

Announcement

Seminar on Deformation Quantization and Geometry

15.09.2026 at 11:00 s.t.

Seminarroom SE 31

RUBEN LOUIS (UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN)

Lie-Rinehart and Poisson algebras over C^∞ -rings
(Joint work with E. Lerman)

We study Lie-Rinehart algebras in the context of differential geometry. A fundamental issue arises when relating vector bundles and modules. By Serre-Swan theorem the module of sections of a vector bundle $E \rightarrow M$ is a finitely generated projective module over $C^\infty(M)$. And conversely any finitely generated projective module over $C^\infty(M)$ is the module of global sections of some vector bundle $E \rightarrow M$. It is not clear how to handle geometrically the modules that are not projective or not finitely generated. Such modules arise, for example, in the study of singular foliations.

Closely related to the issue of modules is the correspondence between Lie algebroids and Lie-Rinehart algebras. Recall that any Lie algebroid $E \rightarrow TM$ over a manifold M gives rise to a Lie-Rinehart algebra $\rho : \Gamma(E) \rightarrow \text{Der}(C^\infty(M))$ at the level of sections. On the other hand there are naturally occurring Lie-Rinehart algebras (e.g., in Poisson geometry), particularly in situations involving singularities, where the corresponding modules are not sections of vector bundle (i.e., they are not finitely generated projective modules)

In this talk, we approach these questions using the framework of C^∞ -rings, into which the category of smooth manifolds/Poisson manifolds embeds fully faithfully. This framework offers a natural geometric interpretation of arbitrary modules over C^∞ -rings. Furthermore, it allows us to establish results that cannot be obtained using commutative algebra alone.

We introduce Lie-Rinehart algebras over C^∞ -rings and show that, for any Poisson C^∞ -ring A , the module of C^∞ -Kähler differentials Ω_A^1 naturally carries such a structure. Conversely, given a Lie-Rinehart algebra over A , we construct a natural Poisson bracket on the C^∞ -ring $F(\mathcal{M})$ associated with the underlying module $\mathcal{M}$, and the latter determines the Lie-Rinehart algebra structure on $\mathcal{M}$. In the case where A is the C^∞ -ring of smooth functions on a manifold M and $\mathcal{M}$ is the module $\Gamma(E)$ of sections of a Lie algebroid $E \rightarrow M$, the C^∞ -ring $F(\Gamma(E))$ is the ring of functions $C^\infty(E^*)$ on the total space of the vector bundle $E^* \rightarrow M$ dual to the vector bundle E .

Several examples will be given.

Reference: <https://doi.org/10.48550/arXiv.2606.01388>

Invited by Madeleine Jotz