invertible and positive matrix-valued function on M by Proposition 6.2.9. Thus at every point $p \in M$ the matrix $H_0(p)$ is positive definite. From the spectral calculus we know that there exists a global unique Hermitian logarithm $\log(H_0)$ of H_0 which is still smooth since H_0 is invertible. From here we can use the well-known features of the $\star$-exponential from Exercise 6.4.15, see also e.g. [116, Thm. 6.3.4 and Lem. 6.3.5]. □

This is of course a much stronger statement than the previous property **(H)** which would follow from the general rigidity arguments from Proposition 6.2.12. The existence of the $\star$-logarithm and the square root $\sqrt[\star]{\cdot}$ with respect to $\star$ will provide additional options for star products which are typically not available in general. Note however, that the assumption $H_0 > 0$ is crucial in order to guarantee smoothness of the logarithm and the square root.

6.3 Deformations of Modules

After algebras and their states we now deform modules over algebras. In this section we stick to the ring-theoretic situation where the algebra will not carry any additional structure. Thus we consider an associative algebra $\mathscr{A}$ over a ring R of scalars for which we assume to have a formal associative deformation $\star$.

6.3.1 Deformation and classical limit of modules

Suppose that $\mathscr{M}_{\mathscr{A}}$ is a right $\mathscr{A}$-module with module multiplication $(x, a) \mapsto x \cdot a$. In general, this will no longer be a module structure for a deformation $\star$ of $\mathscr{A}$. To cure this, we need to add higher orders. This motivates the following definition:

Definition 6.3.1 (Module deformation) *Let* $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ *be a formal associative deformation of an associative algebra* $\mathscr{A}$ *over a ring* R*. Moreover, let* $\mathscr{M}_{\mathscr{A}}$ *be a right* $\mathscr{A}$*-module.*

i.) A right module deformation $\bullet$ *of* $\mathscr{M}_{\mathscr{A}}$ *with respect to* $\star$ *consists of an* $\mathsf{R}[\![\lambda]\!]$*-bilinear right module structure* $\bullet\colon \mathscr{M}_{\mathscr{A}}[\![\lambda]\!] \times \mathscr{A}[\![\lambda]\!] \longrightarrow \mathscr{M}_{\mathscr{A}}[\![\lambda]\!]$ *such that*

$$x \bullet a = x \cdot a + \sum_{r=1}^{\infty} \lambda^r \varrho_r(x, a) \tag{6.3.1}$$

for all $x \in \mathscr{M}_{\mathscr{A}}[\![\lambda]\!]$ *and* $a \in \mathscr{A}[\![\lambda]\!]$ *where*

$$\varrho_r\colon \mathscr{M}_{\mathscr{A}} \times \mathscr{A} \longrightarrow \mathscr{M}_{\mathscr{A}} \tag{6.3.2}$$

are R*-bilinear maps extended to* $\mathscr{M}_{\mathscr{A}}[\![\lambda]\!] \times \mathscr{A}[\![\lambda]\!]$ *as usual. If* $\mathscr{A}$ *is unital and also* $\mathscr{M}_{\mathscr{A}}$ *is unital, then we require in addition*

$$x \bullet \mathbb{1} = x \tag{6.3.3}$$

for all $x \in \mathscr{M}_{\mathscr{A}}[\![\lambda]\!]$*.*

ii.) Two right module deformations $\bullet$ *and* $\tilde{\bullet}$ *are called equivalent if there exists a formal series* $T = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ *of* R*-linear maps* $T_r\colon \mathscr{M}_{\mathscr{A}} \longrightarrow \mathscr{M}_{\mathscr{A}}$ *such that*

$$T(x \bullet a) = T(x) \mathbin{\tilde{\bullet}} a \tag{6.3.4}$$

for all $x \in \mathscr{M}_{\mathscr{A}}[\![\lambda]\!]$ *and* $a \in \mathscr{A}[\![\lambda]\!]$*. In this case* T *is called an equivalence transformation.*

Recall that the $\mathsf{R}[\![\lambda]\!]$-bilinearity of $\bullet$ implies the existence of the R-bilinear maps $\varrho_r\colon \mathscr{M}_{\mathscr{A}} \times \mathscr{A} \longrightarrow \mathscr{M}_{\mathscr{A}}$ with (6.3.1).

In a completely analogous fashion one defines deformations and equivalences of left $\mathscr{A}$-modules. As usual, it suffices to check the above conditions on elements of $\mathscr{M}_{\mathscr{A}}$ and $\mathscr{A}$ only. It is clear that for a fixed deformation $\star$ of the underlying algebra the equivalence of module deformations is an equivalence relation. Conversely, given a module deformation $\bullet$ and an arbitrary series $T = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ we get by (6.3.4) again a module deformation $\tilde{\bullet}$ which is then equivalent to $\bullet$.

If we have a fixed module deformation $\bullet$ we also write $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} = (\mathscr{M}_{\mathscr{A}}[\![\lambda]\!], \bullet)$ for the resulting right $\boldsymbol{\mathscr{A}}$-module.

In general, the deformed algebra $\boldsymbol{\mathscr{A}}$ can have modules which are not of the form $(\mathscr{M}_{\mathscr{A}}[\![\lambda]\!], \bullet)$. The following example is rather trivial but of some importance:

Example 6.3.2 Let $\mathscr{M}_{\mathscr{A}}$ be a right $\mathscr{A}$-module and let $\star$ be an associative deformation of $\mathscr{A}$. Then $\mathscr{M}_{\mathscr{A}}$ becomes a right $\boldsymbol{\mathscr{A}}$-module by

$$x \circ a = x \cdot \mathrm{cl}(a) \tag{6.3.5}$$

for all $x \in \mathscr{M}_{\mathscr{A}}$ and $a \in \boldsymbol{\mathscr{A}}$. Indeed, this follows directly from $\mathrm{cl}(a \star b) = \mathrm{cl}(a)\mathrm{cl}(b)$ and the module property of the original module multiplication. Note that this is *not* a module deformation in the sense of Definition 6.3.1 since the underlying space $\mathscr{M}_{\mathscr{A}}$ is equipped with the $\mathsf{R}[\![\lambda]\!]$-module structure where $\lambda x = 0$ for all $x \in \mathscr{M}_{\mathscr{A}}$. Thus we have an enormous amount of torsion. Now if in addition $\bullet$ is a module deformation for $\mathscr{M}_{\mathscr{A}}$ then we can consider the classical limit map

$$\mathrm{cl}\colon \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} = \mathscr{M}_{\mathscr{A}}[\![\lambda]\!] \ni x \ \mapsto\ \mathrm{cl}(x) = x_0 \in \mathscr{M}_{\mathscr{A}} \tag{6.3.6}$$

as usual. It then turns out that this is a module morphism of right $\boldsymbol{\mathscr{A}}$-modules if we equip $\mathscr{M}_{\mathscr{A}}$ with the right $\boldsymbol{\mathscr{A}}$-module structure (6.3.5). Indeed, this is clear since

$$\mathrm{cl}(x \bullet a) = \mathrm{cl}(x) \cdot \mathrm{cl}(a) = \mathrm{cl}(x) \circ a \tag{6.3.7}$$

for all $x \in \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ and $a \in \boldsymbol{\mathscr{A}}$.

As we discussed implicitly already in Section 6.2.2 we can always define a classical limit of a right $\boldsymbol{\mathscr{A}}$-module by dividing by the multiples of λ: if we have no other structure to take care of, this is the ring-theoretic construction of the classical limit:

Proposition 6.3.3 (Ring-theoretic classical limit of modules) *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be an associative deformation of an algebra $\mathscr{A}$ over a ring of scalars R.*

i.) If $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ is a right $\boldsymbol{\mathscr{A}}$-module then $\mathrm{cl}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) = \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} / \lambda \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ becomes a right $\mathscr{A}$-module by

$$\mathrm{cl}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \times \mathscr{A} \ni (\mathrm{cl}(\boldsymbol{x}), a) \;\mapsto\; \mathrm{cl}(\boldsymbol{x}) \cdot a = \mathrm{cl}(\boldsymbol{x} \bullet a) \in \mathrm{cl}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}), \tag{6.3.8}$$

where $\bullet$ denotes the module multiplication of $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$.

ii.) If $\boldsymbol{T} \colon \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} \longrightarrow \boldsymbol{\mathscr{M}}'_{\boldsymbol{\mathscr{A}}}$ is a module morphism of right $\boldsymbol{\mathscr{A}}$-modules then $\mathrm{cl}(\boldsymbol{T}) \colon \mathrm{cl}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \longrightarrow \mathrm{cl}(\boldsymbol{\mathscr{M}}'_{\boldsymbol{\mathscr{A}}})$ defined by

$$\mathrm{cl}(\boldsymbol{T})\mathrm{cl}(\boldsymbol{x}) = \mathrm{cl}(\boldsymbol{T}(\boldsymbol{x})) \tag{6.3.9}$$

is a module morphism of right $\mathscr{A}$-modules.

iii.) The classical limit of right modules and their morphisms yields a functor

$$\mathrm{cl} \colon \mathsf{mod}_{\boldsymbol{\mathscr{A}}} \longrightarrow \mathsf{mod}_{\mathscr{A}}. \tag{6.3.10}$$

PROOF: A simple verification shows that the map (6.3.8) is well-defined, R-bilinear, and yields a right $\mathscr{A}$-module structure. For the second part we note that $\mathrm{cl}(\boldsymbol{T})$ is again well-defined since $\boldsymbol{T}$ is $\mathsf{R}[\![\lambda]\!]$-linear. Then it is easy to see that $\mathrm{cl}(\boldsymbol{T})$ is right $\mathscr{A}$-linear indeed. The functoriality (6.3.10) is a straightforward computation. □

This version of the classical limit is functorial but not very suited to modules carrying additional structures like inner products. While in the present ring-theoretic situation this definition of cl is completely appropriate, we shall need a more sophisticated one later. For the case of a module deformation $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} = (\mathscr{M}_{\mathscr{A}}[\![\lambda]\!], \bullet)$ this version of the classical limit reproduces the original module $\mathscr{M}_{\mathscr{A}}$ up to a natural isomorphism.

On the other hand, it may also happen that the classical limit erases quite a bit of information about the module: if the multiplication by λ is invertible in the module then the classical limit is trivial. Here a typical examples can be obtained as follows:

Example 6.3.4 (Classical limit of formal Laurent series) We consider the ring R as algebra over itself and take the trivial deformation $\mathsf{R}[\![\lambda]\!]$. Moreover, we consider the formal Laurent series $\mathsf{R}(\!(\lambda)\!)$ with coefficients in R. Recall that they consist of formal series where now a finite number of negative powers of λ is allowed. The multiplication is still given by the Cauchy product formula. Then every element in $\mathsf{R}(\!(\lambda)\!)$ is a multiple of λ since we have $\lambda^{-1} \in \mathsf{R}(\!(\lambda)\!)$. Viewing $\mathsf{R}(\!(\lambda)\!)$ as a $\mathsf{R}[\![\lambda]\!]$-module we therefore find $\lambda\mathsf{R}(\!(\lambda)\!) = \mathsf{R}(\!(\lambda)\!)$ and hence

$$\mathrm{cl}(\mathsf{R}(\!(\lambda)\!)) = \{0\}. \tag{6.3.11}$$

6.3.2 Hochschild Cohomology II

As for algebras, also module deformations allow for a formulation of obstructions by means of Hochschild cohomological techniques. In this section we briefly outline to generalization needed to incorporate modules. The module condition for a deformation $\bullet$ explicitly means that

$$(x \bullet a) \bullet b = x \bullet (a \star b) \tag{6.3.12}$$

holds for all $x \in \mathscr{M}_{\mathscr{A}}[\![\lambda]\!]$ and $a, b \in \mathscr{A}[\![\lambda]\!]$. It is sufficient to consider x and a, b without powers of λ. Then we can evaluate this condition order by order to obtain the infinite system of equations

$$\sum_{r=0}^{k} \varrho_r(\varrho_{k-r}(x, a), b) = \sum_{r=0}^{k} \varrho_r(x, \mu_{k-r}(a, b)) \tag{6.3.13}$$

for $x \in \mathscr{M}_{\mathscr{A}}$ and $a, b \in \mathscr{A}$. Here μ_r is the given r-th order term of the deformation $\star$ as in (6.1.17) and $\mu_0(a, b) = ab$ as well as $\varrho_0(x, a) = x \cdot a$. Then (6.3.13) is understood as a recursive conditions for the maps $\varrho_r \colon \mathscr{M}_{\mathscr{A}} \times \mathscr{A} \longrightarrow \mathscr{M}_{\mathscr{A}}$ for all $r \geq 1$ which encode the module deformation $\bullet$ as in (6.3.2).

As in the case of an associative deformation of $\mathscr{A}$ we can interpret this system of conditions using an appropriate Hochschild differential. To find the corresponding complex, we first note that the endomorphisms $\operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ are an $(\mathscr{A}, \mathscr{A})$-bimodule in the following canonical way. For $A \in \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ and $a \in \mathscr{A}$ we define $a \cdot A$ and $A \cdot a$ on elements $x \in \mathscr{M}_{\mathscr{A}}$ by

$$(a \cdot A)(x) = A(x \cdot a) \quad \text{and} \quad (A \cdot a)(x) = (A(x)) \cdot a. \tag{6.3.14}$$

This determines new endomorphisms $a \cdot A, A \cdot a \in \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ and it is easily shown that we obtain a bimodule structure as claimed, see also Exercise 6.4.17. Note that we consider only R-linear endomorphisms but not necessarily $\mathscr{A}$-linear ones: the R-submodule $\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}) \subseteq \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ is typically *not* a sub-bimodule for this $(\mathscr{A}, \mathscr{A})$-bimodule structure.

We re-interpret the maps ϱ_r now as R-linear maps

$$\varrho_r \colon \mathscr{A} \longrightarrow \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}), \tag{6.3.15}$$

using the same symbol. This motivates to consider the space of all multilinear maps from copies of the algebra into the bimodule of endomorphisms of $\mathscr{M}_{\mathscr{A}}$. More generally, for an $(\mathscr{A}, \mathscr{A})$-bimodule ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ one defines the Hochschild complex with values in ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ as follows:

Definition 6.3.5 (Hochschild complex II) *Let $\mathscr{A}$ be an associative algebra over a ring R and let ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ be an $(\mathscr{A}, \mathscr{A})$-bimodule.*

i.) The Hochschild complex $\mathrm{HC}^\bullet(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})$ of $\mathscr{A}$ with coefficients in ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ is

$$\mathrm{HC}^\bullet(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) = \bigoplus_{n=0}^{\infty} \mathrm{HC}^n(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) \quad \textit{with} \quad \mathrm{HC}^n(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) = \operatorname{Hom}_{\mathsf{R}}\big(\underbrace{\mathscr{A}, \ldots, \mathscr{A}}_{n \textit{ times}}; {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}\big), \tag{6.3.16}$$

equipped with the Hochschild differential $\delta\colon \mathrm{HC}^\bullet(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) \longrightarrow \mathrm{HC}^{\bullet+1}(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})$ defined by

$$\begin{aligned}(\delta\phi)(a_1, \ldots, a_{n+1}) &= a_1 \cdot (\phi(a_2, \ldots, a_{k+1})) \\ &\quad + \sum_{i=1}^{k} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{k+1}) + (-1)^{k+1} (\phi(a_1, \ldots, a_k)) \cdot a_{k+1}.\end{aligned} \tag{6.3.17}$$

ii.) The Hochschild cohomology of $\mathscr{A}$ with values in the bimodule ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ is defined by

$$\mathrm{HH}^\bullet(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) = \bigoplus_{k=0}^{\infty} \mathrm{HH}^k(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) \quad \textit{with} \quad \mathrm{HH}^k(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) = \frac{\ker\big(\delta\big|_{\mathrm{HC}^k(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})}\big)}{\delta(\mathrm{HC}^{k-1}(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}))}. \tag{6.3.18}$$

It is clear from the definition that we recover our previous definition of the Hochschild complex for the algebra if we take as bimodule ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$: this explains the notation in (6.1.28).

Of course, we have to check that δ is indeed a differential. We collect this and a few other immediate properties of the Hochschild complex in the following proposition:

Proposition 6.3.6 *Let $\mathscr{A}$ be an associative algebra over a ring R and let ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ be an $(\mathscr{A}, \mathscr{A})$-bimodule.*

i.) The Hochschild differential satisfies

$$\delta^2 = 0. \tag{6.3.19}$$

ii.) For $n = 0$ we have

$$\mathrm{HH}^0(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) = \ker\left(\delta\big|_{\mathrm{HC}^0(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})}\right) = \left\{x \in {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \;\middle|\; a \cdot x = x \cdot a \textit{ for all } a \in \mathscr{A}\right\}. \tag{6.3.20}$$

iii.) For $n = 1$ we have

$$\ker\left(\delta\big|_{\mathrm{HC}^1(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})}\right) = \left\{D \in \mathrm{Hom}_{\mathsf{R}}(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}) \;\middle|\; D(ab) = a \cdot D(b) + D(a) \cdot b \textit{ for all } a, b \in \mathscr{A}\right\}. \tag{6.3.21}$$

while $(\delta x)(a) = a \cdot x - x \cdot a$ for all $x \in {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ and $a \in \mathscr{A}$.

PROOF: The first part is an immediate verification, see Exercise 6.4.10. In degree $n = 0$ there are not yet exact terms and thus (6.3.20) follows at once. In degree $n = 1$ the condition $\delta D = 0$ immediately gives the Leibniz rule $D(ab) = a \cdot D(b) + D(a) \cdot b$. ☐

Thus the zeroth Hochschild cohomology of a bimodule ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ is given by the *central elements* of the bimodule, generalizing the center of the algebra in the case ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} = {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$ from (6.1.32). Hence the zeroth Hochschild cohomology is a measure for the difference of the left and right module structures on ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$. In the particular case of the $(\mathscr{A}, \mathscr{A})$-bimodule $\mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ with some right $\mathscr{A}$-module $\mathscr{M}_{\mathscr{A}}$ this gives

$$\mathrm{HH}^0(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})) = \mathrm{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}), \tag{6.3.22}$$

see also Exercise 6.4.17, *ii.)*. The first Hochschild cohomology $\mathrm{HH}^1(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})$ can be interpreted as the ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$-valued derivations modulo the inner derivations: we get the *outer derivations* with values in ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$, see also (6.1.34).

For the deformation problem (6.3.13) we have now the following result:

Proposition 6.3.7 (Existence of right module deformations) *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be an associative deformation of an associative algebra $\mathscr{A}$ over a ring R. Moreover, let $\mathscr{M}_{\mathscr{A}}$ be a right $\mathscr{A}$-module and denote the right module structure by $x \cdot a = \varrho_0(a)x$. Finally, suppose that $\varrho_0, \ldots, \varrho_r \colon \in \mathrm{HC}^1(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$ are given such that $\varrho^{(r)} = \varrho_0 + \lambda\varrho_1 + \cdots + \lambda^r\varrho_r$ is a right module deformation up to order r for some $r \in \mathbb{N}_0$.*

i.) The map $R_r \in \mathrm{HC}^2(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$ defined by

$$R_r(a, b) = \sum_{s=0}^{r} \varrho_s(\mu_{r+1-s}(a, b)) - \sum_{s=1}^{r} \varrho(b) \circ \varrho_{r+1-s}(a) \tag{6.3.23}$$

for $a, b \in \mathscr{A}$ is a δ-cocycle, i.e. $\delta R_r = 0$.

ii.) There exists a $\varrho_{r+1} \in \mathrm{HC}^1(\mathscr{A}, {}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}})$ such that $\varrho^{(r+1)} = \varrho^{(r)} + \lambda^{r+1}\varrho_{r+1}$ is a right module deformation up to order $r + 1$ iff

$$\delta\varrho_{r+1} = R_r, \tag{6.3.24}$$

iff the cohomology class $[R_r] \in \mathrm{HH}^2(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$ is trivial.

PROOF: The proof is essentially a direct computation and can be done in Exercise 6.4.18. ☐

Corollary 6.3.8 *If $\mathrm{HH}^2(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})) = \{0\}$ then there exists a deformation $\bullet$ of $\mathscr{M}_{\mathscr{A}}$ as a right module for every associative deformation $\star$ of $\mathscr{A}$.*

For the equivalence of two module deformations $\bullet$ and $\tilde{\bullet}$ we get also a cohomological description:

Proposition 6.3.9 *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[[\lambda]], \star)$ be an associative deformation of an associative algebra $\mathscr{A}$ over a ring R. Moreover, let $\mathscr{M}_{\mathscr{A}}$ be a right $\mathscr{A}$-module and denote the right module structure by $x \cdot a = \varrho_0(a)x$. Suppose that $\bullet = \varrho_0 + \sum_{r=1}^{\infty} \lambda^r \varrho_r$ and $\tilde{\bullet} = \varrho_0 + \sum_{r=1}^{\infty} \lambda^r \tilde{\varrho}_r$ are two module deformations of $\mathscr{M}_{\mathscr{A}}$ with respect to $\star$. Furthermore, suppose that $T^{(r)} = \mathrm{id} + \lambda^1 T_1 + \cdots + \lambda^r T_r$ with $T_r \in \mathrm{HC}^0(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})) = \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ constitutes an equivalence from $\bullet$ to $\tilde{\bullet}$ up to order r, i.e. $T^{(r)}(x \bullet a) = T^{(r)}(x) \mathbin{\tilde{\bullet}} a$ holds up to order n for all $a \in \mathscr{A}$ and $x \in \mathscr{M}_{\mathscr{A}}$.*

i.) The map $E_r \in \mathrm{HC}^1(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$, defined by

$$E_r(a) = \sum_{s=0}^{r} (\tilde{\varrho}_{r+1-s} \circ T_s - T_s \circ \varrho_{r+1-s}(a)) \tag{6.3.25}$$

for $a \in \mathscr{A}$, satisfies $\delta E_r = 0$.

ii.) There exists a $T_{r+1} \in \mathrm{HC}^0(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$ such that $T^{(r+1)} = T^{(r)} + \lambda^{r+1} T_{r+1}$ is an equivalence up to order $r+1$ iff

$$\delta T_{r+1} = E_r \tag{6.3.26}$$

iff the class $[E_r] \in \mathrm{HH}^1(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$ is trivial.

PROOF: Again, this is a direct computation discussed in Exercise 6.4.18. □

Corollary 6.3.10 *If $\mathrm{HH}^1(\mathrm{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})) = \{0\}$ then any two deformations $\bullet$ and $\tilde{\bullet}$ of $\mathscr{M}_{\mathscr{A}}$ as a right module for an associative deformation $\star$ of $\mathscr{A}$ are equivalent.*

Note, however, that the proposition gives only an absolute obstruction in first order as in higher orders we could in principle still allow for equivalence transformations T of more general type. It only gives an obstruction to *continue* a given equivalence transformation to the next order: there might be equivalence transformations after changing also the lower order terms.

In many situations the Hochschild cohomologies $\mathrm{HH}^k(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$ are nontrivial. In this situation, the above cohomological analysis of the deformation problem will not help much: there is the possibility for obstructions but whether one actually can avoid them or not is not answered by the two propositions. If, however, the cohomologies are trivial then we get an easy existence and uniqueness of module deformations.

Note also that a similar analysis can be done for left modules instead of right modules: the formulas for the error terms change slightly but the cohomological obstructions are again in the first and second Hochschild cohomology of the bimodule of the endomorphisms of the left module, see also Exercise 6.4.18.

Finally, we note that in many cases one wants the components ϱ_r of the module structure to be more specific maps than just arbitrary elements in $\mathrm{HC}^1(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}))$. E.g. in differential geometric contexts one is interested in differential operators etc. In these cases it is often useful to pass to certain sub-complexes which take into account the desired properties, see Exercise 6.4.19 for a refined version of the above two propositions.

6.3.3 Deformation of Bimodules

Since ultimately we are interested in Morita theory we need to investigate the deformation theory of bimodules. The strategy will be to consider a bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ first as a right $\mathscr{A}$-module and investigate its deformations as a right $\boldsymbol{\mathscr{A}}$-module: this will hopefully allow to determine the module endomorphisms into which we need to map the deformed algebra $\boldsymbol{\mathscr{B}}$ in a second step. Alternatively, one can try to deform the bimodule structure directly, see Exercise 6.4.20. However, it turns out that this is typically much more complicated in actual examples.

Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[[\lambda]], \star)$ be a given associative deformation of $\mathscr{A}$. We start with the following simple observation that module endomorphisms of deformed modules have classical limits:

Lemma 6.3.11 *Let $\mathscr{M}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with a given module deformation $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} = (\mathscr{M}_{\mathscr{A}}[[\lambda]], \bullet)$ as a right $\boldsymbol{\mathscr{A}}$-module.*

i.) The module endomorphisms $\operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ form a unital $\mathsf{R}[[\lambda]]$-subalgebra of $\operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})[[\lambda]]$.

ii.) The classical limit map induces a unital algebra homomorphism

$$\mathrm{cl}\colon \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \ni A \mapsto \mathrm{cl}(A) = A_0 \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}). \tag{6.3.27}$$

PROOF: First we note that by our conventions any module endomorphism is linear over the underlying scalars. Hence for $A \in \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ we have $A \in \operatorname{End}_{\mathsf{R}[[\lambda]]}(\mathscr{M}_{\mathscr{A}}[[\lambda]]) = \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})[[\lambda]]$ using Proposition 6.1.1. Now clearly, the module endomorphisms form a unital algebra over $\mathsf{R}[[\lambda]]$, completing the proof of the first part. This allows to consider the classical limit of A since by the first part $A = \sum_{r=0}^{\infty} \lambda^r A_r$ with $A_r \in \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$. Hence $\mathrm{cl}(A) = A_0$ gives the map needed for (6.3.27). Being a subalgebra of $\operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})[[\lambda]]$, the classical limit preserves the algebraic structures as usual. Finally, $A_0 = \mathrm{cl}(A)$ is right $\mathscr{A}$-linear since for $a \in \mathscr{A}$ and $A \in \operatorname{End}_{\mathsf{R}[[\lambda]]}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ we have for all $x = \sum_{r=0}^{\infty} \lambda^r x_r \in \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} = \mathscr{M}_{\mathscr{A}}[[\lambda]]$ the relation $A(x \bullet a) = A(x) \bullet a$. Taking classical limits gives $\mathrm{cl}(A(x \bullet a)) = A_0(x_0 \cdot a)$ and $\mathrm{cl}(A(x) \bullet a) = A_0(x_0) \cdot a$. □

As usual, the classical limit map

$$\mathrm{cl}\colon \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \longrightarrow \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}) \tag{6.3.28}$$

is not injective since we loose the higher order terms of the endomorphisms of the deformed module: it is only injective for the zeroth order. Moreover, in general it is also not surjective: surjectivity would mean that we can *quantize* every classical endomorphism A_0 into a quantum endomorphism $A = A_0 + \lambda A_1 + \cdots$ by finding appropriate higher order terms $A_1, \ldots \in \operatorname{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ to achieve $A \in \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$. Note that in general we can not expect that $A_0 \in \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ is already $\boldsymbol{\mathscr{A}}$-linear without such corrections:

Example 6.3.12 We consider a unital algebra $\mathscr{A}$ as a right $\mathscr{A}$-module. In this case we know that $\operatorname{End}_{\mathscr{A}}(\mathscr{A}_{\mathscr{A}})$ is isomorphic to $\mathscr{A}$ acting via left-multiplications. Given a deformation $\star$ of $\mathscr{A}$ then the undeformed left multiplications on $\mathscr{A}[[\lambda]]$ will not be right $\boldsymbol{\mathscr{A}}$-linear in general. Instead, we have to pass to the left multiplications with respect to $\star$: in the map $x \mapsto a \star x = a \cdot x + \cdots$ we typically have higher orders needed to turn this into a right $\boldsymbol{\mathscr{A}}$-linear map.

Without the classical limit being surjective, not much can be said. But even then, yet another difficulty is the following: when working with rings instead of fields for the scalars then a surjective map might or might not split in an R-linear way. To avoid the resulting subtleties we will even assume to have such a splitting which in the case of a field R is automatic:

Proposition 6.3.13 *Let $\mathscr{M}_{\mathscr{A}}$ be a right $\mathscr{A}$-module with a given deformation $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} = (\mathscr{M}_{\mathscr{A}}[[\lambda]], \bullet)$ as a right $\boldsymbol{\mathscr{A}}$-module. Suppose that in addition there exists a R-linear map*

$$\mathfrak{q}\colon \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}) \longrightarrow \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \tag{6.3.29}$$

with $\mathrm{cl} \circ \mathfrak{q} = \mathrm{id}_{\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})}$.

i.) The $\mathsf{R}[[\lambda]]$-linear extension of $\mathfrak{q}$ yields an isomorphism

$$\mathfrak{q}\colon \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[[\lambda]] \longrightarrow \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \tag{6.3.30}$$

of $\mathsf{R}[[\lambda]]$-modules.

ii.) *There exists an associative unital deformation* $\star'$ *of* $\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ *together with a left module structure* $\bullet'$ *on* $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ *such that* $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ *becomes a* $((\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!], \star'), \boldsymbol{\mathscr{A}})$*-bimodule with respect to* $\bullet'$ *and* $\bullet$.

iii.) *If* $\tilde{\star}'$ *and* $\tilde{\bullet}'$ *are other such deformations of* $\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ *and the left module structure then* $\tilde{\star}'$ *and* $\star'$ *are equivalent via some uniquely determined equivalence transformation* S *such that*

$$A \mathbin{\tilde{\bullet}'} x = S(A) \bullet' x \tag{6.3.31}$$

for all $A \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!]$ *and* $x \in \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$.

Proof: First we note that the existence of $\mathfrak{q}$ implies that cl is surjective and in the case where R is a field the converse would be true as well: surjectivity of cl implies the existence of $\mathfrak{q}$. Since $\mathrm{cl} \circ \mathfrak{q} = \mathrm{id}$ we see that the $\mathsf{R}[\![\lambda]\!]$-linear extension $\mathfrak{q}$ as in (6.3.29) is injective in zeroth order and hence injective. We have to show that (6.3.29) is surjective. Thus let $A \in \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ be given with $A_0 = \mathrm{cl}(A)$. Then $\mathfrak{q}(A_0) \in \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ and $\mathrm{cl}(\mathfrak{q}(A_0) - A) = 0$ shows that there is a $A_1 = \mathrm{cl}(\frac{1}{\lambda}(A - \mathfrak{q}(A_0))) \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$. We continue with $\mathfrak{q}(A_0 + \lambda A_1) \in \operatorname{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}})$ and a simple induction completes the proof that we can construct $A_1, A_2, \ldots$ such that $A = \mathfrak{q}(A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots)$. This gives the first part. Then the second part is easy since for $A, B \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!]$ we can set

$$A \bullet' x = \mathfrak{q}(A)x \quad \text{and} \quad A \star' B = \mathfrak{q}^{-1}(\mathfrak{q}(A)\mathfrak{q}(B)),$$

which are now easily verified to do the job. Note that $\bullet'$ and $\star'$ are indeed deformations, i.e. reproduce the classical left module structure and the classical algebra multiplication in zeroth order of λ. Suppose now that $\tilde{\star}'$ and $\tilde{\bullet}'$ are other such deformations. Then $x \mapsto A \mathbin{\tilde{\bullet}'} x$ is right $\boldsymbol{\mathscr{A}}$-linear by assumption. By the first and second part there exists a unique $S(A) \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!]$ with $A \mathbin{\tilde{\bullet}'} x = S(A) \bullet' x$ for all $x \in \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$. From the uniqueness one concludes immediately that the resulting map $S\colon \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!] \longrightarrow \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!]$ is $\mathsf{R}[\![\lambda]\!]$-linear. Moreover, since both $\bullet'$ and $\tilde{\bullet}'$ deform the usual action of endomorphism, taking the classical limit of $A \mathbin{\tilde{\bullet}'} x = S(A) \bullet' x$ gives $\mathrm{cl}(A)\mathrm{cl}(x) = \mathrm{cl}(S)(\mathrm{cl}(A))\mathrm{cl}(x)$ and hence $S = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r S_r$ follows. Finally, we have

$$S(A \mathbin{\tilde{\star}'} B) \bullet' x = (A \mathbin{\tilde{\star}'} B) \mathbin{\tilde{\bullet}'} x = A \mathbin{\tilde{\bullet}'} (B \mathbin{\tilde{\bullet}'} x) = S(A) \bullet' (S(B) \bullet' x) = (S(A) \star' S(B)) \bullet' x$$

for all $x \in \boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ showing $S(A \mathbin{\tilde{\star}'} B) = S(A) \star' S(B)$ since the map $A \mapsto (x \mapsto A \bullet' x)$ is injective. This shows that S is an equivalence between $\star'$ and $\tilde{\star}'$. ☐

Remark 6.3.14 (Deformation of module endomorphisms) In many interesting situations the surjectivity of cl and (up to technical questions) hence the existence of a quantization map $\mathfrak{q}$ is actually fulfilled. In this situation, we call the deformation $\star'$ of $\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ the *induced deformation* of the endomorphism. This is unique up to equivalence and depends on the deformation $\star$ of $\mathscr{A}$ and on the right module deformation $\bullet$. Note that if we pass to an equivalent deformation $\tilde{\star}$ with an equivalent module deformation $\tilde{\bullet}$ then we also get equivalent induced deformations for $\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ and the corresponding left module structure. Moreover, in many cases we will encounter, the right module $\mathscr{M}_{\mathscr{A}}$ can be deformed only in a unique way up to equivalence: in this case, the deformation $\star'$ and $\bullet'$ are already fixed up to equivalence by $\star$ and the classical right module $\mathscr{M}_{\mathscr{A}}$. A large class of such examples arising from principal bundles in differential geometry can be found in [16], see also [64].

The problem of finding deformations of *bimodules* can now be rephrased in the following way:

Corollary 6.3.15 *Let* ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ *be a* $(\mathscr{B}, \mathscr{A})$*-bimodule and let* $\boldsymbol{\mathscr{B}} = (\mathscr{B}[\![\lambda]\!], \star')$ *and* $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ *be associative deformations such that there exists a right module deformation* $\bullet$ *of* $\mathscr{M}_{\mathscr{A}}$ *together with a quantization map* $\mathfrak{q}$ *as* (6.3.29). *Then there exists a left* $\boldsymbol{\mathscr{B}}$*-module structure* $\bullet'$ *on* $\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ *turning it into* *a* $(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$*-bimodule via* $\bullet'$ *and* $\bullet$ *iff the algebra homomorphism* $\mathscr{B} \ni b \mapsto (x \mapsto b \cdot x) \in \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ *can be deformed into an algebra homomorphism*

$$(\mathscr{B}[\![\lambda]\!], \star') \longrightarrow (\operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})[\![\lambda]\!], \star'), \tag{6.3.32}$$

where on the right hand side we use the induced deformation $\star'$ *of the endomorphisms.*

Note that thanks to the uniqueness it will not be important which of the equivalent deformations $\star'$ for the endomorphisms we actually use: either we can find the deformation of the algebra homomorphism for all or for none.

Remark 6.3.16 (Obstructions to bimodule deformations) In general, we will encounter *obstructions* for bimodule deformations: suppose we are in the situation of Corollary 6.3.15 with the additional assumption that $\mathscr{B} = \operatorname{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ simply coincides with the classical endomorphisms. This will be the case of e.g. Morita equivalence bimodules. Then a given deformation $\star''$ of $\mathscr{B}$ will allow for a bimodule deformation iff $\star''$ is in the equivalence class of the induced deformation $\star'$. If $\mathscr{B}$ has non-equivalent deformations we obtain hard obstructions, see again [16, 64].

6.3.4 Deformation of Projective Modules

In view of Morita theory it is clear that projective modules will play a particular role also with respect to deformations. We investigate the deformation theory of this class of modules in some detail. As before we consider an associative deformation $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ of an associative algebra $\mathscr{A}$ over a ring of scalars R. For simplicity we assume that $\mathscr{A}$ is unital and the deformation respects the unit. The following fundamental result clarifies the deformations of projective right $\mathscr{A}$-modules completely:

Theorem 6.3.17 (Deformation of projective modules) *Let* $\mathscr{E}_{\mathscr{A}}$ *be a finitely generated projective right module over* $\mathscr{A}$.

i.) *There exists a deformation* $\bullet$ *of* $\mathscr{E}_{\mathscr{A}}$ *into a right* $\boldsymbol{\mathscr{A}}$*-module* $(\mathscr{E}_{\mathscr{A}}[\![\lambda]\!], \bullet)$.

ii.) *Any two deformations of* $\mathscr{E}_{\mathscr{A}}$ *are equivalent.*

iii.) *The deformed module* $(\mathscr{E}_{\mathscr{A}}[\![\lambda]\!], \bullet)$ *is again finitely generated and projective, now over* $\boldsymbol{\mathscr{A}}$*, with the same number of generators.*

iv.) *If* $\mathscr{E}'_{\mathscr{A}}$ *is another finitely generated projective right module over* $\mathscr{A}$ *then the classical limit map*

$$\operatorname{cl}\colon \operatorname{Hom}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}, \boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}}) \tag{6.3.33}$$

is split surjective and induces an $\mathsf{R}[\![\lambda]\!]$*-linear isomorphism*

$$\operatorname{Hom}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}, \boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}) \cong \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{E}'_{\mathscr{A}})[\![\lambda]\!]. \tag{6.3.34}$$

v.) *The deformation* $\bullet$ *induces a deformation* $\star'$ *with a corresponding deformed left-module structure* $\bullet'$ *for the endomorphisms* $\operatorname{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$. *The equivalence class of the deformation* $\star'$ *depends only on the equivalence class of* $\star$ *and the isomorphism class of* $\mathscr{E}_{\mathscr{A}}$.

vi.) *Every dual basis* $\{e_i, e^i\}_{i\in I}$ *of* $\mathscr{E}_{\mathscr{A}}$ *can be deformed into a dual basis* $\{\boldsymbol{e}_i, \boldsymbol{e}^i\}_{i\in I}$ *for* $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$*, i.e. we have* $\boldsymbol{e}_i \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ *and* $\boldsymbol{e}^i \in \operatorname{Hom}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}, \boldsymbol{\mathscr{A}})$ *with*

$$\operatorname{cl}(\boldsymbol{e}_i) = e_i \quad \textit{and} \quad \operatorname{cl}(\boldsymbol{e}^i) = e^i \tag{6.3.35}$$

as well as

$$x = \sum_{i\in I} \boldsymbol{e}_i \bullet \boldsymbol{e}^i(x) \tag{6.3.36}$$

for all $x \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$.

Proof: We mainly follow the arguments from [23]. For the first part we know that up to isomorphism we can write $\mathscr{E}_{\mathscr{A}}$ as $e_0\mathscr{A}^n$ with some idempotent $e_0 = e_0^2 \in \mathrm{M}_n(\mathscr{A})$ and some $n \in \mathbb{N}$. From Proposition 6.1.30, *iv.)*, we know that we can deform e_0 into an idempotent $e = e_0 + \cdots \in \mathrm{M}_n(\mathscr{A})[\![\lambda]\!]$ with respect to $\star$, i.e. we have $e \star e = e$. Thus we obtain a finitely generated projective module $e \star \boldsymbol{\mathscr{A}}^n$. Now again Proposition 6.1.30, *ii.)*, shows that the classical limit map $\mathrm{cl}\colon e \star \boldsymbol{\mathscr{A}}^n \longrightarrow e_0\mathscr{A}^n$ is split surjective via the map I. It induces an $\mathsf{R}[\![\lambda]\!]$-linear isomorphism $e \star \boldsymbol{\mathscr{A}}^n \cong (e_0\mathscr{A}^n)[\![\lambda]\!]$: indeed, by choosing f_0 in Proposition 6.1.30, *ii.)*, to be a projection onto a single column this follows directly. Transferring the right $\boldsymbol{\mathscr{A}}$-module structure of $e \star \boldsymbol{\mathscr{A}}^n$ back to $(e_0\mathscr{A}^n)[\![\lambda]\!]$ and thus to $\mathscr{E}_{\mathscr{A}}[\![\lambda]\!]$ yields a right $\boldsymbol{\mathscr{A}}$-module structure $\bullet$ which is the deformation of $\mathscr{E}_{\mathscr{A}}$ we are looking for. By construction, the deformation is a finitely generated projective module over $\boldsymbol{\mathscr{A}}$ with the number n of generators being the same as classically. To prove the uniqueness assume that $\tilde{\bullet}$ is another deformation of $\mathscr{E}_{\mathscr{A}}$ as a right $\boldsymbol{\mathscr{A}}$-module, not necessarily projective. Then we have the diagram

$$\begin{array}{ccccc} & & (\mathscr{E}_{\mathscr{A}}[\![\lambda]\!], \bullet) & & \\ & \overset{T}{\swarrow} & \Big\downarrow \mathrm{cl} & & \\ (\mathscr{E}_{\mathscr{A}}[\![\lambda]\!], \tilde{\bullet}) & \xrightarrow[\mathrm{cl}]{\qquad} & \mathscr{E}_{\mathscr{A}} & \longrightarrow & 0 \end{array}$$

with the classical limit maps cl for both deformations. According to Example 6.3.2 we can view the classical limit maps as morphisms of $\boldsymbol{\mathscr{A}}$-modules if we equip the classical right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ with the trivial $\boldsymbol{\mathscr{A}}$-module structure (6.3.5). Since the classical limit is obviously surjective, we get a morphism T of $\boldsymbol{\mathscr{A}}$-modules such that the diagram commutes by the very properties of a projective module from Proposition 2.3.3, *iv.)*. Thus we have $\mathrm{cl} \circ T = \mathrm{cl}$ which simply means that $T = \mathrm{id}_{\mathscr{E}_{\mathscr{A}}} + \cdots$ starts with the identity in zeroth order. But then T is invertible and $\tilde{\bullet}$ and $\bullet$ turn out to be equivalent via T. For the next part we can assume that the two projective modules are given as $\mathscr{E}_{\mathscr{A}} = e_0\mathscr{A}^n$ and $\mathscr{E}'_{\mathscr{A}} = f_0\mathscr{A}^m$ with idempotents $e_0 \in \mathrm{M}_n(\mathscr{A})$ and $f_0 \in \mathrm{M}_m(\mathscr{A})$ for some $n, m \in \mathbb{N}$. From Exercise 2.4.14 we know that the module morphisms from $\mathscr{E}_{\mathscr{A}}$ to $\mathscr{E}'_{\mathscr{A}}$ are then given by the matrices $f_0\mathrm{M}_{m\times n}(\mathscr{A})e_0$ acting on $e_0\mathscr{A}^n$ by matrix multiplication as usual. Correspondingly, the $\boldsymbol{\mathscr{A}}$-linear maps from $e \star \boldsymbol{\mathscr{A}}^n$ to $f \star \boldsymbol{\mathscr{A}}^m$ are given by $f \star \mathrm{M}_{m\times n}(\boldsymbol{\mathscr{A}}) \star e$. Then Proposition 6.1.30, *ii.)*, shows that the classical limit map is split surjective and induces an isomorphism also in this case. This shows the fourth part. Using this, the fifth then follows from Proposition 6.3.13 at once. For the last part, we first note that we can apply the fourth part to $\mathscr{E}_{\mathscr{A}}$ and $\mathscr{E}'_{\mathscr{A}} = \mathscr{A}$ to get the corresponding statement for the dual module $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{A})$. This means that we have for $e^i \in \mathrm{Hom}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{A})$ an element $\boldsymbol{e}^i \in \mathrm{Hom}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}, \boldsymbol{\mathscr{A}})$ with $\mathrm{cl}(\boldsymbol{e}^i) = e^i$. Choosing such quantizations we consider the map

$$A\colon \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \ni x \;\mapsto\; A(x) = \sum\nolimits_{i\in I} e_i \bullet \boldsymbol{e}^i(x).$$

By construction, A is right $\boldsymbol{\mathscr{A}}$-linear and hence $A \in \mathrm{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$. Since we started with a dual basis we have $\mathrm{cl}(A) = \mathrm{id}_{\mathscr{E}_{\mathscr{A}}}$. This means that A is invertible by the usual geometric series. Setting $\boldsymbol{e}_i = A^{-1}(e_i)$ will then give the desired elements in $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ with $\mathrm{cl}(\boldsymbol{e}_i) = e_i$ and (6.3.36). □

This theorem has several important corollaries. As a first application we note that the classical limit functor on projective modules is essentially surjective and full. Note that it is *not* faithful as we typically have e.g. many nontrivial endomorphisms of the deformed projective modules which are the identity in zeroth order.

Corollary 6.3.18 *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be a unital associative deformation of a unital associative algebra $\mathscr{A}$ over R. Then the classical limit functor of modules restricts to a functor*

$$\mathrm{cl}\colon \underline{\mathsf{Proj}}(\boldsymbol{\mathscr{A}}) \longrightarrow \underline{\mathsf{Proj}}(\mathscr{A}), \tag{6.3.37}$$

which is essentially surjective, injective on objects up to isomorphism, and full.

Proof: Indeed, the classical limit of a finitely generated projective module over $\boldsymbol{\mathscr{A}}$ is again finitely generated and projective over $\mathscr{A}$. The functoriality is clear already in the larger context of Proposition 6.3.3, *iii.)*. Since we can deform every projective module over $\mathscr{A}$ in a unique way up to isomorphism, the essential surjectivity and injectivity statement follows. Finally, the fullness follows from the fourth part of the theorem. □

As an immediate consequence, we see that the K_0-theory of a deformed algebra is isomorphic to the one of the undeformed algebra:

Corollary 6.3.19 (Rosenberg [100]) *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be a unital associative deformation of a unital associative algebra $\mathscr{A}$ over R. Then the classical limit induces an isomorphism of semigroups*

$$\mathrm{cl}_*\colon \mathsf{Proj}(\boldsymbol{\mathscr{A}}) \xrightarrow{\cong} \mathsf{Proj}(\mathscr{A}) \tag{6.3.38}$$

and an isomorphism of groups

$$\mathrm{cl}_*\colon \mathsf{K}_0(\boldsymbol{\mathscr{A}}) \xrightarrow{\cong} \mathsf{K}_0(\mathscr{A}). \tag{6.3.39}$$

We can give also a different interpretation of the rigidity of projective modules. The fact that they always can be deformed in a unique way up to equivalence also follows from the computation of their Hochschild cohomology:

Proposition 6.3.20 *Let $\mathscr{E}_{\mathscr{A}}$ be a finitely generated projective module over a unital associative algebra $\mathscr{A}$ over R. Then one has*

$$\mathrm{HH}^k(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}})) = \begin{cases} \mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}) & \textit{for } k = 0 \\ \{0\} & \textit{for } k \neq 0. \end{cases} \tag{6.3.40}$$

Proof: For $k = 0$ we always have that the zeroth Hochschild cohomology coincides with the right $\mathscr{A}$-linear endomorphisms, see Proposition 6.3.6, *ii.)*, and in particular (6.3.22). To show that the higher Hochschild cohomologies are trivial, we follow [120, Prop. 2.7.1] and consider a dual basis $\{e_i, e^i\}_{i\in I}$ for $\mathscr{E}_{\mathscr{A}}$, see Proposition 2.3.3, *iii.)*. For a Hochschild cochain $\phi \in \mathrm{HC}^k(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}}))$ we can then define

$$((h\phi)(a_1, \ldots, a_{k-1}))x = \sum_{i\in I} \phi(e^i(x), a_1, \ldots, a_{k-1})e_i,$$

where $a_1, \ldots, a_{k-1} \in \mathscr{A}$ and $x \in \mathscr{E}_{\mathscr{A}}$. This defines an element $h\phi \in \mathrm{HC}^{k-1}(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}}))$ and hence a map

$$h\colon \mathrm{HC}^\bullet(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}})) \longrightarrow \mathrm{HC}^{\bullet-1}(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}})),$$

if we set h to be zero on $\mathrm{HC}^0(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}}))$. We claim that h is a homotopy of the Hochschild differential, i.e. for $k \geq 1$ we have

$$(h\delta + \delta h)\Big|_{\mathrm{HC}^k(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}}))} = \mathrm{id}_{\mathrm{HC}^k(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}}))}. \tag{$*$}$$

Indeed, this is just a computation. We have for $\phi \in \mathrm{HC}^k(\mathscr{A}, \mathrm{End}_{\mathsf{R}}(\mathscr{E}_{\mathscr{A}}))$

$$\begin{aligned} &((\delta h\phi)(a_1, \ldots, a_k))(x) \\ &\quad = ((h\phi)(a_2, \ldots, a_k))(x \cdot a_1) \\ &\qquad + \sum_{r=0}^{k-1} (-1)^r ((h\phi)(a_1, \ldots, a_r a_{r+1}, a_k))(x) + (-1)^k (((h\phi)(a_1, \ldots, a_{k-1}))(x)) \cdot a_k \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in I} \big(\phi(e^i(x\cdot a_1), a_2, \ldots, a_k)\big)(e_i) \\
&\quad + \sum_{i\in I}\sum_{r=0}^{k-1} (-1)^r \big(\phi(e^i(x), a_1, \ldots, a_r a_{r+1}, a_k)\big)(e_i) + (-1)^k \sum_{i\in I} \big((\phi(e^i(x), a_1, \ldots, a_{k-1}))(e_i)\big)\cdot a_k,
\end{aligned}
$$

where $a_1, \ldots, a_k$ and $x \in \mathscr{E}_{\mathscr{A}}$. For the other contribution we get

$$
\begin{aligned}
&((h\delta\phi)(a_1, \ldots, a_k))(x) \\
&\quad = \sum_{i\in I} \big((\delta\phi)(e^i(x), a_1, \ldots, a_k)\big)(e_i) \\
&\quad = \sum_{i\in I} (\phi(a_1, \ldots, a_k))(e_i \cdot e^i(x)) - \sum_{i\in I} \big(\phi(e^i(x)a_1, a_2, \ldots, a_k)\big)(e_i) \\
&\qquad + \sum_{i\in I}\sum_{r=1}^{k-1} (-1)^{r+1} \big(\phi(e^i(x), a_1, \ldots, a_r a_{r+1}, \ldots, a_k)\big)(e_i) \\
&\qquad + (-1)^{k+1} \sum_{i\in I} \big((\phi(e^i(x), a_1, \ldots, a_{k-1}))(e_i)\big)\cdot a_k.
\end{aligned}
$$

Now the property

$$
x = \sum\nolimits_{i\in I} e_i \cdot e^i(x)
$$

of a dual basis together with the right $\mathscr{A}$-linearity $e^i(x\cdot a) = e^i(x)a$ of the maps $e^i \in \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}}, \mathscr{A})$ shows that $(*)$ holds, i.e. the identity is homotopic to zero. With this homotopy equation we get (6.3.40) for $k \geq 1$ at once. □

With this result we can now apply our considerations from Section 6.3.2 and conclude that deformations of projective modules always exist and are unique up to equivalence: Proposition 6.3.7 and Proposition 6.3.9 can be applied to this case.

6.4 Exercises

Exercise 6.4.1 (Proof of Proposition 6.1.1) Prove Proposition 6.1.1.

Hint: Let Φ be given and define ϕ_0 to be the map

$$
\phi_0(v_1, \ldots, v_n) = \operatorname{cl}(\Phi(v_1, \ldots, v_n))
$$

for $v_1 \in V_1, \ldots, v_n \in V_n$. Show that this is an R-multilinear map. Extend this map now to formal power series and consider $\Phi - \phi_0$. Why is this the starting point for an induction?

Exercise 6.4.2 (Classical limit of insertions) Let V and W be modules over R. Moreover, let $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$, as well as U be modules over R and consider the formal power series $V[\![\lambda]\!]$ etc. as modules over $\mathsf{R}[\![\lambda]\!]$ as usual.

i.) Let $\Phi\colon V_1[\![\lambda]\!] \times \cdots \times V_n[\![\lambda]\!] \longrightarrow W_{i+1}[\![\lambda]\!]$ be an $\mathsf{R}[\![\lambda]\!]$-multilinear map with $i = 0, \ldots, m-1$. Moreover, let also $\Psi\colon W_1[\![\lambda]\!] \times \cdots \times W_m[\![\lambda]\!] \longrightarrow U[\![\lambda]\!]$ be $\mathsf{R}[\![\lambda]\!]$-multilinear. Show that for the insertion after the i-th position we have

$$
\operatorname{cl}(\Psi \circ_i \Phi) = \operatorname{cl}(\Psi) \circ_i \operatorname{cl}(\Phi). \tag{6.4.1}
$$

ii.) Let $\Phi\colon V[\![\lambda]\!] \longrightarrow W[\![\lambda]\!]$ and $\Psi\colon W[\![\lambda]\!] \longrightarrow U[\![\lambda]\!]$ be $\mathsf{R}[\![\lambda]\!]$-linear maps. Show that for their composition one has

$$
\operatorname{cl}(\Psi \circ \Phi) = \operatorname{cl}(\Psi) \circ \operatorname{cl}(\Phi). \tag{6.4.2}
$$

iii.) Show that the classical limit gives a monoid morphism

$$\mathrm{cl}\colon \operatorname{End}_{\mathsf{R}[\![\lambda]\!]}(V[\![\lambda]\!]) \longrightarrow \operatorname{End}_{\mathsf{R}}(V), \tag{6.4.3}$$

which restricts to a group morphism

$$\mathrm{cl}\colon \mathrm{Gl}(V[\![\lambda]\!]) \longrightarrow \mathrm{Gl}(V). \tag{6.4.4}$$

Determine the kernel and the image of this group morphism explicitly.

Exercise 6.4.3 (The λ-adic topology) Give a proof of the statements in Proposition 6.1.2.

Hint: All statements except the completeness are rather straightforward. For the completeness rewrite the condition of being a Cauchy sequence using the order. The main step consists now in showing that for the Cauchy sequence and for every $k \in \mathbb{N}_0$ the k-th order of the members of the sequence becomes *constant* after finitely many terms.

Exercise 6.4.4 (Topologically free modules)

Exercise 6.4.5 (Banach's fixed point theorem) Let V be a module over a ring R and consider $V[\![\lambda]\!]$ as $\mathsf{R}[\![\lambda]\!]$-module as usual.

i.) Recall Banach's fixed point theorem for contracting maps on complete metric spaces.

ii.) Let $T\colon V[\![\lambda]\!] \longrightarrow V[\![\lambda]\!]$ be a (not necessarily linear) map. Show that T is contracting with respect to the λ-adic metric iff there is a $k \in \mathbb{N}$ with

$$o(T(x) - T(y)) \geq o(x - y) + k \tag{6.4.5}$$

for all $x, y \in V[\![\lambda]\!]$. In this case $q = 2^{-k}$ is a Lipschitz constant for T.

This gives a very simple criterion for contracting maps as one only has to count orders in λ correctly.

Exercise 6.4.6 (The induced Poisson bracket) Let $(\mathscr{A}, \mu_0)$ be an associative commutative algebra with two formal associative deformations $\mu = \mu_0 + \lambda\mu_1 + \cdots$ and $\tilde{\mu} = \mu_0 + \lambda\tilde{\mu}_1 + \cdots$. Show that the induced Poisson brackets according to Proposition 6.1.8 coincide if the deformations μ and $\tilde{\mu}$ are equivalent.

Hint: Compute explicitly how the first order term of the deformation changes if one passes from μ to an equivalent deformation by means of an equivalence transformation $S = \mathrm{id} + \lambda S_1 + \cdots$.

Exercise 6.4.7 (Gerstenhaber deformation) Let $\mathscr{A}$ be an associative algebra over R where $\mathbb{Q} \subseteq \mathsf{R}$.

i.) Consider the formal power series $\mathscr{A}[\![\lambda]\!]$ equipped with the undeformed product inherited from $\mathscr{A}$. Show that in this case any derivation D of $\mathscr{A}[\![\lambda]\!]$ is a formal power series of derivations of $\mathscr{A}$ and conclude

$$\operatorname{Der}(\mathscr{A}[\![\lambda]\!]) = \operatorname{Der}(\mathscr{A})[\![\lambda]\!]. \tag{6.4.6}$$

ii.) Consider $2n$ pairwise commuting derivations $D_1, \ldots, D_n, E_1, \ldots, E_n \in \operatorname{Der}(\mathscr{A}[\![\lambda]\!])$. Show that Gerstenhaber's construction as in Proposition 6.1.12 still yields an associative deformation $\star$ of $\mathscr{A}$.

iii.) Assume that $\mathscr{A}$ is in addition commutative. Compute for this deformation $\star$ the induced Poisson bracket according to Proposition 6.1.8.

Exercise 6.4.8 (Deformation by commuting derivations) Inspired by Gerstenhaber's construction from Proposition 6.1.12 one can consider the following slight but very useful generalization. Suppose one has an associative algebra $\mathscr{A}$ and a linear map

$$P\colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A} \tag{6.4.7}$$

with the following properties: First, P is a biderivation in the sense that

$$P(ab \otimes c) = P(a \otimes c)(b \otimes \mathbb{1}) + (a \otimes \mathbb{1})P(b \otimes c) \tag{6.4.8}$$

and

$$P(a \otimes bc) = P(a \otimes b)(\mathbb{1} \otimes c) + (\mathbb{1} \otimes b)P(a \otimes c) \tag{6.4.9}$$

for all $a, b, c \in \mathscr{A}$. Second, the three maps

$$P_{12} = P \otimes \mathrm{id}, \quad P_{13} = (\mathrm{id} \otimes \tau) \circ P_{12} \circ (\mathrm{id} \otimes \tau), \quad \text{and} \quad P_{23} = \mathrm{id} \otimes P \tag{6.4.10}$$

pairwise commute, where $\tau\colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$ is the canonical flip map sending $a \otimes b$ to $b \otimes a$.

i.) Rewrite the biderivation properties of P without using elements but just the undeformed multiplication μ_0. This will lead to the maps P_{12}, P_{13}, and P_{23}.

ii.) Follow now the proof of Proposition 6.1.12 to conclude that

$$a \star b = \mu_0 \circ \mathrm{e}^{\lambda P}(a \otimes b) \tag{6.4.11}$$

provides an associative deformation.

This is a generalization to situations where one wants to have more than finitely many commuting derivations in the exponent: it is then sometimes not possible to write the map P as a sum (or series) of tensor products of commuting derivations but it may still be possible to directly prove the two above properties, see e.g. [117, Sect. 6] for several applications in quantum field theory.

Exercise 6.4.9 (Gerstenhaber bracket) Show that the identity (6.1.38) implies the graded Jacobi identity of the Gerstenhaber bracket (6.1.39).

Exercise 6.4.10 (The Hochschild differential) Let ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ be an $(\mathscr{A}, \mathscr{A})$-bimodule. Prove that the Hochschild differential δ as in (6.3.17) satisfies $\delta^2 = 0$.

Hint: Start with small n and an explicit computation. Then for arbitrary n the pattern of how the terms cancel becomes clear. This will also give an independent and more direct proof of $\delta^2 = 0$ in the case of the algebra itself, see Remark 6.1.17.

Exercise 6.4.11 (Multiderivations) Let $\mathscr{A}$ be a commutative associative algebra over a ring of scalars R containing $\mathbb{Q}$. Denote by

$$\mathrm{Alt}\colon \mathrm{HC}^k(\mathscr{A}, \mathscr{A}) \longrightarrow \mathrm{HC}^k(\mathscr{A}, \mathscr{A}) \tag{6.4.12}$$

the usual antisymmetrization operator, i.e.

$$\mathrm{Alt}(\phi)(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) \phi(a_{\sigma(1)}, \ldots, a_{\sigma(k)}) \tag{6.4.13}$$

for $a_1, \ldots, a_k \in \mathscr{A}$. For $k = 0$ we set $\mathrm{Alt} = \mathrm{id}$.

i.) Show $\mathrm{Alt} \circ \mathrm{Alt} = \mathrm{Alt}$.

ii.) Show $\mathrm{Alt} \circ \delta = 0$.

Hint: This is a lengthy but elementary computation.

iii.) Let $\phi \in \mathrm{HC}^k(\mathscr{A}, \mathscr{A})$ be an antisymmetric cocycle. Show that ϕ is exact iff $\phi = 0$.

iv.) Let $X \in \mathrm{HC}^k(\mathscr{A}, \mathscr{A})$ be a *multiderivation*, i.e. an antisymmetric map satisfying the Leibniz rule in each argument. Show that $\delta X = 0$.

v.) Conclude that the multiderivations always contribute to the Hochschild cohomology, i.e. the map $X \mapsto [X]$ is injective for multiderivations.

vi.) Extend the discussion to cochains with values in a bimodule ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$. Which properties on ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}$ will be useful to assume?

In many situations one can actually show more: the Hochschild cohomology is given by the multi-derivations. This is the statement of the Hochschild-Kostant-Rosenberg theorem in its various forms for many interesting algebras, see e.g. [34, 112] for the case of smooth functions on a manifold as well as [63] for the original setting.

Exercise 6.4.12 (Gerstenhaber product in low degrees) Let $\mathscr{A}$ be a module over a commutative ring R of scalars. Let $a, b \in \mathrm{HC}^0(\mathscr{A}, \mathscr{A})$, $A, B \in \mathrm{HC}^1(\mathscr{A}, \mathscr{A})$ and $\mu, \nu \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})$. Determine all the Gerstenhaber products $a \circ b$, $a \circ A$, $a \circ \mu$, etc. explicitly by evaluating them on elements of $\mathscr{A}$. Which known operations show up?

Exercise 6.4.13 (Hochschild cohomology of a *-algebra) Let $\mathscr{A}$ be a module over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and let ${}^*\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a C-antilinear involution. The following results are based on [24].

i.) Let $\phi \in \mathrm{HC}^k(\mathscr{A}, \mathscr{A})$. Show that ϕ^* defined by

$$(\phi^*)(a_1, \ldots, a_k) = \phi(a_k^*, \ldots, a_1^*)^* \tag{6.4.14}$$

yields a C-multilinear map $\phi^* \in \mathrm{HC}^k(\mathscr{A}, \mathscr{A})$ such that the map $\phi \mapsto \phi^*$ is C-antilinear and involutive.

ii.) Show that for $\phi \in \mathrm{HC}^k(\mathscr{A}, \mathscr{A})$ and $\psi \in \mathrm{HC}^\ell(\mathscr{A}, \mathscr{A})$ one has

$$(\phi \circ \psi)^* = (-1)^{(k-1)(\ell-1)} \phi^* \circ \psi^*, \tag{6.4.15}$$

and compute also $[\phi, \psi]^*$.

iii.) Show that an associative product $\mu \in \mathrm{HC}^2(\mathscr{A}, \mathscr{A})$ on $\mathscr{A}$ yields a *-algebra structure with respect to * iff $\mu^* = \mu$.

iv.) Suppose now that $\mathscr{A}$ is indeed a *-algebra. Compute $(\delta\phi)^*$ and conclude that the involution passes to the Hochschild cohomology $\mathrm{HH}^\bullet(\mathscr{A}, \mathscr{A})$.

v.) Define a Hochschild cochain ϕ to be *Hermitian* if $\phi^* = \phi$. Show that this allows to define a *Hermitian Hochschild cohomology* of $\mathscr{A}$.

vi.) Show that a formal deformation $\mu = \mu_0 + \lambda\mu_1 + \cdots$ is Hermitian iff its cochains $\mu_r^* = \mu_r$ are Hermitian for all $r \in \mathbb{N}$.

vii.) Repeat the discussion of Proposition 6.1.18 and Proposition 6.1.19 to conclude that in the case of Hermitian deformations the obstructions for existence and equivalence are located in the third and second Hermitian Hochschild cohomology, respectively.

viii.) Show that the Hochschild cohomology $\mathrm{HH}^\bullet(\mathscr{A}, \mathscr{A})$ decomposes canonically into two copies of the Hermitian Hochschild cohomology, provided that $\frac{1}{2} \in \mathsf{R}$, by decomposing a class into its real and imaginary part.

Exercise 6.4.14 (Unitary deformations of $\mathbb{1}$) Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathbb{Q} \subseteq \mathsf{R}$ with a Hermitian deformation $\star$.

i.) Let $a \in \mathscr{A}[[\lambda]]$ be Hermitian. Show that the $\star$-exponential series

$$\mathrm{Exp}(\mathrm{i}\lambda a) = \sum_{r=0}^{\infty} \frac{(\mathrm{i}\lambda)^r}{r!} a^{\star r} \tag{6.4.16}$$

is unitary.

ii.) Conversely, let $u \in \mathscr{A}[\![\lambda]\!]$ be unitary with $\operatorname{cl}(u) = \mathbb{1}$. Prove that there is a unique $a \in \mathscr{A}[\![\lambda]\!]$ with $u = \operatorname{Exp}(\mathrm{i}\lambda a)$. Show that necessarily $a = a^*$ is Hermitian.

Hint: Discuss why the Taylor expansion of $\log(1+x)$ around $x = 0$ will give the existence of a solution for a. Why is it unique?

iii.) Let $u_0 \in \mathscr{A}$ be unitary with respect to the undeformed algebra structure. Let $u, v \in \mathscr{A}[\![\lambda]\!]$ be two deformations of u_0 into unitary elements with respect to $\star$. Show that there exists a unique Hermitian $a \in \mathscr{A}[\![\lambda]\!]$ with $u = v \star \operatorname{Exp}(\mathrm{i}\lambda a)$.

Exercise 6.4.15 (The $\star$-exponential) While the series of the exponential map as in (6.4.16) allows to exponentiate algebra elements with respect to a deformed product $\star$ as soon as they come with (at least) one power of λ, this is no longer that easy if this assumption is not satisfied. Here we extend the definition of the exponential for the case of a star product $\star$ on a manifold M. We base our discussion on [17, 18], see also [116, Sect. 6.3.1].

i.) Let $H = H_0 + \lambda H_1 + \cdots \in \mathscr{C}^\infty(M)[\![\lambda]\!]$ be given. Show that there exists a unique solution $\mathbb{R} \ni t \mapsto f(t) \in \mathscr{C}^\infty(M)[\![\lambda]\!]$ of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = H \star f(t) \tag{6.4.17}$$

with initial condition $f(0) = 1$.

Hint: Factorize $f(t) = \mathrm{e}^{-tH_0} g(t)$ and obtain a differential equation for g which can be rewritten as an integral equation. This you can solve by means of the Banach fixed point theorem based on a simple counting argument as in Exercise 6.4.5.

We denote the above solution by $\operatorname{Exp}(tH) = f(t)$ and call this the $\star$-*exponential* function with respect to H.

ii.) Show that one has $\operatorname{Exp}(tH) \star H = H \star \operatorname{Exp}(tH)$ for all $t \in \mathbb{R}$.

iii.) Show that $t \mapsto \operatorname{Exp}(tH)$ is a one-parameter group, i.e. $\operatorname{Exp}(0) = 1$ and $\operatorname{Exp}(tH) \star \operatorname{Exp}(sH) = \operatorname{Exp}((t+s)H)$.

iv.) Show that for a Hermitian star product one has $\overline{\operatorname{Exp}(tH)} = \operatorname{Exp}(t\overline{H})$.

v.) Show that for $H_0 = 0$ the exponential $\operatorname{Exp}(tH)$ reduces to the series (6.4.16).

vi.) Show that for all $f \in \mathscr{C}^\infty(M)[\![\lambda]\!]$ one has $\operatorname{Exp}(tH) \star f \star \operatorname{Exp}(-tH) = \mathrm{e}^{t \operatorname{ad}(H)}(f)$.

vii.) Let $f, g \in \mathscr{C}^\infty(M)[\![\lambda]\!]$. Show that $[f, g]_\star = 0$ iff $[\operatorname{Exp}(f), g]_\star = 0$ iff $[\operatorname{Exp}(f), \operatorname{Exp}(g)]_\star = 0$.

Hint: This is not completely trivial as it would be to conclude $[f, g]_\star = 0$ from $[\operatorname{Exp}(tf), \operatorname{Exp}(sg)]_\star = 0$ for all $t, s \in \mathbb{R}$ where one simply can differentiate. Instead, one has to rewrite $[\operatorname{Exp}(f), \operatorname{Exp}(g)]_\star = 0$ as a fixed point equation for $\operatorname{ad}(g)(\operatorname{Exp}(f))$ of a *linear* contracting operator to conclude that $\operatorname{ad}(g)(\operatorname{Exp}(f)) = 0$.

viii.) Suppose $f, g \in \mathscr{C}^\infty(M)[\![\lambda]\!]$ commute with respect to $\star$. Show that in this case $\operatorname{Exp}(f) \star \operatorname{Exp}(g) = \operatorname{Exp}(f + g)$.

ix.) Let M be connected. Show that $\operatorname{Exp}(H) = 1$ iff $H = 2\pi\mathrm{i}k$ for some $k \in \mathbb{Z}$.

Exercise 6.4.16 (The Wick star product) Consider again the *-algebra $\mathscr{A} = \mathbb{C}[\![z, \overline{z}]\!]$ with its *-subalgebra $\mathbb{C}[z, \overline{z}]$ from Exercise 1.4.19. Define the Wick star product

$$a \star_{\mathrm{Wick}} b = \mu \circ \exp\left(2\lambda \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}}\right)(a \otimes b), \tag{6.4.18}$$

where $a, b \in \mathscr{A}[\![\lambda]\!]$.

i.) Show that $\mathscr{C}[z, \overline{z}][\![\lambda]\!]$ is a *-subalgebra with respect to $\star_{\mathrm{Wick}}$. Moreover, show that $\mathscr{C}[z, \overline{z}][\lambda]$ is a *-subalgebra with respect to $\star_{\mathrm{Wick}}$, too, when viewed as algebra over $\mathbb{C}[\lambda]$.

ii.) Show that $\star_{\mathrm{Wick}}$ is a completely positive and in fact strongly positive deformation of $\mathscr{A}$. Show that this is also true for the subalgebra $\mathbb{C}[z, \overline{z}]$.

Hint: The first statement is pretty trivial according to Exercise 1.4.19. For the second, use Proposition 6.2.18.

iii.) Compute the Gel'fand ideal of the positive δ-functional $\delta\colon \mathscr{A}[\![\lambda]\!] \longrightarrow \mathbb{C}[\![\lambda]\!]$ with respect to $\star_{\mathrm{Wick}}$ explicitly. Conclude that the GNS pre-Hilbert space can be identified with $\mathcal{H} = (\mathbb{C}[\![\overline{z}]\!])[\![\lambda]\!]$ and find the explicit formula for the inner product.

Hint: This is the analog of the Bargmann-Fock space from complex function theory.

iv.) Show that the deformed algebra $(\mathscr{A}[\![\lambda]\!], \star_{\mathrm{Wick}})$ has a faithful *-representation and hence sufficiently many positive linear functionals, quite contrary to its classical limit $\mathscr{A}$.

v.) Compute explicitly the classical limit of the GNS representation of δ according to the general construction from Proposition 6.2.17 and determine the the corresponding Null space (6.2.9).

Thus deforming a *-algebra can sometimes have the effect that we end up with better properties for the deformed algebra than for the classical one. In the above example all the additional positive functionals have a trivial classical limit except for δ itself. More on this fundamental example can be found in [18, Sect. 6] and [113, Sect. 5] as well as in [116, Sect. 7.2].

Exercise 6.4.17 (Bimodule structure of $\mathrm{Hom}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}})$) As a slight variation of Exercise 4.4.5 we consider an algebra $\mathscr{A}$ over a ring R. Moreover, let $\mathscr{M}_{\mathscr{A}}$ and $\mathscr{M}'_{\mathscr{A}}$ be right $\mathscr{A}$-modules.

i.) Show that $\mathrm{Hom}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}})$ becomes a $(\mathscr{A}, \mathscr{A})$-bimodule by defining $a \cdot A$ and $A \cdot a$ by

$$(a \cdot A)(x) = A(x \cdot a) \quad \text{and} \quad (A \cdot a)(x) = (A(x)) \cdot a \tag{6.4.19}$$

for $a \in \mathscr{A}$, $A \in \mathrm{Hom}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}})$, and $x \in \mathscr{M}_{\mathscr{A}}$.

ii.) Show that for this bimodule structure one obtains

$$\mathrm{HH}^0(\mathscr{A}, \mathrm{Hom}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}})) = \mathrm{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}}). \tag{6.4.20}$$

iii.) Now consider the algebras $\mathrm{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}})$ and $\mathrm{End}_{\mathscr{A}}(\mathscr{M}'_{\mathscr{A}})$. Show that $\mathrm{Hom}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}})$ becomes a $(\mathrm{End}_{\mathscr{A}}(\mathscr{M}'_{\mathscr{A}}), \mathrm{End}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}))$-bimodule in a natural way. Is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M}_{\mathscr{A}}, \mathscr{M}'_{\mathscr{A}})$ a sub-bimodule?

Exercise 6.4.18 (Cohomological approach to deformation of modules) Provide the missing proofs for Proposition 6.3.7 and Proposition 6.3.9. Formulate and prove the analogous results for left modules instead of right modules.

Exercise 6.4.19 (Deformations of a particular type) Let $\mathscr{A}$ be an algebra over a ring R of scalars. We consider a subset of the Hochschild complex which we denote by $\mathrm{HC}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$ where the word "type" should characterize the allows elements compared to general elements of $\mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$. We require that $\mu_0 \in \mathrm{HC}^2_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$, that $\mathrm{HC}^0_{\mathrm{type}}(\mathscr{A}, \mathscr{A}) = \mathscr{A}$, and that $\mathrm{HC}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$ is closed under the insertion operations $\circ_i$ for all i.

i.) Show that under these assumptions $\mathrm{HC}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$ becomes a subcomplex of the usual Hochschild complex $\mathrm{HC}^\bullet(\mathscr{A}, \mathscr{A})$ which is also closed under the Gerstenhaber product and the Gerstenhaber bracket.

ii.) Conclude that the cohomology $\mathrm{HH}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$ of the subcomplex $\mathrm{HC}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$ becomes a graded Lie algebra itself.

iii.) Consider now a formal deformation $\mu = \mu_0 + \lambda\mu_1 + \cdots$ of $\mathscr{A}$ where all terms are required to satisfy $\mu_r \in \mathrm{HC}^2_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$. Formulate and prove the analogous statements to Proposition 6.1.18 and Proposition 6.1.19 for such more specified deformations of a given type.

In a next step we consider a right $\mathscr{A}$-module $\mathscr{M}_{\mathscr{A}}$. Now suppose that we are interested in a subalgebra $\mathscr{D} \subseteq \mathrm{End}_{\mathsf{R}}(\mathscr{M}_{\mathscr{A}})$ of the endomorphisms of $\mathscr{M}_{\mathscr{A}}$. Moreover, we specify also a type of cochains $\mathrm{HC}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$ with values in $\mathscr{D}$ subject to the conditions that $\mathrm{HC}^0_{\mathrm{type}}(\mathscr{A}, \mathscr{D}) = \mathscr{D}$, and the module multiplication ρ_0 of $\mathscr{M}_{\mathscr{A}}$ is an element of $\mathrm{HC}^1_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$. In addition, we want $\phi \circ_i \psi \in \mathrm{HC}^{k+\ell-1}_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$

whenever $\phi \in \mathrm{HC}^k_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$ and $\psi \in \mathrm{HC}^\ell_{\mathrm{type}}(\mathscr{A}, \mathscr{A})$ as well as $\phi \circ \psi \in \mathrm{HC}^{k+\ell}_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$ for all $\phi \in \phi \in \mathrm{HC}^k_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$ and $\psi \in \mathrm{HC}^\ell_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$ where we define

$$(\phi \circ \psi)(a_1, \ldots, a_{k+\ell}) = \phi(a_1, \ldots a_k) \circ \psi(a_{k+1}, \ldots, a_{k+\ell}) \tag{6.4.21}$$

for $a_1, \ldots, a_{k+\ell} \in \mathscr{A}$.

iv.) Show that $\mathrm{HC}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$ is a subcomplex of $\mathrm{HC}^\bullet(\mathscr{A}, \mathscr{D})$ giving a cohomology $\mathrm{HH}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$.

v.) Formulate and prove that also for the module deformations of the specified type we have the analogs of Proposition 6.3.7 and Proposition 6.3.9, now based on the cohomology $\mathrm{HH}^\bullet_{\mathrm{type}}(\mathscr{A}, \mathscr{D})$.

In many situations a more specified type of the cochains in a deformation problem is helpful or even required by other needs. Thus the more restricted Hochschild cohomologies respecting the type will control such deformation problems, see also [16].

Exercise 6.4.20 (Cohomological approach to bimodule deformations) Let $\mathscr{A}$ and $\mathscr{B}$ be unital algebras over a ring R of scalars and let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule.

i.) Consider the opposite algebra $\mathscr{A}^{\mathrm{opp}}$ and the tensor product $\mathscr{A}^{\mathrm{opp}} \otimes_{\mathsf{ext}} \mathscr{B}$. Show that ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ is a $\mathscr{A}^{\mathrm{opp}} \otimes_{\mathsf{ext}} \mathscr{B}$-left module in a natural way. Conversely, show that any $\mathscr{A}^{\mathrm{opp}} \otimes_{\mathsf{ext}} \mathscr{B}$-left module can be viewed as a $(\mathscr{B}, \mathscr{A})$-bimodule.

ii.) Show that the above correspondence of bimodules and left modules is compatible with the usual (bi-)module morphisms and establish an equivalence of categories this way.

Consider now formal deformations $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star_{\mathscr{A}})$ and $\boldsymbol{\mathscr{B}} = (\mathscr{B}[\![\lambda]\!], \star_{\mathscr{B}})$.

iii.) Show that the deformations $\star_{\mathscr{A}}$ and $\star_{\mathscr{B}}$ induce a formal deformation of $\mathscr{A}^{\mathrm{opp}} \otimes_{\mathsf{ext}} \mathscr{B}$, denoted by $\star$ in the following.

iv.) Show that a bimodule deformation of ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ into a $(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$-bimodule corresponds to a left module deformation with respect to $((\mathscr{A}^{\mathrm{opp}} \otimes \mathscr{B})[\![\lambda]\!], \star)$. Show that this correspondence is compatible with equivalence of (bi-) module deformations.

v.) Use this result to formulate a cohomological approach to bimodule deformations based on the Hochschild cohomology of $\mathscr{A}^{\mathrm{opp}} \otimes_{\mathsf{ext}} \mathscr{B}$.

While this approach is perhaps conceptually more clear than the one taken in Section 6.3.3 it typically suffers from the fact that the relevant Hochschild cohomology for the algebra $\mathscr{A}^{\mathrm{opp}} \otimes_{\mathsf{ext}} \mathscr{B}$ is, in many cases of interest, hard to compute.

Exercise 6.4.21 (Deformation of projections) Consider again an idempotent $e_0 \in \mathscr{A}$ and a formal deformation $\star$ for $\mathscr{A}$.

i.) Show that the formal Taylor expansion needed in (6.1.73) converges in the λ-adic topology.

ii.) Verify by an explicit computation that (6.1.73) defines an idempotent.

iii.) Show by an explicit computation of the Taylor expansion that one only has integer coefficients needed in (6.1.73). Hence the assumptions of $\mathbb{Q} \subseteq \mathsf{R}$ are in fact superfluous.

Hint: One needs some properties of the binomial coefficients here, see also the discussion in [40, Theorem 1.54].

Exercise 6.4.22 (Classical limit of a projective module) Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be an associative deformation of a unital algebra $\mathscr{A}$ over R. Moreover, let $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ be a finitely generated projective module over $\boldsymbol{\mathscr{A}}$.

i.) Show that the classical limit module $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ is a finitely generated projective module over $\mathscr{A}$ with the same number of generators.

Hint: Choose an idempotent $e \in \mathrm{M}_n(\mathscr{A})[\![\lambda]\!] = \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ with a module isomorphism $\Psi\colon \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \longrightarrow e \star \boldsymbol{\mathscr{A}}^n$. Show that this induces an isomorphism $\psi\colon \mathscr{E}_{\mathscr{A}} \longrightarrow e_0 \mathscr{A}^n$ for $e_0 = \mathrm{cl}(e)$ such that $\psi \circ \mathrm{cl} = \mathrm{cl} \circ \Psi$.

ii.) Show that there is an $\mathsf{R}[\![\lambda]\!]$-linear isomorphism $\phi\colon \mathscr{E}_{\mathscr{A}}[\![\lambda]\!] \longrightarrow \boldsymbol{\mathscr{E}_{\mathscr{A}}}$ with $\mathrm{cl} \circ \phi = \mathrm{id}_{\mathscr{E}_{\mathscr{A}}}$.

Hint: Let Ψ and ψ be as before. Use then the canonical isomorphism $I\colon e_0 \mathscr{A}^n[\![\lambda]\!] \longrightarrow e \star \mathscr{A}^n[\![\lambda]\!]$ to define $\phi = \Psi^{-1} \circ I \circ \psi$ where we extended ψ to an isomorphism $\psi\colon \mathscr{E}_{\mathscr{A}}[\![\lambda]\!] \longrightarrow e_0 \mathscr{A}^n[\![\lambda]\!]$ as usual and where I is the isomorphism from Corollary 6.1.31.

Chapter 7

Morita Theory of Deformed *-Algebras

7.1 The Ring-Theoretic Classical Limit Homomorphism

In this section we start with the ring-theoretic situation and discuss the classical limit of bimodules. Ultimately, this will be formulated as a homomorphism of bicategories. This point of view will yield many consequences in a clear and simple way as homomorphisms of bicategories lead to various functors between derived concepts. In particular, we will obtain a classical limit as a groupoid morphism on the level of the Picard groupoids. This includes a classical limit group morphism between the corresponding Picard groups. The classical limit morphism completely encodes the question which deformations of algebras are Morita equivalent. We restrict ourselves to the unital case throughout this section for simplicity.

7.1.1 The Classical Limit for $\underline{\mathsf{Bimod}}$

Recall that the bicategory $\underline{\mathsf{Bimod}}$ had unital rings as objects, bimodules as 1-morphisms between them and bimodule morphisms as 2-morphisms. When working with algebras over a specified ring R of scalars we indicate this by writing $\underline{\mathsf{Bimod}}_{\mathsf{R}}$ for the corresponding bicategory. With this notation, the previous bicategory is simply given by

$$\underline{\mathsf{Bimod}} = \underline{\mathsf{Bimod}}_{\mathbb{Z}} . \tag{7.1.1}$$

We will fix now a ring of scalars R and consider $\underline{\mathsf{Bimod}}_{\mathsf{R}}$. Then we can also consider the deformed algebras as algebras over $\mathsf{R}[\![\lambda]\!]$. However, in general, there are more algebras over $\mathsf{R}[\![\lambda]\!]$ than just those which are formal unital associative deformations of algebras over R. Hence we restrict ourselves to the following sub-bicategory:

Definition 7.1.1 (The bicategory $\underline{\boldsymbol{\mathsf{Bimod}}}_{\mathsf{R}[\![\lambda]\!]}$) *The sub-bicategory of $\underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}$ consisting of formal unital associative deformations of algebras over R as objects, all strongly non-degenerate bimodules between them as 1-morphisms, and all bimodule morphisms as 2-morphisms is denoted by $\underline{\boldsymbol{\mathsf{Bimod}}}_{\mathsf{R}[\![\lambda]\!]}$. The classifying category of $\underline{\boldsymbol{\mathsf{Bimod}}}_{\mathsf{R}[\![\lambda]\!]}$ is denoted by $\boldsymbol{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}$.*

Recall that by our usual convention we consider all structure maps of algebras and bimodules to be (multi-) linear for scalars. Since we only consider unital algebras this is automatic since we can consider every scalar $\alpha \in \mathsf{R}$ as element $\alpha\mathbb{1} \in \mathscr{A}$ and analogously for the deformed case. In particular, the formal parameter λ becomes an algebra element this way.

For the deformed algebras we immediately have a definition of the classical limit. For $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ we set $\mathrm{cl}(\boldsymbol{\mathscr{A}}) = \mathscr{A}$ and get a unital algebra morphism

$$\mathrm{cl}\colon \boldsymbol{\mathscr{A}} \longrightarrow \mathrm{cl}(\boldsymbol{\mathscr{A}}) = \mathscr{A} \tag{7.1.2}$$

as usual. We want to extend this map on objects now to 1-morphisms and 2-morphisms.

Thus let $\boldsymbol{\mathscr{A}}$ and $\boldsymbol{\mathscr{B}}$ be two such deformations and let ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ be a $(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$-bimodule. We do *not* assume that ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ is of the form ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}[\![\lambda]\!]$ with some $(\mathscr{B}, \mathscr{A})$-bimodule ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$. Nevertheless, we can define a classical limit by the quotient

$$\mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) = {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} / \lambda {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}, \tag{7.1.3}$$

which on one hand is a R-module. On the other hand, it becomes a $(\mathscr{B}, \mathscr{A})$-bimodule for the classical limits by setting

$$b \cdot \mathrm{cl}(\boldsymbol{x}) = \mathrm{cl}(b \bullet \boldsymbol{x}) \quad \text{and} \quad \mathrm{cl}(\boldsymbol{x}) \cdot a = \mathrm{cl}(\boldsymbol{x} \bullet a) \tag{7.1.4}$$

for $a \in \mathscr{A}$, $b \in \mathscr{B}$, and $\boldsymbol{x} \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$. Here we denote the two module structures on ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ with respect to the deformed algebras by $\bullet$. From Proposition 6.3.3 we know that we obtain a left $\mathscr{B}$-module structure on ${}_{\mathscr{B}}\mathscr{M}_{\mathscr{A}}$ as well as a right $\mathscr{A}$-module structure. Since on ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ we have a bimodule structure for the deformed algebras, it easily follows that (7.1.4) actually gives a $(\mathscr{B}, \mathscr{A})$-bimodule. Note that it is crucial that elements $a \in \mathscr{A}$ can be viewed as elements of the deformed algebra $\boldsymbol{\mathscr{A}}$ in a canonical way.

Finally, assume that $\boldsymbol{T}\colon {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} \longrightarrow {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}'_{\boldsymbol{\mathscr{A}}}$ is a $(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$-bimodule morphism. Then we can again use Proposition 6.3.3 to construct a morphism

$$\mathrm{cl}(\boldsymbol{T})\colon \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \longrightarrow \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}'_{\boldsymbol{\mathscr{A}}}) \tag{7.1.5}$$

between the classical limits by setting

$$\mathrm{cl}(\boldsymbol{T})(\mathrm{cl}(\boldsymbol{x})) = \mathrm{cl}(\boldsymbol{T}(\boldsymbol{x})) \tag{7.1.6}$$

for $\boldsymbol{x} \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$. We know already that $\mathrm{cl}(\boldsymbol{T})$ is a morphism of each of the module structures. Hence we indeed obtain a bimodule morphism this way.

Lemma 7.1.2 *Let* $\boldsymbol{\mathscr{A}}$ *and* $\boldsymbol{\mathscr{B}}$ *be formal unital associative deformations of the algebras* $\mathscr{A}$ *and* $\mathscr{B}$ *over* R. *Then the classical limit yields a functor*

$$\mathrm{cl}\colon \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}}) \longrightarrow \underline{\mathsf{Bimod}}_{\mathsf{R}}(\mathscr{B}, \mathscr{A}). \tag{7.1.7}$$

Proof: The only thing left to show is that cl preserves the identity bimodule morphisms and the composition. But both claims are a simple computation analogous to the one needed for Proposition 6.3.3, *iii.)*. ☐

The idea is now to combine all these individual classical limit functors into a big homomorphism of bicategories. To this end we need to check the compatibility of cl with tensor products of bimodules.

Lemma 7.1.3 *Let* $\boldsymbol{\mathscr{C}}$, $\boldsymbol{\mathscr{B}}$, *and* $\boldsymbol{\mathscr{A}}$ *be formal unital associative deformations of algebras* $\mathscr{C}$, $\mathscr{B}$, *and* $\mathscr{A}$ *over* R. *Moreover, let* ${}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}} \in \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}(\boldsymbol{\mathscr{C}}, \boldsymbol{\mathscr{B}})$ *and* ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} \in \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$ *be bimodules. Then the* R*-linear map determined by*

$$I\colon \mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}}) \otimes_{\mathscr{B}} \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \ni \mathrm{cl}(\boldsymbol{y}) \otimes_{\mathscr{B}} \mathrm{cl}(\boldsymbol{x}) \;\mapsto\; \mathrm{cl}(\boldsymbol{y} \otimes_{\boldsymbol{\mathscr{B}}} \boldsymbol{x}) \in \mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}} \otimes_{\boldsymbol{\mathscr{B}}} {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \tag{7.1.8}$$

for $\boldsymbol{y} \in {}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}}$ *and* $\boldsymbol{x} \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}$ *yields a well-defined* R*-linear* $(\mathscr{C}, \mathscr{A})$*-bimodule isomorphism with inverse determined by*

$$I^{-1}\colon \mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}} \otimes_{\boldsymbol{\mathscr{B}}} {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}) \ni \mathrm{cl}(\boldsymbol{y} \otimes_{\boldsymbol{\mathscr{B}}} \boldsymbol{x}) \;\mapsto\; \mathrm{cl}(\boldsymbol{y}) \otimes_{\mathscr{B}} \mathrm{cl}(\boldsymbol{x}) \in \mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}}) \otimes_{\mathscr{B}} \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}}). \tag{7.1.9}$$

Proof: We first show that the map $(\mathrm{cl}(\boldsymbol{y}), \mathrm{cl}(\boldsymbol{x})) \mapsto \mathrm{cl}(\boldsymbol{y} \otimes_{\boldsymbol{\mathscr{B}}} \boldsymbol{x})$ is well-defined and R-bilinear. Indeed, let $\boldsymbol{y}'$ and $\boldsymbol{x}'$ be other representatives then $\boldsymbol{y} - \boldsymbol{y}' = \lambda \boldsymbol{y}''$ and $\boldsymbol{x} - \boldsymbol{x}' = \lambda \boldsymbol{x}''$ for some $\boldsymbol{y}'' \in {}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathcal{N}}_{\boldsymbol{\mathscr{B}}}$ and $\boldsymbol{x}'' \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathscr{A}}}$, respectively. Since the tensor product over $\boldsymbol{\mathscr{B}}$ is in particular also $\mathsf{R}[\![\lambda]\!]$-bilinear we find

$$\begin{aligned} \mathrm{cl}(\boldsymbol{y}' \otimes \boldsymbol{x}') &= \mathrm{cl}\big((\boldsymbol{y} - \lambda \boldsymbol{y}'') \otimes (\boldsymbol{x} - \lambda \boldsymbol{x}'')\big) \\ &= \mathrm{cl}\big(\boldsymbol{y} \otimes \boldsymbol{x} - \lambda \boldsymbol{y}'' \otimes \boldsymbol{x} + \lambda^2 \boldsymbol{y}'' \otimes \boldsymbol{x}'' - \lambda \boldsymbol{y} \otimes \boldsymbol{x}'\big) \\ &= \mathrm{cl}(\boldsymbol{y} \otimes \boldsymbol{x}). \end{aligned}$$

This shows that the above map is well-defined. The R-bilinearity is clear and hence the map I is well-defined over the R-tensor product. Next we show that I is also well-defined over the tensor product over $\mathscr{B}$. Here we use that $\mathrm{cl}(\boldsymbol{y}) \cdot b = \mathrm{cl}(\boldsymbol{y} \bullet b)$ as well as $b \cdot \mathrm{cl}(\boldsymbol{x}) = \mathrm{cl}(b \bullet \boldsymbol{x})$ for all $b \in \mathscr{B}$ and for $\boldsymbol{y} \in {}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathcal{N}}_{\boldsymbol{\mathscr{B}}}$ and $\boldsymbol{x} \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathscr{A}}}$. This shows that $(\mathrm{cl}(\boldsymbol{y}) \cdot b, \mathrm{cl}(\boldsymbol{x}))$ is mapped to $\mathrm{cl}((\boldsymbol{y} \bullet b) \otimes_{\boldsymbol{\mathscr{B}}} \boldsymbol{x}) = \mathrm{cl}(\boldsymbol{y} \otimes_{\boldsymbol{\mathscr{B}}} b \bullet \boldsymbol{x})$ which is the image of $(\mathrm{cl}(\boldsymbol{y}), b \cdot \mathrm{cl}(\boldsymbol{x}))$. Hence I is well-defined on the tensor product over $\mathscr{B}$. In a second step we consider the map

$${}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathcal{N}}_{\boldsymbol{\mathscr{B}}} \otimes_{\boldsymbol{\mathscr{B}}} {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathscr{A}}} \ni \boldsymbol{y} \otimes_{\boldsymbol{\mathscr{B}}} \boldsymbol{x} \ \mapsto\ \mathrm{cl}(\boldsymbol{y}) \otimes_{\mathscr{B}} \mathrm{cl}(\boldsymbol{x}) \in \mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathcal{N}}_{\boldsymbol{\mathscr{B}}}) \otimes_{\mathscr{B}} \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathscr{A}}}).$$

This is a well-defined R-linear map, now on the tensor product over $\boldsymbol{\mathscr{B}}$. Moreover, since the tensor product $\boldsymbol{\mathscr{B}}$ is also $\mathsf{R}[\![\lambda]\!]$-bilinear we see that multiples of λ are mapped to zero under this map: both classical limit maps cl on the right hand side have the multiples of λ in their kernel. Hence this map passes to the quotient and yields a well-defined map (7.1.9). It is now clear that I is the inverse of this map (7.1.9) and hence both are R-linear isomorphisms. Finally, it is easy to see that both maps I and I^{-1} are left $\mathscr{C}$-linear and right $\mathscr{A}$-linear since we can check this on factorizing tensors. Hence we have $(\mathscr{C}, \mathscr{A})$-bimodule isomorphisms as claimed. □

In this sense, the tensor product commutes with the classical limit. Of course, there is an isomorphism necessary to make this statement correct. In accordance with our bicategory creed it will require particular attention to keep track of all the necessary identifications and their naturalness. Hence we also denote the above isomorphism by $I(\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{M}})$ instead of just I in order to stress the dependence on the bimodules $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$. Then we can formulate the fact that I is natural as follows:

Lemma 7.1.4 *The isomorphisms I are natural. More precisely, for two given bimodule morphisms $\boldsymbol{S}\colon {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathscr{A}}} \longrightarrow {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathcal{M}}'_{\boldsymbol{\mathscr{A}}}$ and $\boldsymbol{T}\colon {}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathcal{N}}_{\boldsymbol{\mathscr{B}}} \longrightarrow {}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathcal{N}}'_{\boldsymbol{\mathscr{B}}}$ we have*

$$\mathrm{cl}(\boldsymbol{T} \otimes \boldsymbol{S}) \circ I(\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{M}}) = I(\boldsymbol{\mathcal{N}}', \boldsymbol{\mathcal{M}}') \circ (\mathrm{cl}(\boldsymbol{T}) \otimes \mathrm{cl}(\boldsymbol{S})). \tag{7.1.10}$$

Proof: It is sufficient to verify (7.1.10) on elementary tensors. Let $\boldsymbol{x} \in \boldsymbol{\mathcal{M}}$ and $\boldsymbol{y} \in \boldsymbol{\mathcal{N}}$ be given. Then

$$\begin{aligned} \big(\mathrm{cl}(\boldsymbol{T} \otimes \boldsymbol{S}) \circ I(\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{M}})\big)(\mathrm{cl}(\boldsymbol{y}) \otimes \mathrm{cl}(\boldsymbol{x})) &= \mathrm{cl}(\boldsymbol{T} \otimes \boldsymbol{S})(\mathrm{cl}(\boldsymbol{y} \otimes \boldsymbol{x})) \\ &= \mathrm{cl}((\boldsymbol{T} \otimes \boldsymbol{S})(\boldsymbol{y} \otimes \boldsymbol{x})) \\ &= \mathrm{cl}(\boldsymbol{T}(\boldsymbol{y}) \otimes \boldsymbol{S}(\boldsymbol{x})) \\ &= (I(\boldsymbol{\mathcal{N}}', \boldsymbol{\mathcal{M}}'))\big(\mathrm{cl}(\boldsymbol{T}(\boldsymbol{y})) \otimes \mathrm{cl}(\boldsymbol{S}(\boldsymbol{x}))\big) \\ &= (I(\boldsymbol{\mathcal{N}}', \boldsymbol{\mathcal{M}}'))\big((\mathrm{cl}(\boldsymbol{T})(\mathrm{cl}(\boldsymbol{y}))) \otimes (\mathrm{cl}(\boldsymbol{S})(\mathrm{cl}(\boldsymbol{x})))\big) \\ &= \big(I(\boldsymbol{\mathcal{N}}', \boldsymbol{\mathcal{M}}') \circ (\mathrm{cl}(\boldsymbol{T}) \otimes \mathrm{cl}(\boldsymbol{S}))\big)(\mathrm{cl}(\boldsymbol{y}) \otimes \mathrm{cl}(\boldsymbol{x})). \end{aligned}$$

Note that we have used the same $\otimes$-symbol for various tensor products in the above computation. □

Thus cl is compatible with tensor products in a natural way. Moreover, the classical limit is also compatible with the unit elements of the tensor product:

Lemma 7.1.5 *The map*

$$I_{\mathscr{A}} \colon {}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}} \ni a \mapsto \operatorname{cl}(a) \in \operatorname{cl}({}_{\boldsymbol{\mathscr{A}}}\boldsymbol{\mathscr{A}}_{\boldsymbol{\mathscr{A}}}) \tag{7.1.11}$$

is an $(\mathscr{A}, \mathscr{A})$*-bimodule isomorphism.*

PROOF: Note that $\operatorname{cl}({}_{\boldsymbol{\mathscr{A}}}\boldsymbol{\mathscr{A}}_{\boldsymbol{\mathscr{A}}}) = {}_{\boldsymbol{\mathscr{A}}}\boldsymbol{\mathscr{A}}_{\boldsymbol{\mathscr{A}}} \big/ \lambda {}_{\boldsymbol{\mathscr{A}}}\boldsymbol{\mathscr{A}}_{\boldsymbol{\mathscr{A}}}$ is not *equal* to ${}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}$, but only isomorphic via $I_{\mathscr{A}}$. Then the statement is clear. □

Collecting all the above isomorphisms shows that the classical limit can be viewed as a homomorphism of bicategories: it only remains to be checked that cl satisfies the coherence conditions of a homomorphism:

Theorem 7.1.6 (Classical limit for $\underline{\mathbf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}$) *Let* R *be a unital commutative ring. Then the classical limit functors* cl *from* (7.1.7) *together with the natural isomorphisms* I *from* (7.1.8) *and the isomorphisms* $I_{\mathscr{A}}$ *from* (7.1.11) *constitute a homomorphism of bicategories*

$$\underline{\operatorname{cl}} \colon \underline{\mathbf{Bimod}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \underline{\mathsf{Bimod}}_{\mathsf{R}}. \tag{7.1.12}$$

PROOF: It remains to check the coherence conditions (5.3.55), (5.3.56), and (5.3.57) from Definition 5.3.27 are fulfilled, see also Remark 5.3.28 for an explanation of the various possibilities of morphisms between bicategories. Indeed, the classical limit of the underlying algebras provides the map between the objects of the bicategories, the classical limit functor cl is the functor needed in (5.3.52). The natural isomorphism I from Lemma 7.1.3 is the natural isomorphism required in (5.3.59) and, finally, the map from Lemma 7.1.5 is the 2-morphism as requested in (5.3.60). Note that we indeed have isomorphisms everywhere and thus will end up with a homomorphism of bicategories. We check the coherence (5.3.55). Let $\boldsymbol{\mathscr{D}}$, $\boldsymbol{\mathscr{C}}$, $\boldsymbol{\mathscr{B}}$, and $\boldsymbol{\mathscr{A}}$, be deformations of $\mathscr{D}$, $\mathscr{C}$, $\mathscr{B}$, and $\mathscr{A}$, respectively. Moreover, let ${}_{\boldsymbol{\mathscr{D}}}\boldsymbol{\mathscr{O}}_{\boldsymbol{\mathscr{C}}} \in \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}(\boldsymbol{\mathscr{D}}, \boldsymbol{\mathscr{C}})$, ${}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{N}}_{\boldsymbol{\mathscr{B}}} \in \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}(\boldsymbol{\mathscr{C}}, \boldsymbol{\mathscr{B}})$, and ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{M}}_{\boldsymbol{\mathscr{A}}} \in \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$ be corresponding bimodules. Finally, let $\boldsymbol{z} \in \boldsymbol{\mathscr{O}}$, $\boldsymbol{y} \in \boldsymbol{\mathscr{N}}$, and $\boldsymbol{x} \in \boldsymbol{\mathscr{M}}$, respectively. For the left path of maps in (5.3.55) applied to $(\operatorname{cl}(\boldsymbol{z}) \otimes \operatorname{cl}(\boldsymbol{y})) \otimes \operatorname{cl}(\boldsymbol{x})$ we get

$$\begin{aligned}
&\big(I(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}} \otimes \boldsymbol{\mathscr{M}}) \circ (\mathrm{id} \otimes I(\boldsymbol{\mathscr{N}} \otimes \boldsymbol{\mathscr{M}})) \circ \mathsf{asso}(\operatorname{cl}(\boldsymbol{\mathscr{O}}), \operatorname{cl}(\boldsymbol{\mathscr{N}}), \operatorname{cl}(\boldsymbol{\mathscr{M}}))\big)((\operatorname{cl}(\boldsymbol{z}) \otimes \operatorname{cl}(\boldsymbol{y})) \otimes \operatorname{cl}(\boldsymbol{x})) \\
&\quad = I(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}} \otimes \boldsymbol{\mathscr{M}})(\operatorname{cl}(\boldsymbol{z}) \otimes (I(\boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}})(\operatorname{cl}(\boldsymbol{y}) \otimes \operatorname{cl}(\boldsymbol{x})))) \\
&\quad = I(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}} \otimes \boldsymbol{\mathscr{M}})(\operatorname{cl}(\boldsymbol{z}) \otimes \operatorname{cl}(\boldsymbol{y} \otimes \boldsymbol{x})) \\
&\quad = \operatorname{cl}(\boldsymbol{z} \otimes (\boldsymbol{y} \otimes \boldsymbol{x})).
\end{aligned}$$

For the right path in the diagram (5.3.55) we have to evaluate the classical limit of the associativity $\mathsf{asso}(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}})$ explicitly. Here we have

$$\big(\operatorname{cl}(\mathsf{asso}(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}}))\big)(\operatorname{cl}((\boldsymbol{z} \otimes \boldsymbol{y}) \otimes \boldsymbol{x})) = \operatorname{cl}\big(\mathsf{asso}(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}})((\boldsymbol{z} \otimes \boldsymbol{y}) \otimes \boldsymbol{x})\big) = \operatorname{cl}(\boldsymbol{z} \otimes (\boldsymbol{y} \otimes \boldsymbol{x}))$$

as expected. Using this we get for the right path of (5.3.55)

$$\begin{aligned}
&\big(\operatorname{cl}(\mathsf{asso}(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}})) \circ I(\boldsymbol{\mathscr{O}} \otimes \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}}) \circ (I(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}) \otimes \mathrm{id})\big)((\operatorname{cl}(\boldsymbol{z}) \otimes \operatorname{cl}(\boldsymbol{y})) \otimes \operatorname{cl}(\boldsymbol{x})) \\
&\quad = \operatorname{cl}(\mathsf{asso}(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}}))\big(I(\boldsymbol{\mathscr{O}} \otimes \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}})(\operatorname{cl}(\boldsymbol{z} \otimes \boldsymbol{y}) \otimes \operatorname{cl}(\boldsymbol{x}))\big) \\
&\quad = \operatorname{cl}(\mathsf{asso}(\boldsymbol{\mathscr{O}}, \boldsymbol{\mathscr{N}}, \boldsymbol{\mathscr{M}}))(\operatorname{cl}((\boldsymbol{z} \otimes \boldsymbol{y}) \otimes \boldsymbol{x})) \\
&\quad = \operatorname{cl}(\boldsymbol{z} \otimes (\boldsymbol{y} \otimes \boldsymbol{x})),
\end{aligned}$$

which proves that (5.3.55) commutes. Next we consider the diagram (5.3.56). Here we compute for $\boldsymbol{a} \in \boldsymbol{\mathscr{A}}$

$$\operatorname{cl}(\mathsf{right}(\boldsymbol{\mathscr{M}}))(\operatorname{cl}(\boldsymbol{x} \otimes \boldsymbol{a})) = \operatorname{cl}(\mathsf{right}(\boldsymbol{\mathscr{M}})(\boldsymbol{x} \otimes \boldsymbol{a})) = \operatorname{cl}(\boldsymbol{x} \bullet \boldsymbol{a}) = \operatorname{cl}(\boldsymbol{x}) \cdot \operatorname{cl}(\boldsymbol{a}).$$

For $a \in \mathcal{A}$ we therefore have

$$\begin{aligned}\big(\mathrm{cl}(\mathsf{right}(\boldsymbol{\mathcal{M}})) \circ I(\boldsymbol{\mathcal{M}}, {}_{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{A}}}) \circ (\mathrm{id} \otimes I_{\boldsymbol{\mathcal{A}}})\big)(\mathrm{cl}(\boldsymbol{x}) \otimes a) &= \mathrm{cl}(\mathsf{right}(\boldsymbol{\mathcal{M}}))\big(I(\boldsymbol{\mathcal{M}}, {}_{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{A}}})(\mathrm{cl}(\boldsymbol{x}) \otimes \mathrm{cl}(a))\big) \\ &= \mathrm{cl}(\mathsf{right}(\boldsymbol{\mathcal{M}}))(\mathrm{cl}(\boldsymbol{x} \otimes a)) \\ &= \mathrm{cl}(\boldsymbol{x}) \cdot \mathrm{cl}(a) \\ &= \mathrm{cl}(\boldsymbol{x}) \cdot a, ,\end{aligned}$$

which coincides with $\mathsf{right}(\mathrm{cl}(\boldsymbol{\mathcal{M}}))(\mathrm{cl}(\boldsymbol{x}) \otimes a) = \mathrm{cl}(\boldsymbol{x}) \cdot a$. This shows that (5.3.56) commutes, too. The remaining diagram (5.3.57) commutes by an analogous computation. Thus $\underline{\mathrm{cl}}$ is indeed a homomorphism of bicategories. □

Before we can discuss the consequences of this theorem we need to extend the results of Proposition 5.3.30 slightly. The following result on homomorphisms of bicategories holds in general and was already used for bigroupoids:

Proposition 7.1.7 *Let $\underline{\Phi}\colon \underline{\mathfrak{B}} \longrightarrow \underline{\mathfrak{C}}$ be a homomorphism of bicategories. If $E \in \mathfrak{B}_1(b, a)$ is an invertible 1-morphism with inverse $E' \in \mathfrak{B}_1(a, b)$ and isomorphisms*

$$\phi\colon E' \otimes_b E \longrightarrow \mathsf{Id}_a \tag{7.1.13}$$

and

$$\psi\colon E \otimes_a E' \longrightarrow \mathsf{Id}_b, \tag{7.1.14}$$

then $\Phi(E) \in \mathfrak{C}_1(\Phi(b), \Phi(a))$ is invertible as well with inverse $\Phi(E')$ and isomorphisms

$$\varphi_a^{-1} \circ \Phi(\phi) \circ \varphi_{aba}(E', E)\colon \Phi(E') \otimes_{\Phi(b)} \Phi(E) \longrightarrow \mathsf{Id}_{\Phi(a)} \tag{7.1.15}$$

and

$$\varphi_b^{-1} \circ \Phi(\psi) \circ \varphi_{bab}(E, E')\colon \Phi(E) \otimes_{\Phi(a)} \Phi(E') \longrightarrow \mathsf{Id}_{\Phi(b)}. \tag{7.1.16}$$

Proof: We extend the argument from Proposition 5.3.30, *ii.)*, to homomorphisms between general bicategories and not just between bigroupoids. Since $\Phi\colon \underline{\mathfrak{B}}(b, a) \longrightarrow \underline{\mathfrak{C}}(\Phi(b), \Phi(a))$ is a functor for all a, b we conclude that

$$\Phi(\phi)\colon \Phi(E' \otimes_b E) \longrightarrow \Phi(\mathsf{Id}_a) \quad \text{and} \quad \Phi(\psi)\colon \Phi(E \otimes_a E') \longrightarrow \Phi(\mathsf{Id}_b)$$

are isomorphism as well. By the very definition of a homomorphism of bicategories we have (even natural) *isomorphisms*

$$\varphi_{aba}(E', E)\colon \Phi(E') \otimes_{\Phi(b)} \Phi(E) \longrightarrow \Phi(E' \otimes_b E)$$

and

$$\varphi_{bab}(E, E')\colon \Phi(E) \otimes_{\Phi(a)} \Phi(E') \longrightarrow \Phi(E \otimes_a E')$$

as well as

$$\varphi_a\colon \mathsf{Id}_{\Phi(a)} \longrightarrow \Phi(\mathsf{Id}_a) \quad \text{and} \quad \varphi_b\colon \mathsf{Id}_{\Phi(b)} \longrightarrow \Phi(\mathsf{Id}_b).$$

Thus also the compositions (7.1.15) and (7.1.16) are isomorphisms. This shows that $\Phi(E)$ is a invertible 1-morphisms. □

Note that here we indeed need the notion of a homomorphism of bicategories as we need that all four morphisms $\varphi_{aba}(E', E)$, $\varphi_{bab}(E, E')$, φ_a, and φ_b are *invertible*. The weaker versions from Remark 5.3.28 will not be sufficient for Proposition 7.1.7. A first application of the above proposition is the following corollary:

Corollary 7.1.8 *If $\underline{\Phi}\colon \mathfrak{B} \longrightarrow \mathfrak{C}$ is a homomorphism of bicategories then $\underline{\Phi}$ restricts to a homomorphism of the corresponding bigroupoids on invertible* 1*-morphisms.*

Using this, we can now consider the situation of Theorem 7.1.6 and conclude that the classical limit restricts to the Picard bigroupoids. We denote the bigroupoid of invertible 1-morphisms in $\underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}$ by $\underline{\mathsf{Pic}}_{\mathsf{R}[\![\lambda]\!]}$: this bigroupoid has the same objects, i.e. the deformed algebras and the 1-morphisms are now the Morita bimodules between the deformed algebras. We indicate the underlying ring $\mathsf{R}[\![\lambda]\!]$ of scalars as before. The purely ring-theoretic version is then obtained by setting $\mathsf{R} = \mathbb{Z}$ as usual. If the reference to the underlying ring of scalars is clear we just omit the explicit mentioning in our notation. The classifying groupoid **Pic** consists then of isomorphism classes of Morita equivalence bimodules as usual. We emphasize once more that $\underline{\mathsf{Pic}}$ has *all* equivalence bimodules between two deformed algebras as 1-morphisms. Compared to $\underline{\mathsf{Pic}}$ we only have restricted the objects: we are interested in deformed algebras. With these notations we get the following result:

Corollary 7.1.9 *Let* R *be a unital commutative ring. Then the classical limit* $\underline{\mathrm{cl}}\colon \underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \underline{\mathsf{Bimod}}_{\mathsf{R}}$ *restricts to a homomorphism of the Picard bigroupoids*

$$\underline{\mathrm{cl}}\colon \underline{\mathsf{Pic}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \underline{\mathsf{Pic}}_{\mathsf{R}}\,. \tag{7.1.17}$$

Passing to the classifying groupoids this induces a groupoid morphism

$$\mathrm{cl}_*\colon \mathsf{Pic}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \mathsf{Pic}_{\mathsf{R}}\,. \tag{7.1.18}$$

PROOF: By definition, the Picard bigroupoid is precisely the bigroupoid of invertible 1-morphisms of $\underline{\mathsf{Bimod}}$, see Definition 5.1.8. Analogously, the sub-bigroupoid $\underline{\mathsf{Pic}}$ of Morita equivalence bimodules over deformed algebras is the bigroupoid of invertible 1-morphisms of $\underline{\mathsf{Bimod}}$. Then the first part is clear by Corollary 7.1.8 and Theorem 7.1.6. The second part holds in general: homomorphisms of bicategories, in our case (7.1.17), become functors of the classifying categories, in our case (7.1.18), see also [7, p. 56]. ☐

Corollary 7.1.10 *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be an associative deformation of a unital algebra $\mathscr{A}$ over* R*. Then the classical limit yields a group morphism*

$$\mathrm{cl}_*\colon \mathsf{Pic}(\boldsymbol{\mathscr{A}}) \longrightarrow \mathsf{Pic}(\mathscr{A}). \tag{7.1.19}$$

PROOF: This is clear as a groupoid morphism induces group morphisms for the isotropy groups. ☐

Remark 7.1.11 The results in Corollary 7.1.9 and Corollary 7.1.10 can also be shown directly, without using the bigroupoid morphism $\underline{\mathrm{cl}}$, see [28, Prop. 3.7 and Lem. 3.8]. However, the above approach seems to be more conceptual.

7.1.2 The Action of Pic on Def

In a next step we want to understand which deformations of Morita equivalent algebras stay Morita equivalent. This will ultimately result in a groupoid action of the Picard groupoid Pic on the deformation theories $\mathrm{Def}(\,\cdot\,)$ of unital algebras over R. We follow closely [22, 28] in this section.

To this end, we consider a deformation $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ of a unital algebra $\mathscr{A}$ over a R and another unital algebra $\mathscr{B}$ over R which is Morita equivalent to $\mathscr{A}$ by means of an equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. For the ring-theoretic Morita theory we know that $\mathscr{E}_{\mathscr{A}}$ is finitely generated and projective with a full projection representing it and $\mathscr{B} \cong \mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ via the left action of $\mathscr{B}$ on $\mathscr{E}_{\mathscr{A}}$, see Theorem 4.3.5. Thus we can assume without restriction that we already implemented the module as $\mathscr{E}_{\mathscr{A}} = e_0\mathscr{A}^n$ for some full idempotent $e_0 \in \mathrm{M}_n(\mathscr{A})$. Moreover, we can assume that $\mathscr{B} = e_0\mathrm{M}_n(\mathscr{A})e_0$ acting on $e_0\mathscr{A}^n$ by matrix multiplication as usual. We start with the following simple observation:

Lemma 7.1.12 *Let $e \in \mathrm{M}_n(\mathscr{A})[[\lambda]]$ be an idempotent deforming $\mathrm{cl}(e) = e_0$. Then e is full iff e_0 is full.*

PROOF: Suppose e is full. Then $\mathbb{1} = \sum_{\alpha,i} a_{\alpha i} \star e_{ij} \star b_{\alpha i}$ for some $a_{\alpha i}, b_{\alpha i} \in \mathscr{A}[[\lambda]]$. Taking the classical limit gives $\mathbb{1} = \sum_{\alpha,i} \mathrm{cl}(a_{\alpha i})(e_0)_{ij}\mathrm{cl}(b_{\alpha i})$, showing that $\mathbb{1}$ is contained in the two-sided ideal generated by the components of e_0. Hence $\mathrm{cl}(e) = e_0$ is full. Conversely, suppose e_0 is full. Hence we find $a_{\alpha,i}, b_{\alpha,i} \in \mathscr{A}$ with $\mathbb{1} = \sum_{\alpha,i} a_{\alpha i}(e_0)_{ij} b_{\alpha,j}$. Then $c = \sum_{\alpha,i} a_{\alpha i} \star e_{ij} \star b_{\alpha,j} = \mathbb{1} + \cdots$ coincides with $\mathbb{1}$ in zeroth order. Thus c is invertible and contained in the ideal $\mathscr{A}[[\lambda]] \star e \star \mathscr{A}[[\lambda]]$ which therefore is $\mathscr{A}[[\lambda]]$. Thus e is full, too. □

Corollary 7.1.13 *Let $e_0 \in \mathrm{M}_n(\mathscr{A})$ be a full projection. For any deformation $\star$ of $\mathscr{A}$ the algebras $\boldsymbol{\mathscr{A}}$ and $e \star \mathrm{M}_n(\mathscr{A}) \star e$ are Morita equivalent via the equivalence bimodule $e \star \boldsymbol{\mathscr{A}}^n$.*

The next observation is now that there are, up to isomorphisms, no other possibilities to get Morita equivalence bimodules for a deformation $\boldsymbol{\mathscr{A}}$ of $\mathscr{A}$. This is in fact a consequence of our much more general consideration on the classical limit of bimodules:

Corollary 7.1.14 *Let $\boldsymbol{\mathscr{A}}$ and $\boldsymbol{\mathscr{B}}$ be deformations of $\mathscr{A}$ and $\mathscr{B}$ with a Morita equivalence bimodule ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$. Then $\mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ is a Morita equivalence bimodule for $\mathscr{B}$ and $\mathscr{A}$.*

PROOF: Indeed, this is clear form Corollary 7.1.9. □

Since we can deform idempotents $e_0 \in \mathrm{M}_n(\mathscr{A})$ into idempotents $e = e_0 + \cdots \in \mathrm{M}_n(\mathscr{A})[[\lambda]] = \mathrm{M}_n(\boldsymbol{\mathscr{A}})$ preserving fullness, we can also construct a Morita equivalent algebra $\boldsymbol{\mathscr{B}}$ to $\boldsymbol{\mathscr{A}}$ out of a classically full idempotent e_0 based on our considerations in Proposition 6.1.30, *iii.)*:

Proposition 7.1.15 *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[[\lambda]], \star)$ be an associative deformation of a unital algebra $\mathscr{A}$ over R. Moreover, let ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ be a Morita equivalence $(\mathscr{B}, \mathscr{A})$-bimodule for another unital algebra $\mathscr{B}$.*

- *i.) There exists an associative deformation $\star'$ of $\mathscr{B}$ and a deformed bimodule structure $\bullet'$ and $\bullet$ of ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ turning ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} = ({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}[[\lambda]], \bullet', \bullet)$ into a $(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$-bimodule.*
- *ii.) The deformations $\star'$, $\bullet'$, and $\bullet$ are uniquely determined up to equivalence by the equivalence class $[\star] \in \mathrm{Def}(\mathscr{A})$ of $\star$ and the isomorphism class $[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}] \in \mathsf{Pic}(\mathscr{B}, \mathscr{A})$.*
- *iii.) The deformed bimodule ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ is a Morita equivalence bimodule.*

PROOF: We know that we can deform the right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ into a right $\boldsymbol{\mathscr{A}}$-module $\mathscr{E}_{\mathscr{A}}[[\lambda]], \bullet)$ according to Theorem 6.3.17, *i.)*, unique up to equivalence by *ii.)*. Moreover, in this case the endomorphisms $\mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ inherit a deformation $\star'$ with a corresponding left module deformation $\bullet'$ according to *v.)* of the same theorem, being again unique up to equivalence. Now finally, $\mathscr{B}$ is isomorphic to $\mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ by the classical Theorem 4.3.5 of Morita via the left action: here we have no choice in implementing $\mathscr{B} \cong \mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ such that the left actions match. Hence we can pull-back to deformation of $\mathrm{End}_{\mathscr{A}}(\mathscr{E}_{\mathscr{A}})$ to $\mathscr{B}$ and get the first part. The uniqueness statements from Theorem 6.3.17 given then the second part. The last part is clear since the deformation of a full projective module is again full by Lemma 7.1.12. □

This proposition allows now to define a map

$$\Phi\colon \mathsf{Pic}(\mathscr{B}, \mathscr{A}) \times \mathrm{Def}(\mathscr{A}) \ni ([{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}], [\star]) \mapsto \Phi_{[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]}([\star]) = [\star'] \in \mathrm{Def}(\mathscr{B}) \tag{7.1.20}$$

where $\star'$ is the deformation of $\mathscr{B}$ induced by the equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ according to Proposition 7.1.15, *i.)*. The second statement of the proposition then ensures that this is indeed well-defined.

The last part of the proposition then shows that the obtained deformation of $\mathscr{B}$ is Morita equivalent to the deformation of $\mathscr{A}$. However, taking classical limits shows immediately that the converse is true as well, leading to the following statement:

Proposition 7.1.16 *Let* $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ *and* $\boldsymbol{\mathscr{B}} = (\mathscr{B}[\![\lambda]\!], \star')$ *be two associative deformations of unital algebras* $\mathscr{A}$ *and* $\mathscr{B}$ *over* R. *Then* $\boldsymbol{\mathscr{A}}$ *and* $\boldsymbol{\mathscr{B}}$ *are Morita equivalent iff there exists a* ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Pic}(\mathscr{B}, \mathscr{A})$ *with*

$$[\star'] = \Phi_{[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]}([\star]). \tag{7.1.21}$$

PROOF: Given the classical equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ and a deformation $\star$ we know from Proposition 7.1.15, *iii.)*, that the deformation $\boldsymbol{\mathscr{B}} = (\mathscr{B}[\![\lambda]\!], \star')$ of $\mathscr{B}$ is Morita equivalent to $\boldsymbol{\mathscr{A}}$. Conversely, suppose ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ is an equivalence bimodule between $\boldsymbol{\mathscr{B}}$ and $\boldsymbol{\mathscr{A}}$. Then ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ is an equivalence bimodule between $\mathscr{B}$ and $\mathscr{A}$ by the general result from Corollary 7.1.9. Moreover, the right $\boldsymbol{\mathscr{A}}$-module structure on ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ is finitely generated and projective. Thus $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ becomes $\mathsf{R}[\![\lambda]\!]$-linearly isomorphic to $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})[\![\lambda]\!]$ via some choice of an isomorphism $\phi\colon \mathscr{E}_{\mathscr{A}}[\![\lambda]\!] \longrightarrow \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$. Moreover, we can adjust ϕ in such a way that $\mathrm{cl} \circ \phi = \mathrm{id}_{\mathscr{E}_{\mathscr{A}}}$ by Exercise 6.4.22, *ii.)*. Then the right $\mathscr{A}$-module structure $\triangleleft$ on $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ induces a deformation $\bullet$ of $\mathscr{E}_{\mathscr{A}}$ such that

$$\phi(x \bullet a) = \phi(x) \triangleleft a$$

for all $x \in \mathscr{E}_{\mathscr{A}}[\![\lambda]\!]$ and $a \in \boldsymbol{\mathscr{A}}$, i.e. ϕ becomes an isomorphism of right $\boldsymbol{\mathscr{A}}$-modules. A different choice of such a ϕ would lead now to a different $\tilde{\bullet}$ which is nevertheless equivalent to $\bullet$ by the uniqueness statement from Theorem 6.3.17, *ii.)*. The left $\boldsymbol{\mathscr{B}}$-module structure on ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ induces an isomorphism $\boldsymbol{\mathscr{B}} \cong \mathrm{End}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ and hence ϕ yields an isomorphism

$$\Phi\colon \boldsymbol{\mathscr{B}} \ni b \;\mapsto\; \Phi(b) \in \mathrm{End}_{\boldsymbol{\mathscr{A}}}(\mathscr{E}_{\mathscr{A}}[\![\lambda]\!], \bullet)$$

given by $\Phi(b)x\phi^{-1}(b \triangleright \phi(x))$ with the left $\boldsymbol{\mathscr{B}}$-module structure $\triangleright$ of ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$. Defining $b \bullet' x = \Phi(b)x$ this yields a bimodule deformation of ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ which in turn induces a deformation of $\mathscr{B}$ according to Proposition 7.1.15. But since $\bullet'$ is the module structure we already had, this induced deformation of $\mathscr{B}$ *coincides* with $\star'$. □

In a last step we want to show that the map Φ from (7.1.21) is actually a groupoid action of the Picard groupoid Pic of the undeformed algebras on their deformation theories, extending the action of the isomorphism groupoid.

Lemma 7.1.17 *Let* $\Psi\colon \mathscr{A} \longrightarrow \mathscr{B}$ *be a unital isomorphism of unital algebras over* R. *Then for the bimodule* ${}_{\mathscr{B}}\mathscr{B}^{\Psi}_{\mathscr{A}}$ *one gets*

$$\Phi_{[{}_{\mathscr{B}}\mathscr{B}^{\Psi}_{\mathscr{A}}]}([\star]) = [\Psi(\star)] \tag{7.1.22}$$

where $\star' = \Psi(\star)$ *is defined by* $b \star' b' = \Psi(\Psi^{-1}b \star \Psi^{-1}b')$ *for* $b, b' \in \mathscr{B}[\![\lambda]\!]$.

PROOF: The definition of $\Psi(\star)$ yields a deformation of $\mathscr{B}$ which gives an isomorphic algebra structure $\star'$. Therefore we have a deformation $\bullet$ of the classical right $\mathscr{A}$-module $\mathscr{B}^{\Psi}_{\mathscr{A}}$ given by $x \bullet a = \Psi(\Psi^{-1}(x) \star a) = x \cdot_{\Psi} a + \cdots$. This gives a bimodule deformation with respect to the deformation $\star'$ of $\mathscr{B}$ directly. □

Corollary 7.1.18 *Let* $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ *be an associative deformation of a unital algebra* $\mathscr{A}$ *over a ring* R. *Then one has*

$$\Phi_{[{}_{\mathscr{A}}\mathscr{A}_{\mathscr{A}}]} = \mathrm{id}\,. \tag{7.1.23}$$

Lemma 7.1.19 *Let* ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$ *and* ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ *be Morita equivalence bimodules for unital algebras* $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{C}$ *over* R. *Then*

$$\Phi_{[{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}]} \circ \Phi_{[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]} = \Phi_{[{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]}. \tag{7.1.24}$$

PROOF: Let $\star$ be an associative deformation of $\mathscr{A}$ and denote by $\star'$ the induced deformation of $\mathscr{B}$ coming from a corresponding bimodule deformation ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} = ({}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}[\![\lambda]\!], \bullet', \bullet)$. Similarly, we have an induced deformation $\star''$ of *algebraC* from $\star'$ via a bimodule deformation ${}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{F}}_{\boldsymbol{\mathscr{B}}} = ({}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}[\![\lambda]\!], \bullet'', \bullet')$ of ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$. From Theorem 7.1.6 we then know that canonically

$$\mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{F}}_{\boldsymbol{\mathscr{B}}} \otimes_{\boldsymbol{\mathscr{B}}} {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) \cong \mathrm{cl}({}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{F}}_{\boldsymbol{\mathscr{B}}}) \otimes_{\mathscr{B}} \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) = {}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}.$$

By the analogous argument as in the proof of Proposition 7.1.16 we see that ${}_{\boldsymbol{\mathscr{C}}}\boldsymbol{\mathscr{F}}_{\boldsymbol{\mathscr{B}}} \otimes_{\boldsymbol{\mathscr{B}}} {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ is a bimodule deformation of ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$. But this particular deformation yields $\star'' = (\Phi_{[{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}]} \circ \Phi_{[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]})(\star)$ on the level of representatives. Hence $\Phi_{[{}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}} \otimes_{\mathscr{B}} {}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]}$ yields the same class $[\star'']$, proving the claim. ☐

The statements can now be combined into one theorem clarifying the role of the map Φ completely, see [22] as well as [28, Thm. 3.13]:

Theorem 7.1.20 (Morita equivalence of deformations) *The map Φ yields a groupoid action of the Picard groupoid* Pic *on the deformation theories* $\mathrm{Def}(\,\cdot\,)$

$$\Phi\colon \mathsf{Pic} \times \mathrm{Def} \longrightarrow \mathrm{Def} \tag{7.1.25}$$

extending the canonical action of Iso*. Two deformations yield Morita equivalent algebras iff they are in the same orbit of this action.*

In the spirit of Section 5.3 we have now found another Morita invariant, the deformation theories:

Corollary 7.1.21 *Morita equivalent algebras have isomorphic deformation theories. In fact, every $(\mathscr{B}, \mathscr{A})$-equivalence bimodule ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$ yields a bijection*

$$\Phi_{[{}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}]}\colon \mathrm{Def}(\mathscr{A}) \longrightarrow \mathrm{Def}(\mathscr{B}). \tag{7.1.26}$$

Corollary 7.1.22 *The Picard group* $\mathsf{Pic}(\mathscr{A})$ *of an associative unital algebra $\mathscr{A}$ acts on its deformation theory* $\mathrm{Def}(\mathscr{A})$ *via Φ. Two deformations $\star$ and $\star'$ of $\mathscr{A}$ are Morita equivalent iff there is a self-equivalence ${}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}} \in \mathsf{Pic}(\mathscr{A})$ with $\Phi_{[{}_{\mathscr{A}}\mathscr{E}_{\mathscr{A}}]}([\star]) = [\star']$.*

7.1.3 Kernel and Image of cl in the Ring-Theoretic Case

In a last step we investigate now the image and the kernel of the (bi-) groupoid morphism $\mathrm{cl}\colon \underline{\boldsymbol{\mathsf{Pic}}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \underline{\mathsf{Pic}}_{\mathsf{R}}$ more closely. As before, we fix a ring of scalars R and consider unital algebras over R and their formal associative deformations as algebra over $\mathsf{R}[\![\lambda]\!]$.

As warming-up we investigate the situation of algebra morphisms first to establish an analog of Theorem 7.1.6 for the categories of unital algebras directly. If $\boldsymbol{\mathscr{A}}$ and $\boldsymbol{\mathscr{B}}$ are deformations of unital algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, and if $\Phi\colon \boldsymbol{\mathscr{A}} \longrightarrow \boldsymbol{\mathscr{B}}$ is a morphism we have

$$\Phi = \sum_{r=0}^{\infty} \lambda^r \Phi_r \tag{7.1.27}$$

with unique linear maps $\Phi_r\colon \mathscr{A} \longrightarrow \mathscr{B}$.

Lemma 7.1.23 *If $\Phi = \sum_{r=0}^{\infty} \lambda^r \Phi_r\colon \boldsymbol{\mathscr{A}} \longrightarrow \boldsymbol{\mathscr{B}}$ is an morphism then*

$$\mathrm{cl}(\Phi) = \Phi_0\colon \mathscr{A} \longrightarrow \mathscr{B} \tag{7.1.28}$$

is a morphism of the undeformed algebras.

PROOF: This is a trivial verification. ☐

This defines a map $\mathrm{cl}\colon \mathrm{Hom}(\boldsymbol{\mathscr{A}}, \boldsymbol{\mathscr{B}}) \longrightarrow \mathrm{Hom}(\mathscr{A}, \mathscr{B})$. It turns out that this combines into a functor on the level of Ring: we specify the categories in question now a bit more precisely and denote by $\mathsf{Ring}_{\mathsf{R}}$ the category of unital algebras over the ring R with usual R-linear algebra morphisms and analogously we have $\mathsf{Ring}_{\mathsf{R}[\![\lambda]\!]}$. Analogously to Definition 7.1.1, we have the full sub-category $\boldsymbol{\mathsf{Ring}}_{\mathsf{R}[\![\lambda]\!]}$ of deformed algebras inside the category $\mathsf{Ring}_{\mathsf{R}[\![\lambda]\!]}$: the objects are now required to be associative deformations of algebras over R while the morphisms are still all $\mathsf{R}[\![\lambda]\!]$-linear algebra morphisms. Then we have the classical limit functor:

Proposition 7.1.24 (Classical limit for $\boldsymbol{\mathsf{Ring}}_{\mathsf{R}[\![\lambda]\!]}$) *The classical limit* cl *of deformations of unital algebras yields a functor*

$$\mathrm{cl}\colon \boldsymbol{\mathsf{Ring}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \mathsf{Ring}_{\mathsf{R}}. \tag{7.1.29}$$

PROOF: We have to check the functoriality as we already know that the classical limit of a morphism is a morphism as wanted. But this is trivial as the classical limit of the identity map is the identity map and since the classical limit of a composition is the composition of the classical limits in general, see also Exercise 6.4.2. □

It follows that the functor cl restricts to a groupoid morphism between the isomorphism groupoids of the deformed algebras and their classical limits, i.e. we have

$$\mathrm{cl}\colon \boldsymbol{\mathsf{Iso}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \mathrm{Iso}_{\mathsf{R}}, \tag{7.1.30}$$

where $\boldsymbol{\mathsf{Iso}}_{\mathsf{R}[\![\lambda]\!]} \subseteq \boldsymbol{\mathsf{Ring}}_{\mathsf{R}[\![\lambda]\!]}$ and $\mathrm{Iso}_{\mathsf{R}} \subseteq \mathsf{Ring}_{\mathsf{R}}$ are the isomorphism groupoids, respectively. As usual, the (non-) injectivity of a groupoid morphism is entirely encoded in the kernels of the corresponding group morphisms between the isotropy groups, see also Exercise 5.4.3. Thus we have to determine the kernel of the group morphisms cl between the automorphism groups:

Proposition 7.1.25 *Let $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ be a formal associative deformation of a unital algebra $\mathscr{A}$ over R. Then the kernel of the classical limit*

$$\mathrm{cl}\colon \mathrm{Aut}(\boldsymbol{\mathscr{A}}) \longrightarrow \mathrm{Aut}(\mathscr{A}) \tag{7.1.31}$$

is given by the self-equivalences

$$\ker(\mathrm{cl}) = \mathrm{Equiv}(\star). \tag{7.1.32}$$

PROOF: This is obvious since for $\Phi = \sum_{r=0}^{\infty} \lambda^r \Phi_r \in \mathrm{Aut}(\boldsymbol{\mathscr{A}})$ we have $\mathrm{cl}(\Phi) = \Phi_0$. Thus $\Phi \in \ker(\mathrm{cl})$ is equivalent to say that $\Phi_0 = \mathrm{id}_{\mathscr{A}}$ which is precisely the definition of a self-equivalence according to Definition 6.1.4. □

Remark 7.1.26 Of course, the *image* of cl is much more complicated to describe as it encodes a deformation problem: which classical automorphism $\Phi_0 \in \mathrm{Aut}(\mathscr{A})$ allows for a deformation as an algebra homomorphism? Note that it will necessarily be an automorphism of $\boldsymbol{\mathscr{A}}$ since as usual the invertibility is decided in zeroth order. In general, one can not say much and the answers will strongly depend on the algebra $\mathscr{A}$ and the deformation $\star$, see also Exercise 7.4.2 and Exercise 7.4.3 for some further considerations.

Since we can take classical limits of algebra morphisms as well as of bimodules, we can now relate the functors ℓ from Proposition 4.3.3 for the deformed and undeformed situation. We first note that the image of ℓ always is in the sub-category $\boldsymbol{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}$ when starting in $\boldsymbol{\mathsf{Ring}}_{\mathsf{R}[\![\lambda]\!]}$:

Lemma 7.1.27 *The functor ℓ from Proposition 4.3.3 restricts to a functor*

$$\ell\colon \boldsymbol{\mathsf{Ring}}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \boldsymbol{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}. \tag{7.1.33}$$

Proof: Since the restriction to the sub-categories in both cases is just on the objects and since it is in both cases the same sub-class of objects, the deformations of algebras over R, the statement is clear. ☐

It turns out that the functors ℓ and the classical limit functors are compatible in the following sense:

Proposition 7.1.28 *The functor ℓ commutes with the classical limit, i.e.*

$$\begin{array}{ccc} \mathbf{Bimod}_{\mathsf{R}[\![\lambda]\!]} & \xrightarrow{\ \mathrm{cl}\ } & \mathsf{Bimod}_{\mathsf{R}} \\ \ell \Big\uparrow & & \Big\uparrow \ell \\ \mathbf{Ring}_{\mathsf{R}[\![\lambda]\!]} & \xrightarrow{\ \mathrm{cl}\ } & \mathsf{Ring}_{\mathsf{R}} \end{array} \tag{7.1.34}$$

commutes.

Proof: Let $\Phi\colon \boldsymbol{\mathscr{A}} \longrightarrow \boldsymbol{\mathscr{B}}$ be a morphism between two deformed algebras $\boldsymbol{\mathscr{A}}$ and $\boldsymbol{\mathscr{B}}$. Then $\ell(\Phi)$ is represented by the $(\boldsymbol{\mathscr{B}}, \boldsymbol{\mathscr{A}})$-bimodule ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{B}}^{\Phi}_{\boldsymbol{\mathscr{A}}}$ according to the definition of ℓ in (4.3.2). As a $\mathsf{R}[\![\lambda]\!]$-module, this is just $\boldsymbol{\mathscr{B}} = \mathscr{B}[\![\lambda]\!]$ and thus its classical limit is R-linearly isomorphic to $\mathscr{B}$. The induced $(\mathscr{B}, \mathscr{A})$-bimodule structure on this classical limit is then the usual left $\mathscr{B}$-module structure as we did not twist this on the deformed side. For the right $\mathscr{A}$-module structure $\mathrm{cl}(\cdot_\Phi)$ we obtain for $x \in \mathscr{B}$ and $a \in \mathscr{A}$

$$x\,\mathrm{cl}(\cdot_\Phi)\,a = \mathrm{cl}(x \cdot_\Phi a) = \mathrm{cl}(x\Phi(a)) = \mathrm{cl}(x)\mathrm{cl}(\Phi(a)) = x\Phi_0(a) = x \cdot_{\Phi_0} a,$$

where $\Phi = \sum_{r=0}^{\infty} \lambda^r \Phi_r$ with R-linear maps $\Phi_r\colon \mathscr{A} \longrightarrow \mathscr{B}$ as usual. This shows that the induced right $\mathscr{A}$-module structure is precisely the one from ${}_{\mathscr{B}}\mathscr{B}^{\Phi_0}_{\mathscr{A}}$ with the algebra morphism $\Phi_0 = \mathrm{cl}(\Phi)\colon \mathscr{A} \longrightarrow \mathscr{B}$. Since in the definition of morphisms of $\mathsf{Bimod}_{\mathsf{R}}$ and $\mathbf{Bimod}_{\mathsf{R}[\![\lambda]\!]}$ we work with isomorphism classes of bimodules, this is all we need to show. ☐

7.2 Classical Limit of *-Representations

After the ring-theoretic situation we want to incorporate now the inner products as well. Thus we consider inner products on (bi-)modules over deformed algebras and construct their classical limits in a similar fashion as for the underlying module structures. However, if we want to guarantee a non-degenerate inner product also in the classical limit, we need to modify the naive quotient procedure as we did this already for *-representations on pre-Hilbert spaces. Once this is accomplished, the next difficulty is to guarantee the complete positivity of the classical limit once we started with a completely positive inner product. Here we need to focus on completely positive deformations of *-algebras instead of general Hermitian deformations. In the end, we will find a classical limit as a homomorphism of bicategories also for the bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$.

7.2.1 Classical Limit of Inner Products and *-Representations

Let $\mathscr{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with a Hermitian deformation $\star$ into a *-algebra $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$. Moreover, let $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ be a right $\boldsymbol{\mathscr{A}}$-module. If we have an $\boldsymbol{\mathscr{A}}$-valued inner product $\langle\,\cdot\,,\,\cdot\,\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}$ on $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ then we can define its classical limit as in the scalar case in Proposition 6.2.14, see also [26, Sect. 9]. To this end we consider the following subset

$$\boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}} = \left\{ x \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \;\middle|\; \mathrm{cl}\big(\langle y, x\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}\big) = 0 \text{ for all } y \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \right\} \tag{7.2.1}$$

of the module $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$. Then we obtain the following result:

Proposition 7.2.1 *Let $\boldsymbol{\mathscr{A}}$ be a Hermitian deformation of a *-algebra $\mathscr{A}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Let $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ be a right $\boldsymbol{\mathscr{A}}$-module with $\boldsymbol{\mathscr{A}}$-valued inner product $\langle \cdot\,, \cdot \rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}$.*

i.) The subset $\boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}} \subseteq \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ is a right $\boldsymbol{\mathscr{A}}$-submodule containing $\lambda\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \subseteq \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$.

ii.) The quotient $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) = \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} / \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ becomes a right $\mathscr{A}$-module by

$$\mathrm{cl}(x) \cdot a = \mathrm{cl}(x \cdot a), \tag{7.2.2}$$

where $\mathrm{cl}\colon \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \longrightarrow \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} / \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ denotes the quotient map and where $a \in \mathscr{A} \subseteq \boldsymbol{\mathscr{A}}$ and $x \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$.

iii.) On the quotient right $\boldsymbol{\mathscr{A}}$-module $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) = \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} / \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ one has an $\mathscr{A}$-valued non-degenerate inner product defined by

$$\big\langle \mathrm{cl}(x), \mathrm{cl}(y) \big\rangle^{\mathrm{cl}(\boldsymbol{\mathscr{E}})}_{\mathscr{A}} = \mathrm{cl}\big(\langle x, y \rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}\big), \tag{7.2.3}$$

where $x, y \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$.

iv.) Let $\boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}$ be another right $\boldsymbol{\mathscr{A}}$-module with inner product and let $T\colon \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \longrightarrow \boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}$ be an adjointable map. Then $T(\boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}) \subseteq \boldsymbol{\mathscr{E}}'^{,\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ and

$$\mathrm{cl}(T)\colon \mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) \ni \mathrm{cl}(\phi) \;\mapsto\; \mathrm{cl}(T\phi) \in \mathrm{cl}(\boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}) \tag{7.2.4}$$

is a well-defined adjointable map again.

v.) The classical limit map on adjointable maps is C-linear and satisfies

$$\mathrm{cl}(T)^* = \mathrm{cl}(T^*), \tag{7.2.5}$$

$$\mathrm{cl}(\mathrm{id}_{\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}}) = \mathrm{id}_{\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})}, \tag{7.2.6}$$

and

$$\mathrm{cl}(S \circ T) = \mathrm{cl}(S) \circ \mathrm{cl}(T). \tag{7.2.7}$$

Proof: The strategy is very much analogous to the one in Proposition 6.2.14, which can be also seen as a special case. First, let $x \in \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ and $y \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ be given. Then for $a \in \boldsymbol{\mathscr{A}}$ we have $\mathrm{cl}(\langle a, x \cdot a\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}) = \mathrm{cl}(\langle y, x\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}} \star a) = \mathrm{cl}(\langle y, x\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}})\mathrm{cl}(a) = 0$. From this the first part follows at once. The the second part is clear as we take the quotient by a $\boldsymbol{\mathscr{A}}$-submodule resulting in a right $\boldsymbol{\mathscr{A}}$-module. Since $\lambda\mathrm{cl}(x) = \mathrm{cl}(\lambda x) = 0$ we obtain an induced $\mathscr{A}$-module by (7.2.2). Next, we note that the classical limit (7.2.3) gives indeed a well-defined inner product with values in the undeformed algebra $\mathscr{A}$ since for $x \in \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ and $y \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ arbitrary we have $\langle y, x\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}} = 0$ by the very definition of $\boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$. The properties of an inner product can then be checked easily on representatives and follows directly from those of $\langle \cdot\,, \cdot \rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}$. As with this definition $\boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ is precisely the degeneracy space of $\mathrm{cl}(\langle \cdot\,, \cdot \rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}})$, it follows that (7.2.3) is non-degenerate. Now let $\boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}$ be another inner-product right $\boldsymbol{\mathscr{A}}$-module and let $T\colon \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \longrightarrow \boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}$ be adjointable. Then for $x \in \boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$ and $y \in \boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}$ we have

$$\mathrm{cl}\big(\langle y, T(x)\rangle^{\boldsymbol{\mathscr{E}}'}_{\boldsymbol{\mathscr{A}}}\big) = \mathrm{cl}\big(\langle T^*(y), x\rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}\big) = 0,$$

implying $T(\boldsymbol{\mathscr{E}}^{\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}) \subseteq \boldsymbol{\mathscr{E}}'^{,\mathrm{Null}}_{\boldsymbol{\mathscr{A}}}$. But then $\mathrm{cl}(T)$ is well-defined on the classical limits and yields an adjointable map between them. The last statement can then be checked on representatives where it is a trivial consequence from properties of adjointable maps. ☐

Note that a possible degenerate inner product on $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ becomes automatically non-degenerate in the classical limit: this is a nice side-effect of the quotient procedure above. As a first application we can take now classical limits of *-representations of a *-algebra $\mathscr{B}$ on an inner-product right $\mathscr{A}$-module. This gives the following classical limit functor:

Corollary 7.2.2 *Let $\boldsymbol{\mathscr{A}}$ and $\boldsymbol{\mathscr{B}}$ be Hermitian deformations of *-algebras $\mathscr{A}$ and $\mathscr{B}$ over $\mathsf{C} = \mathsf{R}(\mathrm{i})$. Moreover, let ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ be a *-representation of $\boldsymbol{\mathscr{B}}$ on an inner-product right $\boldsymbol{\mathscr{A}}$-module.*

i.) *The classical limit* $\mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) = \mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ *becomes a* **-representation of* $\mathscr{B}$ *on the inner-product right* $\mathscr{A}$*-module* $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ *via*

$$b \cdot \mathrm{cl}(x) = \mathrm{cl}(b \cdot x) \tag{7.2.8}$$

for $b \in \mathscr{B}$ *and* $x \in \mathscr{E}_{\mathscr{A}}$.

ii.) *If* $T\colon {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}} \longrightarrow {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}}$ *is an adjointable intertwiner then* $\mathrm{cl}(T)\colon \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}) \longrightarrow \mathrm{cl}({}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}'_{\boldsymbol{\mathscr{A}}})$ *is an adjointable intertwiner, too.*

iii.) *Taking the classical limit gives a functor*

$$\mathrm{cl}\colon {}^*\text{-}\mathsf{mod}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{B}}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathscr{A}}(\mathscr{B}), \tag{7.2.9}$$

which restricts to a functor

$$\mathrm{cl}\colon {}^*\text{-}\mathsf{Mod}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{B}}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathscr{A}}(\mathscr{B}). \tag{7.2.10}$$

PROOF: Since for a *-representation the action of $b \in \boldsymbol{\mathscr{B}}$ on ${}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ is by adjointable maps with respect to the inner product of $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$, the first part follows from Proposition 7.2.1, *v.)*. An adjointable intertwiner T is in particular adjointable with respect to the $\boldsymbol{\mathscr{A}}$-valued inner product of $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$. Thus it induces an adjointable right $\mathscr{A}$-linear map in the classical limit by Proposition 7.2.1, *iv.)*. Since $T(b \cdot x) = b \cdot T(x)$ for all $b \in \boldsymbol{\mathscr{B}}$ and $x \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ we also get $b \cdot \mathrm{cl}(T)(\mathrm{cl}(x)) = \mathrm{cl}(T)(b \cdot \mathrm{cl}(x))$ for all $b \in \mathscr{B}$, i.e. $\mathrm{cl}(T)$ is an intertwiner again. The functoriality of the classical limit is a consequence of (7.2.6) and (7.2.7). Finally, if the *-representation was strongly non-degenerate then the classical limit is again strongly non-degenerate since for $x = \sum_i b_i \cdot y_i \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ with $b_i \in \boldsymbol{\mathscr{B}}$ and $y_i \in {}_{\boldsymbol{\mathscr{B}}}\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ we get $\mathrm{cl}(x) = \sum_i \mathrm{cl}(b_i) \cdot \mathrm{cl}(y_i)$. □

While the classical limit of *-representation on inner-product modules enjoys good functorial properties, the case of pre-Hilbert modules is slightly more complicated. The difficulty is that the classical limit of a completely positive inner product has no reason to be completely positive in general. However, if the deformation $\boldsymbol{\mathscr{A}}$ is completely positive, then the classical limit of positive matrices with values in $\mathrm{M}_n(\boldsymbol{\mathscr{A}})$ is again positive: this gives directly the following result:

Proposition 7.2.3 *Let* $\boldsymbol{\mathscr{A}}$ *be a completely positive deformation of a* **-algebra* $\mathscr{A}$ *over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$. *For a right* $\boldsymbol{\mathscr{A}}$*-module* $\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ *with completely positive inner product the classical limit* $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ *is a pre-Hilbert right* $\mathscr{A}$*-module.*

PROOF: We know already that the classical limit is an inner-product right $\mathscr{A}$-module by Proposition 7.2.1, *iii.)*. For $x_1, \ldots, x_n \in \boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}}$ we know $(\langle x_i, x_j \rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}}) \in \mathrm{M}_n(\boldsymbol{\mathscr{A}})^+$. Since the deformation is completely positive,

$$(\langle \mathrm{cl}(x_i), \mathrm{cl}(x_j) \rangle^{\mathrm{cl}(\boldsymbol{\mathscr{E}})}_{\mathscr{A}}) = (\mathrm{cl}(\langle x_i, x_j \rangle^{\boldsymbol{\mathscr{E}}}_{\boldsymbol{\mathscr{A}}})) \in \mathrm{M}_n(\mathscr{A})^+$$

follows from the very definition of completely positive deformations. Hence $\mathrm{cl}(\boldsymbol{\mathscr{E}}_{\boldsymbol{\mathscr{A}}})$ is a pre-Hilbert module as claimed. □

Corollary 7.2.4 *Let* $\boldsymbol{\mathscr{A}}$ *be a completely positive deformation of a* **-algebra* $\mathscr{A}$ *over* $\mathsf{C} = \mathsf{R}(\mathrm{i})$. *Then for a Hermitian deformations* $\boldsymbol{\mathscr{B}}$ *of a* **-algebra* $\mathscr{B}$ *over* C *the classical limit restricts functors*

$$\mathrm{cl}\colon {}^*\text{-}\mathsf{rep}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{B}}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathscr{A}}(\mathscr{B}), \tag{7.2.11}$$

and

$$\mathrm{cl}\colon {}^*\text{-}\mathsf{Rep}_{\boldsymbol{\mathscr{A}}}(\boldsymbol{\mathscr{B}}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathscr{A}}(\mathscr{B}). \tag{7.2.12}$$

Remarkably, only the deformation $\boldsymbol{\mathscr{A}}$ has to be completely positive in order to guarantee the above compatibility of the classical limit with positivity. However, this will change when we move to the bicategory picture.

7.2.2 The Classical Limit for $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$

7.3 Classical Limit and Strong Morita Equivalence

7.4 Exercises

Exercise 7.4.1 (The category $\mathsf{Bimod}_{\mathsf{R}[\![\lambda]\!]}$) Show that the classifying category $\mathsf{Bimod}_{\mathsf{R}[\![\lambda]\!]}$ of the bicategory $\underline{\mathsf{Bimod}}_{\mathsf{R}[\![\lambda]\!]}$ can be identified with a full sub-category of $\mathsf{Bimod}_{\mathsf{R}[\![\lambda]\!]}$, i.e. the full sub-category of Bimod consisting of unital algebras over $\mathsf{R}[\![\lambda]\!]$.

Exercise 7.4.2 (Deformation of automorphisms) Let $\mathscr{A}$ be a unital associative algebra over R with a formal associative deformation $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$ such that the center of $\boldsymbol{\mathscr{A}}$ is trivial.

i.) Suppose that $D_0 \in \mathrm{InnDer}(\mathscr{A})$ is an inner derivation of the undeformed algebra. Show that there exists a $D \in \mathrm{InnDer}(\boldsymbol{\mathscr{A}})$ with $\mathrm{cl}(D) = D_0$. Show that one can arrange the choice of D in such a way that this induces a $\mathsf{R}[\![\lambda]\!]$-linear isomorphism

$$\mathrm{Der}(\mathscr{A})[\![\lambda]\!] \longrightarrow \mathrm{Der}(\boldsymbol{\mathscr{A}}). \tag{7.4.1}$$

ii.) Let $T \in \mathrm{Equiv}(\star)$. Show that $T = \mathbb{1} + \lambda T_1 + \cdots$ with a derivation T_1 of the undeformed algebra $\mathscr{A}$.

iii.) Suppose that $\mathrm{InnDer}(\mathscr{A}) = \mathrm{Der}(\mathscr{A})$, i.e. every derivation of the undeformed algebra $\mathscr{A}$ is inner. Show that in this case

$$\ker(\mathrm{cl}) \cong \mathrm{Der}(\mathscr{A})[\![\lambda]\!] \tag{7.4.2}$$

as sets. The group structure induced on the right hand side might be very much non-abelian.

Hint: First use (7.4.1) and then Proposition 6.1.6.

iv.) Use the Baker-Campbell-Hausdorff series to describe the induced group law on the right hand side of (7.4.2).

Exercise 7.4.3 (Deforming automorphisms of Poisson algebras) Let $\mathscr{A}$ be a unital associative commutative algebra over R with a formal associative deformation $\boldsymbol{\mathscr{A}} = (\mathscr{A}[\![\lambda]\!], \star)$, not necessarily commutative. Let $\{\,\cdot\,,\,\cdot\,\}$ denote the induced Poisson structure on $\mathscr{A}$ according to Proposition 6.1.8.

i.) Let $T = \mathbb{1} + \lambda T_1 + \cdots \in \mathrm{Equiv}(\star)$ be a self-equivalence of $\star$. Show that T_1 is a *Poisson derivation*, i.e. a derivation $T_1 \in \mathrm{Der}(\mathscr{A})$ with

$$T_1(\{a, b\}) = \{T_1(a), b\} + \{a, T_1(b)\} \tag{7.4.3}$$

for all $a, b \in \mathscr{A}$.

We denote the set of Poisson derivations of $\mathscr{A}$ by

$$\mathrm{PDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\}) = \big\{ D \in \mathrm{Der}(\mathscr{A}) \;\big|\; D \text{ is a Poisson derivation} \big\}. \tag{7.4.4}$$

A Poisson derivation D is called *inner Poisson derivation* if there exists a $a \in \mathscr{A}$ with $D = \{a, \cdot\}$. Moreover, a Poisson derivation is called *integral Poisson derivation* if there exists an invertible $u \in \mathscr{A}$ with $D = u^{-1}\{u, \cdot\}$, see [28, Def. 4.9]. The sets of inner Poisson derivations and integral Poisson derivations are denoted by $\mathrm{InnPDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\})$ and $\mathrm{IntPDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\})$.

i.) Show that inner Poisson derivations are indeed Poisson derivations. In fact, show that

$$\mathrm{InnPDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\}) \subseteq \mathrm{PDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\}) \tag{7.4.5}$$

is a Lie ideal in the Lie algebra of all Poisson derivations.

ii.) Show that also integral Poisson derivations are Poisson derivations. Show that the integral Poisson derivations form an abelian group under addition. Morally, $u^{-1}\{u, \cdot\}$ is the inner Poisson derivation with the (hypothetical) logarithm $\log(u)$.

iii.) Consider the map

$$s\colon \operatorname{Equiv}(\star) \longrightarrow \operatorname{PDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\}) \tag{7.4.6}$$

sending T to T_1. Show that s is a group morphisms (with the additive group structure from $+$ on the right hand side).

iv.) Show that the image of the inner self-equivalences $\operatorname{InnEquiv}(\star) = \operatorname{Equiv}(\star) \cap \operatorname{InnAut}(\boldsymbol{\mathscr{A}})$ under s is given by $\operatorname{IntPDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\})$.

Hint: Use that an invertible $u_0 \in \mathscr{A}$ is also invertible with respect to $\star$.

Exercise 7.4.4 (Algebras with exponential map) Let $\mathscr{A}$ be a unital algebra over R. We call a map $\exp\colon \mathscr{Z}(\mathscr{A}) \longrightarrow \mathscr{Z}(\mathscr{A})$ an *exponential map* if

$$\exp(a + b) = \exp(a)\exp(b) \quad \text{and} \quad \exp(0) = \mathbb{1}, \tag{7.4.7}$$

as well as

$$D(\exp(a)) = \exp(a) D(a) \tag{7.4.8}$$

for all $a, b \in \mathscr{Z}(\mathscr{A})$ and $D \in \operatorname{Der}(\mathscr{A})$, see [115, Def. 5.3]. In case of a *-algebra one requires in addition $\exp(a^*) = \exp(a)^*$ for all $a \in \mathscr{Z}(\mathscr{A})$.

i.) Show that $D \in \operatorname{Der}(\mathscr{A})$ restricts to a derivation of $\mathscr{Z}(\mathscr{A})$.

ii.) Suppose now that $(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\})$ is a (commutative) Poisson algebra. Show that if $\mathscr{A}$ has an exponential map then

$$\operatorname{InnPDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\}) \subseteq \operatorname{IntPDer}(\mathscr{A}, \{\,\cdot\,,\,\cdot\,\}). \tag{7.4.9}$$

iii.) Show that $\mathscr{C}^\infty(M)$ for a smooth manifold M has an exponential map.

Bibliography

[1] Ara, P.: *Morita equivalence for rings with involution.* Alg. Rep. Theo. **2** (1999), 227–247. 80, 83, 87, 128

[2] Ara, P.: *Morita equivalence and Pedersen Ideals.* Proc. AMS **129**.4 (2000), 1041–1049. 80

[3] Aschieri, P., Dimitrijević, M., Kulish, P., Lizzi, F., Wess, J.: *Noncommutative spacetimes*, vol. 774 in *Lecture Notes in Physics*. Springer-Verlag, Berlin, 2009. Symmetries in noncommutative geometry and field theory. 10

[4] Bass, H.: *Algebraic K-Theory.* W. A. Benjamin, Inc., New York, Amsterdam, 1968. 46, 48, 122, 125, 127, 128, 130, 139

[5] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Ann. Phys. **111** (1978), 61–151. 11, 160, 171

[6] Beer, W.: *On Morita equivalence of nuclear C^*-algebras.* J. Pure Appl. Algebra **26**.3 (1982), 249–267. 135

[7] Bénabou, J.: *Introduction to Bicategories.* In: *Reports of the Midwest Category Seminar*, 1–77. Springer-Verlag, 1967. 100, 105, 150, 152, 208

[8] Bertelson, M., Bieliavsky, P., Gutt, S.: *Parametrizing Equivalence Classes of Invariant Star Products.* Lett. Math. Phys. **46** (1998), 339–345. 165

[9] Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of Star Products.* Class. Quant. Grav. **14** (1997), A93–A107. 165

[10] Blackadar, B.: *Operator algebras*, vol. 122 in *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, Heidelberg, 2006. 10, 13

[11] Bony, J.-M.: *Sommes de carrés de fonctions dérivables.* Bull. Soc. Math. France **133**.4 (2005), 619–639. 27

[12] Bony, J.-M., Broglia, F., Colombini, F., Pernazza, L.: *Nonnegative functions as squares or sums of squares.* J. Funct. Anal. **232**.1 (2006), 137–147. 27

[13] Bordemann, M., Neumaier, N., Pflaum, M. J., Waldmann, S.: *On representations of star product algebras over cotangent spaces on Hermitian line bundles.* J. Funct. Anal. **199** (2003), 1–47. 18

[14] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation.* Commun. Math. Phys. **198** (1998), 363–396. 18

[15] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications.* J. Geom. Phys. **29** (1999), 199–234. 18, 178

[16] Bordemann, M., Neumaier, N., Waldmann, S., Weiss, S.: *Deformation quantization of surjective submersions and principal fibre bundles.* Crelle's J. reine angew. Math. **639** (2010), 1–38. 191, 192, 201

[17] Bordemann, M., Römer, H., Waldmann, S.: *A Remark on Formal KMS States in Deformation Quantization.* Lett. Math. Phys. **45** (1998), 49–61. 199

[18] Bordemann, M., Waldmann, S.: *Formal GNS Construction and States in Deformation Quantization.* Commun. Math. Phys. **195** (1998), 549–583. 18, 176, 199, 200

[19] Bratteli, O., Robinson, D. W.: *Operator Algebras and Quantum Statistical Mechanics I: C^*- and W^*-Algebras. Symmetry Groups. Decomposition of States.* Springer-Verlag, New York, Heidelberg, Berlin, 2. edition, 1987. 10

[20] Bratteli, O., Robinson, D. W.: *Operator Algebras and Quantum Statistical Mechanics II: Equilibrium States. Models in Quantum Statistical Mechanics.* Springer-Verlag, New York, Heidelberg, Berlin, 2. edition, 1997. 10

[21] Breuer, H.-P., Petruccione, F.: *The theory of open quantum systems.* Oxford University Press, New York, 2002. 8, 75

[22] Bursztyn, H.: *Semiclassical geometry of quantum line bundles and Morita equivalence of star products.* Int. Math. Res. Not. **2002**.16 (2002), 821–846. 208, 211

[23] Bursztyn, H., Waldmann, S.: *Deformation Quantization of Hermitian Vector Bundles.* Lett. Math. Phys. **53** (2000), 349–365. 171, 174, 193

[24] Bursztyn, H., Waldmann, S.: *On Positive Deformations of *-Algebras.* In: Dito, G., Sternheimer, D. (eds.): *Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries*, *Mathematical Physics Studies* no. **22**, 69–80. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000. 170, 171, 177, 183, 198

[25] Bursztyn, H., Waldmann, S.: **-Ideals and Formal Morita Equivalence of *-Algebras.* Int. J. Math. **12**.5 (2001), 555–577. 19, 24, 25, 80, 142, 144, 158

[26] Bursztyn, H., Waldmann, S.: *Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization.* J. Geom. Phys. **37** (2001), 307–364. 19, 21, 30, 35, 61, 80, 180, 213

[27] Bursztyn, H., Waldmann, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds.* Commun. Math. Phys. **228** (2002), 103–121. 20, 163, 172

[28] Bursztyn, H., Waldmann, S.: *Bimodule deformations, Picard groups and contravariant connections.* K-Theory **31** (2004), 1–37. 208, 211, 216

[29] Bursztyn, H., Waldmann, S.: *Completely positive inner products and strong Morita equivalence.* Pacific J. Math. **222** (2005), 201–236. 30, 35, 53, 61, 65, 66, 75, 80, 83, 91, 93, 110, 122, 134, 136, 173, 179

[30] Bursztyn, H., Waldmann, S.: *Hermitian star products are completely positive deformations.* Lett. Math. Phys. **72** (2005), 143–152. 177, 183

[31] Bursztyn, H., Waldmann, S.: *Induction of Representations in Deformation Quantization.* In: Maeda, Y., Tose, N., Miyazaki, N., Watamura, S., Sternheimer, D. (eds.): *Noncommutative Geometry and Physics*, 65–76. World Scientific, Singapore, 2005. Proceedings of the CEO International Workshop. 69

[32] Bursztyn, H., Weinstein, A.: *Poisson geometry and Morita equivalence.* In: Gutt, S., Rawnsley, J., Sternheimer, D. (eds.): *Poisson Geometry, Deformation Quantisation and Group Representations*, vol. 323 in *London Mathematical Society Lecture Note Series*, 1–78. Cambridge University Press, Cambridge, 2005. 129

[33] Burzstyn, H., Dolgushev, V., Waldmann, S.: *Morita equivalence and characteristic classes of star products.* Crelle's J. reine angew. Math. **662** (2012), 95–163. 20

[34] Cahen, M., Gutt, S., DeWilde, M.: *Local Cohomology of the Algebra of C^∞ Functions on a Connected Manifold.* Lett. Math. Phys. **4** (1980), 157–167. 198

[35] Cannas da Silva, A., Weinstein, A.: *Geometric Models for Noncommutative Algebras. Berkeley Mathematics Lecture Notes.* AMS, 1999. 10

[36] Cartan, H., Eilenberg, S.: *Homological Algebra.* Princeton University Press, Princeton, New Jersey, 1999. Thirteenth printing, originally published in 1956. 166

[37] Chari, V., Pressley, A.: *A Guide to Quantum Groups.* Cambridge University Press, Cambridge, 1994. 11

[38] Connes, A.: *Noncommutative Geometry.* Academic Press, San Diego, New York, London, 1994. 10

[39] Connes, A., Marcolli, M.: *Noncommutative geometry, quantum fields and motives*, vol. 55 in *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2008. 10

[40] Cuntz, J., Meyer, R., Rosenberg, J. M.: *Topological and Bivariant K-Theory.* Birkhäuser, Basel, 2007. 201

[41] Davidson, K. R.: *C^*-Algebras by Example*, vol. 6 in *Fields Institute Monographs.* American Mathematical Society, Providence, Rhode Island, 1996. 10

[42] Deligne, P.: *Déformations de l'Algèbre des Fonctions d'une Variété Symplectique: Comparaison entre Fedosov et DeWilde, Lecomte.* Sel. Math. New Series **1**.4 (1995), 667–697. 165

[43] DeWilde, M., Lecomte, P. B. A.: *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds.* Lett. Math. Phys. **7** (1983), 487–496. 165

[44] Dito, G., Sternheimer, D.: *Deformation quantization: genesis, developments and metamorphoses.* In: Halbout, G. (eds.): *Deformation quantization*, vol. 1 in *IRMA Lectures in Mathematics and Theoretical Physics*, 9–54. Walter de Gruyter, Berlin, New York, 2002. 160

[45] Dixmier, J.: *C^*-Algebras.* North-Holland Publishing Co., Amsterdam, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15. 10

[46] Emmrich, C., Weinstein, A.: *Geometry of the Transport Equation in Multicomponent WKB Approximations.* Comm. Math. Phys. **176** (1996), 701–711. 174

[47] Esposito, C.: *Formality theory. From Poisson structures to deformation quantization.* Springer-Verlag, Heidelberg, Berlin, 2015. 11, 159, 160, 166

[48] Esposito, C., Schnitzer, J., Waldmann, S.: *A universal construction of universal deformation formulas, Drinfeld twists and their positivity.* Pacific J. Math. **291**.2 (2017), 319–358. 9, 182, 184

[49] Fedosov, B. V.: *Quantization and the Index.* Sov. Phys. Dokl. **31**.11 (1986), 877–878. 165

[50] Fedosov, B. V.: *A Simple Geometrical Construction of Deformation Quantization.* J. Diff. Geom. **40** (1994), 213–238. 165

[51] Fedosov, B. V.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996. 174

[52] Gerstenhaber, M.: *Cohomology Structure of an associative Ring.* Ann. Math. **78** (1963), 267–288. 159, 166, 167, 168

[53] Gerstenhaber, M.: *On the Deformation of Rings and Algebras.* Ann. Math. **79** (1964), 59–103. 20, 159, 162, 166

[54] Gerstenhaber, M.: *On the Deformation of Rings and Algebras II.* Ann. Math. **84** (1966), 1–19. 20, 159, 166

[55] Gerstenhaber, M.: *On the Deformation of Rings and Algebras III.* Ann. Math. **88** (1968), 1–34. 20, 159, 165, 166

[56] Gerstenhaber, M.: *On the Deformation of Rings and Algebras IV.* Ann. Math. **99** (1974), 257–276. 20, 159, 166

[57] Gerstenhaber, M., Schack, S. D.: *Algebraic Cohomology and Deformation Theory.* In: Hazewinkel, M., Gerstenhaber, M. (eds.): *Deformation Theory of Algebras and Structures and Applications*, 13–264. Kluwer Academic Press, Dordrecht, 1988. 174

[58] Grabowski, J.: *Isomorphisms of algebras of smooth functions revisited.* Arch. Math. (Basel) **85**.2 (2005), 190–196. 131

[59] Gracia-Bondía, J. M., Várilly, J. C., Figueroa, H.: *Elements of noncommutative geometry. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks].* Birkhäuser Boston Inc., Boston, MA, 2001. 10, 166

[60] Gutt, S.: *Variations on deformation quantization.* In: Dito, G., Sternheimer, D. (eds.): *Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries*, *Mathematical Physics Studies* no. **21**, 217–254. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000. 160

[61] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes.* J. Geom. Phys. **29** (1999), 347–392. 165

[62] Gutt, S., Waldmann, S.: *Involutions and Representations for Reduced Quantum Algebras.* Adv. Math. **224** (2010), 2583–2644. 20

[63] Hochschild, G., Kostant, B., Rosenberg, A.: *Differential Forms on regular affine Algebras.* Trans. Am. Math. Soc. **102** (1962), 383–408. 198

[64] Hurle, B.: *Bimodule deformation of fibered manifolds and the HKR theorem.* Preprint **arXiv:1806.01131** (2018), 33 pages. 191, 192

[65] Jacobson, N.: *Basic Algebra I.* Freeman and Company, New York, 2. edition, 1985. 22

[66] Jacobson, N.: *Basic Algebra II.* Freeman and Company, New York, 2. edition, 1989. 59, 166

[67] Jansen, S., Neumaier, N., Schaumann, G., Waldmann, S.: *Classification of Invariant Star Products up to Equivariant Morita Equivalence on Symplectic Manifolds.* Lett. Math. Phys. **100** (2012), 203–236. 20

[68] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras. Volume I: Elementary Theory*, vol. 15 in *Graduate Studies in Mathematics.* American Mathematical Society, Providence, 1997. 10, 22

[69] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras. Volume II: Advanced Theory*, vol. 16 in *Graduate Studies in Mathematics.* American Mathematical Society, Providence, 1997. 10, 13

[70] Kaplansky, I.: *Rings of operators.* W. A. Benjamin, Inc., New York-Amsterdam, 1968. 52

[71] Karoubi, M.: *K-Theory*, vol. 226 in *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1978. 50

[72] Kashiwara, M., Schapira, P.: *Categories and sheaves*, vol. 332 in *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, Heidelberg, 2006. 94

[73] Kassel, C.: *Quantum Groups*, vol. 155 in *Graduate Texts in Mathematics.* Springer-Verlag, New York, Berlin, Heidelberg, 1995. 11, 102

[74] Khalkhali, M.: *Basic noncommutative geometry. EMS Series of Lectures in Mathematics.* European Mathematical Society (EMS), Zürich, 2009. 10

[75] Khalkhali, M., Marcolli, M. (eds.): *An invitation to noncommutative geometry.* World Scientific Publishing, Hackensack, NJ, 2008. Lectures from the International Workshop on Noncommutative Geometry held in Tehran, 2005. 10

[76] Klimyk, A., Schmüdgen, K.: *Quantum Groups and Their Representations. Texts and Monographs in Physics.* Springer-Verlag, Heidelberg, Berlin, New York, 1997. 11

[77] Kontsevich, M.: *Deformation Quantization of Poisson manifolds.* Lett. Math. Phys. **66** (2003), 157–216. 11, 165

[78] Lam, T. Y.: *Lectures on Modules and Rings*, vol. 189 in *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1999. 29, 46, 96

[79] Lance, E. C.: *Hilbert C^*-modules. A Toolkit for Operator algebraists*, vol. 210 in *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1995. 29, 30, 35, 55, 68, 88, 110

[80] Landi, G.: *An Introduction to Noncommutative Spaces and Their Geometries. Lecture Notes in Physics* no. **m51**. Springer-Verlag, Heidelberg, Berlin, New York, 1997. 10

[81] Landsman, N. P.: *Mathematical Topics between Classical and Quantum Mechanics. Springer Monographs in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1998. 10, 30, 68, 80

[82] LANDSMAN, N. P.: *Quantized reduction as a tensor product.* In: LANDSMAN, N. P., PFLAUM, M., SCHLICHENMAIER, M. (EDS.): *Quantization of Singular Symplectic Quotients*, 137–180. Birkhäuser, Basel, Boston, Berlin, 2001. 110

[83] LEINSTER, T.: *Higher Operads, Higher Categories*, vol. 298 in *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2004. Available on the arXiv: `math.CT/0305049`. 101, 150

[84] LODAY, J.-L., VALLETTE, B.: *Algebraic Operads*, vol. 346 in *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin, Heidelberg, New York, 2012. 166, 168

[85] MACLANE, S.: *Categories for the Working Mathematician*, vol. 5 in *Graduate Texts in Mathematics.* Springer-Verlag, New York, Berlin, 2. edition, 1998. 150

[86] MADORE, J.: *An Introduction to Noncommutative Differential Geometry and its Physical Applications. London Mathematical Society Lecture Note Series* no. **257**. Cambridge University Press, Cambridge, UK, 2. edition, 1999. 10

[87] MANUILOV, V. M., TROITSKY, E. V.: *Hilbert C^*-modules*, vol. 226 in *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 2005. 29, 55

[88] MORITA, K.: *Duality for modules and its applications to the theory of rings with minimum condition.* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142. 95

[89] MOTZKIN, T. S.: *The arithmetic-geometric inequality.* In: *Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*, 205–224. Academic Press, New York, 1967. 26

[90] MRČUN, J.: *On isomorphisms of algebras of smooth functions.* Proc. Amer. Math. Soc. **133**.10 (2005), 3109–3113 (electronic). 131

[91] NEST, R., TSYGAN, B.: *Algebraic Index Theorem.* Commun. Math. Phys. **172** (1995), 223–262. 165

[92] NIELSEN, M. A., CHUANG, I. L.: *Quantum computation and quantum information.* Cambridge University Press, Cambridge, 2000. 8, 75

[93] OKAYASU, T.: *Polar Decomposition for Isomorphisms of C^*-Algebras.* Tôhoku Math. J. **26** (1974), 541–554. 173

[94] OMORI, H., MAEDA, Y., YOSHIOKA, A.: *Weyl Manifolds and Deformation Quantization.* Adv. Math. **85** (1991), 224–255. 165

[95] RAEBURN, I., WILLIAMS, D. P.: *Morita Equivalence and Continuous-Trace C^*-Algebras*, vol. 60 in *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998. 30, 55, 68, 80

[96] RIEFFEL, M. A.: *Induced representations of C^*-algebras.* Bull. Amer. Math. Soc. **78** (1972), 606–609. 68, 80

[97] RIEFFEL, M. A.: *Induced representations of C^*-algebras.* Adv. Math. **13** (1974), 176–257. 61, 68, 80

[98] RIEFFEL, M. A.: *Morita equivalence for C^*-algebras and W^*-algebras.* J. Pure. Appl. Math. **5** (1974), 51–96. 61, 80

[99] ROSENBERG, J.: *Algebraic K-Theory and Its Applications.* Springer-Verlag, Berlin, Heidelberg, New York, 1994. 46, 48, 139

[100] ROSENBERG, J.: *Rigidity of K-theory under deformation quantization.* Preprint **q-alg/9607021** (July 1996). 174, 175, 194

[101] RUDIN, W.: *Real and Complex Analysis.* McGraw-Hill Book Company, New York, 3. edition, 1987. 26

[102] RUDIN, W.: *Functional Analysis.* McGraw-Hill Book Company, New York, 2. edition, 1991. 13

[103] SAKAI, S.: *C^*-Algebras and W^*-Algebras*, vol. 60 in *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin, Heidelberg, New York, 1971. 134, 173

[104] SCHMÜDGEN, K.: *Unbounded Operator Algebras and Representation Theory*, vol. 37 in *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, Boston, Berlin, 1990. 7, 8, 13, 16, 19, 38

[105] SCHMÜDGEN, K.: *Unbounded Self-adjoint Operators on Hilbert Space*, vol. 265 in *Graduate Texts in Mathematics.* Springer-Verlag, Heidelberg, Berlin, New York, 2012. 7

[106] SCHWEIZER, J.: *Crossed products by equivalence bimodules.* Preprint Univ. Tübingen (1999). 110

[107] SERRE, J.-P.: *Faisceaux algébriques cohérents.* Ann. of Math. (2) **61** (1955), 197–278. 48

[108] SPEYER, D.: *Hilbert's 17th Problem for smooth functions*, 2011. Answer to MathOverflow question `https://mathoverflow.net/questions/53876`. 26

[109] STERNHEIMER, D.: *Deformations and quantizations, an introductory overview.* In: DITO, G., GARCÍA-COMPEÁN, H., LUPERCIO, E., TURRUBIATES, F. J. (EDS.): *Non-commutative geometry in mathematics and physics*, vol. 462 in *Contemporary Mathematics*, 41–54. American Mathematical Society, Providence, 2008. Papers from the 11th Solomon Lefschetz Memorial Lectures and the Satellite Conference: Topics in Deformation Quantization and Non-commutative Structures held in Mexico City, September 7–9, 2005. 160

[110] SWAN, R. G.: *Vector bundles and projective modules.* Trans. Amer. Math. Soc. **105** (1962), 264–277. 48

[111] VÁRILLY, J. C.: *An introduction to noncommutative geometry. EMS Series of Lectures in Mathematics.* European Mathematical Society (EMS), Zürich, 2006. 10

[112] VEY, J.: *Déformation du Crochet de Poisson sur une Variété symplectique.* Comm. Math. Helv. **50** (1975), 421–454. 198

[113] WALDMANN, S.: *A Remark on the Deformation of GNS Representations of *-Algebras.* Rep. Math. Phys. **48** (2001), 389–396. 182, 200

[114] WALDMANN, S.: *The Picard Groupoid in Deformation Quantization.* Lett. Math. Phys. **69** (2004), 223–235. 36, 38

[115] WALDMANN, S.: *The covariant Picard groupoid in differential geometry.* Int. J. Geom. Methods Mod. Phys. **3**.3 (2006), 641–654. 217

[116] WALDMANN, S.: *Poisson-Geometrie und Deformationsquantisierung. Eine Einführung.* Springer-Verlag, Heidelberg, Berlin, New York, 2007. 11, 18, 33, 49, 159, 160, 166, 168, 184, 199, 200

[117] WALDMANN, S.: *A nuclear Weyl algebra.* J. Geom. Phys. **81** (2014), 10–46. 197

[118] WEIBEL, C. A.: *An introduction to homological algebra*, vol. 38 in *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994. 166

[119] WEINSTEIN, A., XU, P.: *Hochschild cohomology and characteristic classes for star-products.* In: KHOVANSKIJ, A., VARCHENKO, A., VASSILIEV, V. (EDS.): *Geometry of differential equations. Dedicated to V. I. Arnold on the occasion of his 60th birthday*, 177–194. American Mathematical Society, Providence, 1998. 165

[120] WEISS, S.: *Deformation Quantization of Principal Fibre Bundles and Classical Gauge Theories.* PhD thesis, Fakultät für Mathematik und Physik, Physikalisches Institut, Albert-Ludwigs-Universität, Freiburg, 2009. Available at `https://freidok.uni-freiburg.de/data/7184`. 194

[121] WELLS, R. O.: *Differential Analysis on Complex Manifolds*, vol. 65 in *Graduate Texts in Mathematics.* Springer-Verlag, New York, Berlin, Heidelberg, 1980. 49, 133

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