1

An Introduction to Deformation Quantization

Stefan Waldmann
Faculty of Mathematics and Physics
Institute of Physics, Hermann Herder Strasse 3
79104 Freiburg, Germany

1 Introduction and Motivation

Quantum theory celebrates its 100th birthday and we should still think about quantization? Indeed, there are many good reasons to do so and many unsolved questions some of them I want to discuss in this article.

The quantum physics of finitely many non-relativistic particles moving in Euclidean space is well understood. Nevertheless, physical reality forces us to gain a deeper understanding of more general situations.

Taking into account the theory of special relativity one is led to quantum field theories where in most cases only perturbative formulations are known for the interacting case. Beside the difficulties of infinitely many degrees of freedom the problem arising in this context is the presence of *gauge degrees of freedom*. For the classical description one uses additional degrees of freedom which do not have an immediate physical relevance. As example I would like to mention Maxwell's theory of electrodynamics: the potentials $(\phi, \vec{A})$ are only needed for simplification but are physically unobservable. The observable content of the theory are the electric and magnetic fields $\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. If one wants to describe the true, physical degrees of freedom one has to consider gauge equivalence classes of the potentials, i.e. $(\phi, \vec{A}) \sim (\phi', \vec{A}')$ if they yield the same $\vec{E}$ and $\vec{B}$. This passage to a gauge invariant description on the classical side is known as *phase space reduction*, since typically the dimension of the classical phase space descreases. However, the geometry of the classical phase space usually becomes more complicated: one may obtain 'holes', the reduced phase space is curved and there are no global canonically conjugate coordinates (q, p). Therefor a naive 'canonical quantization' of the reduced phase space becomes impossible.

As toy models for this situation in field theories with gauge degrees of freedom one considers finite-dimensional phase spaces with non-trivial geometry to study the phenomena which are also expected in the (certainly not easier) infinite-dimensional cases.

Beside the questions on the relations between gauge degrees of freedom, phase space reduction and quantization there are more reasons to investigate the quantization of finite-dimensional phase spaces with non-trivial geometry. One can try to understand the quantum theory of a particle moving in a curved background as this is to be expected from general relativity. Also the quantization in presence of background fields, like the magnetic field of a Dirac monopole, can be considered. Furthermore, it turns out that the question of quantization of phase spaces is, from a technical point of view, deeply related to the question of quantization of geometry itself leading to the notion of non-commutative geometry. Applications of such a quantized geometry may arise from a theory of quantum gravity.

In the following, such more speculative aspects may be taken as motivation. However, I will take a more conservative point of view and consider mainly the quantization of non-relativistic classical mechanics with finitely many degrees of freedom. Here I will mainly focus on a conceptually clear and mathematically rigorous treatment.

In order to approach the question of quantization appropriately, I shall first recall the fundamental structures in classical and quantum mechanics. Here one needs a formulation which is most suited to the problem. In particular, the notions of 'observables', 'states', 'time development', etc. are to be clarified. Starting from the so-called canonical quantization, as it can be found in text books, I will motivated the notion of a *star product*. This basic notion of deformation quantization shall be discussed in detail in order to compare the results of deformation quantization with the original aims of the quantization program in a critical manner. In a concluding section I shall discuss two examples where the techniques of deformation quantization can be applied beyond the original quantization problem.

2 Classical and Quantum Mechanics

2.1 Formulation of classical and quantum mechanics

In this section I shall briefly recall the usual notions in Hamiltonian mechanics and quantum mechanics as it can be found in text books.

2.1.1 *Hamiltonian mechanics: First Version*

The playing ground of classical mechanics is the *phase space* M which in the simplest case is just $M = \mathbb{R}^{2n}$ where n is the number of degrees of freedom.

The *pure states* are the points $x \in M$. They can be denoted by use of the canonical coordinates $x = (q, p) \in \mathbb{R}^{2n}$. Mixed states will be described later. The *observables* are the real-valued functions $f : M \to \mathbb{R}$ which in addition are subject to further analytical conditions: of particular interest are the continuous functions $C(M)$, the smooth functions $C^\infty(M)$, the real-analytical functions $C^\omega(M)$ and the polynomials $\mathrm{Pol}(\mathbb{R}^{2n})$. Note that the notion $x = (q, p)$ has two interpretations: on one hand it denotes a point (state) in phase space, on the other hand it denotes the coordinate functions (observable).

The *expectation value* $E_x(f)$ of an observable f in a state x is the value $f(x)$

at x. The *possible values of a measurement* of the observable f are the values $f(M)$ of f. The *variation* $\mathrm{Var}_x(f)$ of an observable f in a pure state x under repeated measurement is

$$\mathrm{Var}_x(f) = E_x(f^2) - (E_x(f))^2 = 0. \tag{2.1}$$

The *time evolution* is governed by a particular observable, the Hamiltonian $H : M \to \mathbb{R}$. The time evolution of a state x is the unique curve $t \mapsto x(t) = (q(t), p(t)) \in M$ through $x(0) = x$ satisfying Hamilton's equations of motion

$$\dot{q}^i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t)) \quad \text{and} \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q^i}(q(t), p(t)) \quad \text{for} \quad i = 1, \ldots, n. \tag{2.2}$$

2.1.2 *Quantum mechanics: First Version*

The analog of the phase space in quantum mechanics is a complex *Hilbert space* $\mathfrak{H}$. For reasonable physical systems the Hilbert space $\mathfrak{H}$ has a countable infinite Hilbert basis, for certain simplified models also finite-dimensional Hilbert spaces play a role.

The *pure states* are complex rays in $\mathfrak{H}$, i.e. equivalence classes of vectors $\psi \in \mathfrak{H} \setminus \{0\}$ where $\psi \sim \psi'$ if $\psi = z\psi'$ with some $z \in \mathbb{C} \setminus \{0\}$. The *observables* are described by operators on $\mathfrak{H}$. On one hand the *bounded* (continuous) operators $\mathfrak{B}(\mathfrak{H})$, on the other hand the densely defined, *self-adjoint* operators are of interest. Strictly speaking, only the *Hermitian* operators in $\mathfrak{B}(\mathfrak{H})$ correspond to observables.

The *expectation value* $E_\psi(A)$ of an observable A in a state ψ is given by

$$E_\psi(A) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}. \tag{2.3}$$

Note that $E_\psi(A)$ depends on the equivalence class of ψ only. The *possible values of a measurement* of the observable A are the spectral values $\mathrm{spec}(A)$. In order to have a reasonable spectrum the observable A has to be a self-adjoint operator. The spectrum may consist of eigenvalues as for the harmonic oscillator. There are also observables, like the momentum operator which only have spectral values which are no eigenvalues. After repeated measurement of the observable A in the state ψ the *variation*

$$\mathrm{Var}_\psi(A) = E_\psi(A^2) - E_\psi(A)^2 \geq 0 \tag{2.4}$$

is in general *different* from zero: For any state ψ one can find an observable A with $\mathrm{Var}_\psi(A) > 0$. The reason is the non-commutativity of the observables. Physically measurable bounds for $\mathrm{Var}_\psi(A)$ can be obtained from *Heisenberg's uncertainty relations*. Hence the non-commutativity of the quantum observables is crucial in order to implement the uncertainty relations.

The 'size' of the uncertainty, as predicted by Heisenberg's uncertainty relations, is controlled by the 'size' of Planck's constant $\hbar$ as it can be seen from the canonical commutation relations $[\hat{Q}, \hat{P}] = \mathrm{i}\hbar$. Note however, that $\hbar$ is not dimensionless, so it has no intrinsic 'size'. In particular, one can always find a unit system where $\hbar = 1$. In order to obtain small quantum effects one has to compare $\hbar$ with other quantities (expectation values) of the system which also have the physical dimension of an action.

The *time evolution* $\psi(t)$ of a state $\psi = \psi(0)$ is again induced by a certain observable, the Hamilton operator $\hat{H}$, as the unique solution of the *Schrödinger equation*

$$\mathrm{i}\hbar \frac{d}{dt}\psi(t) = \hat{H}\psi(t) \quad \psi(0) = \psi. \tag{2.5}$$

2.2 Generalizations

In the following I shall generalize the above formulations of classical and quantum mechanics in an *algebraic* way. This will allow a more direct comparison. The formulation will not depend much on the number of degrees of freedom whence many of the following will still be valid for field theories or systems in the thermodynamical limit.

For the quantum mechanical description the functional-analytical questions and subtleties shall be postponed in order to focus on the algebraic point of view. On the classical side the geometric properties of the phase space M shall be replaced and encoded in algebraic properties of the function space $C^\infty(M)$. Thus I will be able to avoid to speak too much about differential geometry and analysis in Hilbert spaces but focus on the algebraic properties of both theories instead. This way one finds a language to treat classical and quantum mechanics on the same footing.

2.2.1 *Quantum mechanics: Second Version*

In this second formulation of quantum mechanics I shall start with the properties of the *observable algebra*. Then the states and the time evolution will be derived concepts. See e.g. [22] for further reading.

The central object of a quantum mechanical system is its algebra of observables. As the 'example' $\mathfrak{B}(\mathfrak{H})$ indicates one asks for an *associative algebra* $\mathcal{A}_{\mathrm{QM}}$ over the complex numbers $\mathbb{C}$. Because of the uncertainty relations $\mathcal{A}_{\mathrm{QM}}$ will be non-commutative. To specify the observable elements in $\mathcal{A}_{\mathrm{QM}}$ we have to make sense out of the notion of *Hermitian* elements, whence we require $\mathcal{A}_{\mathrm{QM}}$ to have a *-involution. This is a $\mathbb{C}$-antilinear map $A \mapsto A^*$ such that

$$(AB)^* = B^* A^* \quad \text{and} \quad (A^*)^* = A \tag{2.6}$$

for all $A, B \in \mathcal{A}_{\mathrm{QM}}$. In the case of $\mathcal{A}_{\mathrm{QM}} = \mathfrak{B}(\mathfrak{H})$ the algebra element A^* is just the adjoint operator of A. Such a *-algebra structure allows to tranfer the usual notions of *Hermitian*, *isometric*, and *unitary* elements from the known case of $\mathfrak{B}(\mathfrak{H})$ to the abstract case of $\mathcal{A}_{\mathrm{QM}}$.

Thus the quantum mechanical observable algebra $\mathcal{A}_{\mathrm{QM}}$ shall be a *-algebra over $\mathbb{C}$. Further topological requirements as e.g. a C^*-norm, completeness etc.

shall not be considered at the present stage in order to incorporate also 'unbounded' observables.

The *states* of $\mathcal{A}_{\mathrm{QM}}$ are now identified with the *expectation value functionals*. An expectation value functional of a *-algebra $\mathcal{A}_{\mathrm{QM}}$ is defined to be a linear functional $\omega : \mathcal{A}_{\mathrm{QM}} \to \mathbb{C}$ satisfying

$$\omega(A^*A) \geq 0, \tag{2.7}$$

for all $A \in \mathcal{A}_{\mathrm{QM}}$. Hence ω is a *positive linear functional*. If $\mathcal{A}_{\mathrm{QM}}$ has a unit element $\mathbb{1}$ then one requires in addition the normalization condition $\omega(\mathbb{1}) = 1$. It is an easy exercise to check that the expectation values $A \mapsto E_\psi(A)$ as in (2.3) are indeed positive linear functionals of $\mathfrak{B}(\mathfrak{H})$.

This notion of states allows for a simple characterisation of pure and mixed states. If ω_1 and ω_2 are two states of $\mathcal{A}_{\mathrm{QM}}$ then their *convex combination*

$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2 \tag{2.8}$$

is again a state for $\lambda \in (0,1)$. Now ω is called a *mixed state* if it can be decomposed as in (2.8) in a non-trivial way, i.e. $\omega_1 \neq \omega_2$. A state is called *pure* if such a decomposition is not possible. As an example for a mixed state I shall mention the *thermodynamical state* $A \mapsto \mathsf{tr}(\varrho A)$ of $\mathfrak{B}(\mathfrak{H})$ where $\varrho = \frac{1}{Z}\mathrm{e}^{-\beta H}$ is the density operator for a Hamiltonian H and inverse temperatur β. Here H has to satisfy certain technical conditions in order to make ϱ a trace class operator.

Finally, let me mention the following properties of positive linear functionals: A state $\omega : \mathcal{A}_{\mathrm{QM}} \to \mathbb{C}$ is real

$$\omega(A^*B) = \overline{\omega(B^*A)} \tag{2.9}$$

and satisfies the *Cauchy Schwarz inequality*

$$\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\omega(B^*B). \tag{2.10}$$

Exercise 2.1 *Prove (2.9) and (2.10) by considering the quadratic form* $p(z) = \omega((zA+B)^*(zA+B)) \geq 0$ *for all* $z \in \mathbb{C}$.

The *possible values of a measurement* of an observable A are again given by the spectrum $\mathrm{spec}(A)$. Here one is faced with a technical problem: in order to have a physically reasonable notion of a spectrum one needs more than just a *-algebra. The spectrum should be defined *intrinically*, i.e. as a property of algebraic relations in $\mathcal{A}_{\mathrm{QM}}$ alone. Moreover, taking $\mathfrak{B}(\mathfrak{H})$ as example, one would like to have

- $\mathrm{spec}(A) \subseteq \mathbb{R}$ for $A = A^*$,
- $\mathrm{spec}(A^*A) \subseteq \mathbb{R}^+$ for all A,
- $\mathrm{spec}(p(A)) = p(\mathrm{spec}(A))$ for all polynomials p and all A.

Finally, for a given Hermitian element A and a given state ω one would like to have a spectral measure $d\omega$ on $\mathrm{spec}(A)$ in such a way that

$$\omega(A) = \int_{a \in \mathrm{spec}(A)} a \, d\omega(a). \tag{2.11}$$

Then $\int_{a\in[a_1,a_2]} a \, d\omega(a)$ is interpreted as the probability to obtain a spectral value in the interval $[a_1, a_2]$ when measuring the observable A in the state ω.

For $\mathfrak{B}(\mathfrak{H})$ (or any other C^*-algebra) one defines $\lambda \in \mathrm{spec}(A)$ if $A - \lambda\mathbb{1}$ does not have an inverse in $\mathfrak{B}(\mathfrak{H})$. Then the above properties are guaranteed by the (non-trivial!) spectral theorem. It turns out that a reasonable notion of spectrum is very hard to get *without* a C^*-norm. For a discussion of the spectral theorem in C^*-algebras see e.g. [8, 19].

However, this analytical aspect shall not be considered in the following. Instead we shall assume that $\mathcal{A}_{\mathrm{QM}}$ can be embedded (in some reasonable way) into a C^*-algebra where a good notion of spectrum is available. Typically, it will depend strongly on the example how this can be done. Nevertheless, I shall emphasize that for the interpretation of $\mathcal{A}_{\mathrm{QM}}$ as quantum mechanical observables such a notion of spectrum is *crucial*.

The *time evolution* can be formulated in the following way. First we consider the Heisenberg picture of the time evolution of A in the case of $\mathfrak{B}(\mathfrak{H})$, i.e. the Heisenberg equation

$$\frac{d}{dt} A(t) = \frac{\mathrm{i}}{\hbar}[H, A(t)], \tag{2.12}$$

where $t \mapsto A(t)$ is the unique solution with $A(0) = A$. The solution is of the form

$$A(t) = U_t^* A(0) U_t \tag{2.13}$$

with a *one-parameter group* U_t of unitary operators $U_t \in \mathfrak{B}(\mathfrak{H})$ obying the Schrödinger equation

$$\mathrm{i}\hbar \frac{d}{dt} U_t = H U_t. \tag{2.14}$$

Here once again one needs to specify some more functional-analytical details in order to give life to (2.12) and (2.14) such as strong continuity of $t \mapsto U_t$, see e.g. [26, Sect. VIII.4]. However, I shall again focus on the algebraic properties. The notion of a unitary one-parameter group means

$$U_0 = \mathsf{id}, \quad U_t U_s = U_{t+s} = U_s U_t \quad \text{and} \quad U_t^* = U_{-t} = U_t^{-1}. \tag{2.15}$$

Then the algebraic content of (2.13) can be reformulated in the following way. The linear map $\Phi_t : \mathcal{A}_{\mathrm{QM}} \to \mathcal{A}_{\mathrm{QM}}$ defined by $A \mapsto \Phi_t(A) = U_t^* A U_t$ defines a *one-parameter group of* **-automorphisms* of the observable algebra $\mathcal{A}_{\mathrm{QM}}$: Indeed, one has the property of a one-parameter group

$$\Phi_0 = \mathsf{id} \quad \text{and} \quad \Phi_t \Phi_s = \Phi_{t+s} = \Phi_s \Phi_t, \tag{2.16}$$

and each Φ_t is a *-automorphism of $\mathcal{A}_{\mathrm{QM}}$, i.e.

$$\Phi_t(AB) = \Phi_t(A)\Phi_t(B) \quad \text{and} \quad \Phi_t(A^*) = \Phi_t(A)^* \tag{2.17}$$

for $A, B \in \mathcal{A}_{\mathrm{QM}}$. Thus the quantum mechanical time evolution will be a one-parameter group of *-automorphisms of the observable algebra $\mathcal{A}_{\mathrm{QM}}$. Note that in general Φ_t may have a more complicated form as above.

2.2.2 *Classical mechanics: Second Version*

As already mentioned in the introduction there are physical situations where the classical phase space has a much more complicated geometry than just $\mathbb{R}^{2n}$. The principal structure is then a *differentiable manifold* M together with a *Poisson structure*. Roughly speaking, a differentiable manifold is a geometric object which allows for *local coordinates* in such a way, that passing from one to another coordinate system is a smooth map. As example one may think of the 2-sphere S^2 or the torus T^2. For a differentiable manifold one has a notion of smooth functions $f : M \to \mathbb{C}$.

As *observables* we take again the smooth functions $C^\infty(M)$ where we now allow for *complex-valued* functions. Thus we obtain a *commutative associative* **-algebra* where the product is the pointwise product and the *-involution is the pointwise complex-conjugation of functions. On the first sight it seems unnecessary to consider complex-valued functions in classical mechanics, but one obtains a higher structural similarity to the quantum mechanical observable algbera.

However, there is an additional structure, namely the *Poisson bracket* $\{f, g\}$ for functions on the phase space. This is a bilinear bracket for the smooth functions obying the following properties:

- Antisymmetry: $\{f, g\} = -\{g, f\}$.
- Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.
- Jacobi identity: $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$.

A differentiable manifold with such a Poisson structure for the functions is called a *Poisson manifold*. One can show that in local coordinates $(x^1, \ldots, x^m)$ the Poisson bracket takes the form

$$\{f, g\}(x) = \sum_{i,j} \alpha^{ij}(x) \frac{\partial f}{\partial x^i}(x) \frac{\partial g}{\partial x^j}(x), \tag{2.18}$$

where $\alpha^{ij} = -\alpha^{ji}$ are locally defined functions satisfying the quadratic partial differential equation

$$\sum_{\ell=1}^{m} \left(\alpha^{i\ell} \frac{\partial \alpha^{jk}}{\partial x^\ell} + \alpha^{j\ell} \frac{\partial \alpha^{ki}}{\partial x^\ell} + \alpha^{k\ell} \frac{\partial \alpha^{ij}}{\partial x^\ell} \right) = 0 \tag{2.19}$$

for $i, j, k = 1, \ldots, m$. The last property of the Poisson bracket is its compatibility with the *-involution, i.e. the complex conjugation. We require it to be *real* in the sense that

$$\overline{\{f, g\}} = \{\overline{f}, \overline{g}\}, \tag{2.20}$$

which is equivalent to $\overline{\alpha^{ij}} = \alpha^{ij}$.

The Poisson bracket is called *symplectic* if in any coordinate system and at any point $x \in M$ the matrix $(\alpha^{ij}(x))$ of local functions is invertible. In this case the Poisson manifold is called a *symplectic manifold*.

Exercise 2.2 *Verify that the Jacobi identity is equivalent to (2.19). Moreover, verify that the canonical Poisson bracket (2.21) on* $\mathbb{R}^{2n}$ *defined by*

$$\{f, g\} = \sum_{r=1}^{n} \left(\frac{\partial f}{\partial q^r} \frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r} \frac{\partial g}{\partial q^r} \right) \tag{2.21}$$

indeed is a symplectic Poisson bracket.

A commutative associative algebra $\mathcal{A}$ with a Poisson bracket is called a Poisson algebra. If $\mathcal{A}$ in addition has a *-involution compatible with the Poisson bracket as in (2.20) the $\mathcal{A}$ is called a *Poisson* **-algebra*. Thus we arrive at the following picture: the observables in classical mechanics have the structure of a Poisson *-algebra $\mathcal{A}_{\mathrm{class}}$. Indeed, this characterization is much more general and is valid beyond classical mechanics. It can be seen as the general structure for any classical theory.

For the description of the *states* we have two possibilities: on one hand the pure states are again the points in phase space, on the other hand states can be viewed as positive linear functionals, since we have a *-algebra. It turns out that both approaches are consistent in the following sense: the points $x \in M$ can be identified with the δ-functionals $\delta_x : C^\infty(M) \to \mathbb{C}$ which are positive functionals since

$$\delta_x(\overline{f} f) = \overline{f(x)} f(x) \geq 0. \tag{2.22}$$

Thus the second characterization is indeed a generalization of the previous one. Moreover, there are other positive linear functionals which can be described by integrations with respect to other positive measures μ

$$f \mapsto \int_M f d\mu. \tag{2.23}$$

In particular we obtain the *thermodynamical states* with a integration density given by $\frac{1}{Z} \mathrm{e}^{-\beta H}$.

The *Riesz representation theorem* then shows that indeed all positive linear functionals of $C^\infty(M)$ are of this form: they can be obtained by integration with respect to a positive Borel measure (with compact support). Moreover, the pure states are precisely the δ-functionals as desired.

The *time evolution* can be described algebraically as follows. Again a Hamilton function determines the Hamilton equation of motion whose solutions are curves $t \mapsto x(t)$ through a given initial condition $x = x(0)$. Thus one can define the *flow* $\phi_t : M \to M$ by mapping $x \in M$ to the point $x(t)$ if $x(t)$ is the unique solution with $x(0) = x$. Since we have an autonomous differential equation for

$x(t)$ the flow gives a *one-parameter group*

$$\phi_0 = \mathsf{id} \quad \text{and} \quad \phi_t \circ \phi_s = \phi_{t+s} \tag{2.24}$$

of maps $M \to M$. It turns out that each ϕ_t is even a smooth map whence we have a one-parameter group of *diffeomorphisms* of M. Given an observable f one defines its *pull-back* $\phi_t^* f \in C^\infty(M)$ to be the observable

$$\phi_t^* f = f \circ \phi_t \tag{2.25}$$

and obtains again an observable for all t. Clearly the pull-backs give a one-parameter group of linear maps $\phi_t^* : C^\infty(M) \to C^\infty(M)$ satisfying in addition

$$\phi_t^*(fg) = \phi_t^* f \, \phi_t^* g \quad \text{and} \quad \phi_t^*(\overline{f}) = \overline{\phi_t^*(f)}. \tag{2.26}$$

Thus the time evolution of the observables is a one-parameter group of *-automorphisms as in the quantum case. Since the time evolution is induced by Hamilton's equation of motion one can show in addition that

$$\frac{d}{dt}\phi_t^* f = -\{H, \phi_t^* f\} \tag{2.27}$$

and

$$\phi_t^*(\{f, g\}) = \{\phi_t^* f, \phi_t^* g\}. \tag{2.28}$$

Then (2.27) can be interpreted as infinitesimal time evolution and it is the immediate analog of Heisenberg's equation of motion (2.12). As (2.28) shows the time evolution is compatible with the Poisson brackets.

Summarizing, the time evolution in classical mechanics is a one-parameter group of Poisson *-automorphisms of the observable algebra $\mathcal{A}_{\mathrm{class}}$.

2.2.3 *The algebraic structures*

To summarize to above analysis we find the relevant structural difference between classical and quantum physics: The quantum observable algebra $\mathcal{A}_{\mathrm{QM}}$ is *non-commutative* while the classical observable algebra $\mathcal{A}_{\mathrm{class}}$ is *commutative* but has an additional structure, the *Poisson bracket*. It also became clear that the fundamental object in both cases is given by the observable algebra while the states can be understood as a derived concept. Knowing the *-algebra one also knows its positive linear functionals. Finally, in both cases the time evolution is a one-parameter group of automorphisms the observable algebra.

2.3 What is quantization?

What do we want to achive with 'quantization'? First of all, we have large domains where the classical description of our world is an extraordinarily good approximation, so classical physics is not just 'wrong'. To explain this phenomen of *classical limit* starting with quantum physics is still a delicate and conceptually difficult question.

Nevertheless, to our best present knowledge quantum physics is the more fundamental description of nature. Hence quanization, understood as the passage from classical physics to quantum physics, is *not* a physical phenomen: the world *is* already quantum and quantization is only our poor attempt to find the quantum theoretical description starting with the classical description which we understand better. Apparently we are not able to find the quantum description *a priori* and *intrinsically*, say for the standard model or gravity. Instead we always have to start with the classical analog though we very well know that there should be an a priori quantum description as this is the more fundamental theory.

In the following, I do not want to speculate too much on the question whether or why it seems always to be the case that we have to start with the classical theory. Instead I intend to handle this more pragmatically and take the classical theory in order to find and construct a corresponding quantum theory.

More concrete, quantization shall stand for a construction of a quantum theoretical description of a given physical system starting only with the classical data. According to the discussion in Sect. 2.2 the key role is played by the observable algebra: quantization is a proceedure of constructing $\mathcal{A}_{\mathrm{QM}}$ out of $\mathcal{A}_{\mathrm{class}}$. A priori it is not clear whether such a construction is successful and if so, how unique it will be.

The following *requirements* for such a construction shall not be understood as strict axioms but as ideas and motivations. In concrete examples one typically finds more appropriate and more specific formulations.

(1) The quantum mechanical observable algebra $\mathcal{A}_{\mathrm{QM}}$ should be *as big as* the classical observable algebra. Classical observables are the classical limit of quantum observables, so $\mathcal{A}_{\mathrm{QM}}$ can not be smaller than $\mathcal{A}_{\mathrm{class}}$. On the other hand, if there are quantum observables which do not have a classical counterpart then a quantization is either hopeless or the classical description has to be refined to include more observables.
The spin of an electron provides a (non-)example: there is no classical analog of spin in the usual mechanical description of an electron unless one uses a super-mechanical description. This is indeed possible and a quantization of this super-mechanics yields the correct spin for the electron. Moreover, the *correspondence* between classical and quantum mechanical observables should be sufficiently explicit. One needs a physical interpretation of the elements of $\mathcal{A}_{\mathrm{QM}}$ in order to compare them with the ones in $\mathcal{A}_{\mathrm{class}}$. As the latter are realized as functions on some phase space the interpretation of $\mathcal{A}_{\mathrm{class}}$ usually is obvious.

(2) The quantum observable algebra $\mathcal{A}_{\mathrm{QM}}$ should lead in the classical limit to the classical observable algebra $\mathcal{A}_{\mathrm{class}}$. In order to make such a statement meaningful the notion of classical limit has to be clarified. In particular, one expects that for corresponding classical and quantum observables one

has corresponding algebraic relations

$$\hat{A}^* \rightsquigarrow A*, \quad \hat{A}\hat{B} \rightsquigarrow AB, \quad \text{and} \quad \frac{1}{\mathrm{i}\hbar}[\hat{A},\hat{B}] \rightsquigarrow \{A,B\} \tag{2.29}$$

in the classical limit $\rightsquigarrow$. Here the Poisson bracket of $\mathcal{A}_{\text{class}}$ is seen as 'shadow' of the quantum mechanical non-commutativity. Note that the Poisson bracket in classical mechanics indeed has the dimension of an inverse action while the commutator is dimensionless. Also the i is necessary because of the reality properties of the commutator and the Poisson bracket. Thus (2.29) is consistent from this point of view.

(3) Having constructed $\mathcal{A}_{\text{QM}}$ all the states of $\mathcal{A}_{\text{class}}$ should arise as classical limit of the states of $\mathcal{A}_{\text{QM}}$. Since the quantum description is the more fundamental one this has to be imposed by consistency. Of course the nature of the classical limit of states is delicate and has to be specified in an appropriate way before such a statement can be shown.

(4) The non-commutativity of $\mathcal{A}_{\text{QM}}$ is the manifestation of the uncertainty relations whence their size is controlled by Planck's constant $\hbar$. The classical limit $\rightsquigarrow$ in (2.29) should be understood in such a way, that the typical actions of the system are large compared to the action $\hbar$. Intuitively, one writes $\hbar \to 0$ for the classical limit. However, $\hbar$ is not dimensionless whence a statement about the 'size' can only be *relative* to other quantities of the same physical dimension.

(5) The occurence of a classical limit shows that the corrections which bring us from classical to quantum are not too big, probably even small: One needs quite precise measurements to observe the quantum nature of our world. This aspect will be made more precise in the notion of *deformation of algebraic structures* later.

(6) The quantum observable algebra $\mathcal{A}_{\text{QM}}$ should allow for a reasonable notion of spectra. This can be achieved e.g. by embedding $\mathcal{A}_{\text{QM}}$ into some C^*-algebra.

(7) The construction of $\mathcal{A}_{\text{QM}}$ should be as *explicit* as possible. Moreover, the construction should be conceptually clear: it should be possible to distingiush properties of the construction which are generic from those which use specific features of the example. If there are ad-hoc decisions and choices to be made one should be able to investigate the resulting ambiguities.

3 From canonical quantization to star products

3.1 Canonical quantization and orderings

As first example one investigates the flat phase space $\mathbb{R}^{2n}$ with the canonical Poisson bracket (2.21). For simplicity let $n = 1$ whence we have canonically conjugate coordinates q and p with Poisson bracket

$$\{q,p\} = 1. \tag{3.1}$$

Canonical quantization, as it can be found in textbooks on quantum mechanics, is the replacement of q and p by the quantum mechanical observables $\hat{Q}$ and $\hat{P}$ which are usually realized by *differential operators* acting on wave functions. More specifically, consider the smooth functions with compact support $C_0^\infty(\mathbb{R})$ on the real line and define the linear operators $\hat{Q}, \hat{P} : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})$ by

$$(\hat{Q}\psi)(q) = q\psi(q) \quad \text{and} \quad (\hat{P}\psi)(q) = -\mathrm{i}\hbar \frac{\partial \psi}{\partial q}(q) \tag{3.2}$$

for $\psi \in C_0^\infty(\mathbb{R})$. Then $\hat{Q}$ and $\hat{P}$ are indeed defined on the pre Hilbert space $C_0^\infty(\mathbb{R})$ and obey the commutation relation

$$[\hat{Q}, \hat{P}] = \mathrm{i}\hbar \mathbb{1}. \tag{3.3}$$

According to our general concept we have to quantize the whole observable algebra and not only the two observables q and p. The smallest Poisson *-algebra of functions on the phase space containing q and p are the polynomials $\mathrm{Pol}(\mathbb{R}^2)$. Using this as classical observable algebra we have to give a corresponding operator for all the monomials $q^k p^\ell$, $k, \ell \in \mathbb{N}$. In order to accomplish the correspondence principle (2.29) it is reasonable to use the corresponding monomial in $\hat{Q}$ and $\hat{P}$. Doing so we encounter the following *ordering problem*: while the classical q and p commute we have e.g. $q^k p^\ell = p^\ell q^k$ but the quantum $\hat{Q}$ and $\hat{P}$ do not commute any longer. Thus in $\hat{Q}^k \hat{P}^\ell \neq \hat{P}^\ell \hat{Q}^k$ the ordering plays a crucial role. In order to proceed one has to make a choice how the variables should be ordered. The resulting ambiguities have to be discussed carefully later. To illustrate this by an example, I will discuss two *ordering prescriptions* which are commonly used.

3.1.1 *The standard ordering*

The simplest ordering prescription I shall discuss is the *standard ordering*. Here one writes all momenta to the right before one replaces q by $\hat{Q}$ and p by $\hat{P}$. Thus the *standard representation* ϱ_{S} is the map

$$\varrho_{\mathrm{S}} : \mathrm{Pol}(\mathbb{R}^2) \to \mathrm{DiffOp}(C_0^\infty(\mathbb{R})) \tag{3.4}$$

from the polynomials into the differential operators defined by

$$\varrho_{\mathrm{S}}(q^k p^\ell) = \hat{Q}^k \hat{P}^\ell = \left(\frac{\hbar}{\mathrm{i}}\right)^\ell q^k \frac{\partial^\ell}{\partial q^\ell} \tag{3.5}$$

and linear extension to all polynomials. A simple computation shows that one has the following explicit formula

$$\varrho_{\mathrm{S}}(f) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \left.\frac{\partial^r f}{\partial p^r}\right|_{p=0} \frac{\partial^r}{\partial q^r} \tag{3.6}$$

for any polynomial $f(q,p)$. From this explicit formula we see that $\varrho_{\mathrm{S}}(f)$ is still well-defined if f is only a polynomial function of p and has arbitrary smooth dependence on q since the infinite series terminates after finitely many terms. The smooth functions of q and p depending polynomially on p are denoted by $\mathrm{Pol}(T^*\mathbb{R})$. Note that $\mathrm{Pol}(T^*\mathbb{R})$ indeed is a Poisson *-algebra. Thus we obtain a linear map

$$\varrho_{\mathrm{S}} : \mathrm{Pol}(T^*\mathbb{R}) \to \mathrm{DiffOp}(C_0^\infty(\mathbb{R})) \tag{3.7}$$

into the differential operators acting on $C_0^\infty(\mathbb{R})$ with smooth coefficients. The following statement is an obvious consequence of the explicit formula (3.6).

Lemma 3.1 *The standard representation ϱ_{S} is bijective.*

Using this lemma we obtain the desired correspondence between classical and quantum mechanical observables. This is also known as *symbol calculus for differential operators* and the inverse of ϱ_{S} is also called the *symbol map.*

However, from a physical point of view the standard representation is still unsatisfactory. The reasons comes from the following incompatibility with the *-involutions. The algebra of differential operators has a natural *-involution which is induced by the operator adjoint with respect to the usual L^2-inner product

$$\langle \psi, \phi \rangle = \int_{\mathbb{R}} \overline{\psi(q)} \phi(q) \; dq \tag{3.8}$$

of functions $\psi, \phi \in C_0^\infty(\mathbb{R})$. Clearly this way the smooth functions with compact support become a pre Hilbert space whose completion is the space of square integrable functions $L^2(\mathbb{R}, dq)$. As long as we are working with $C_0^\infty(\mathbb{R})$ the definition of the adjoint operator of a differential operator is trivial: we obtain the ajoint by naive partial integrations. The adjoint of $\varrho_{\mathrm{S}}(f)$ can be computed explicitly for a function $f(q,p) = f_r(q)p^r$ with $f_r \in C^\infty(\mathbb{R})$ and $r \in \mathbb{N}$. We have

$$\begin{aligned}
\langle \psi, \varrho_{\mathrm{S}}(f)\phi \rangle &= \int_{\mathbb{R}} \overline{\psi(q)} \left(\frac{\hbar}{\mathrm{i}}\right)^r f_r(q) \frac{\partial^r \phi}{\partial q^r}(q) \; dq \\
&= \int_{\mathbb{R}} \overline{\left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r}{\partial q^r} \left(\psi(q)\overline{f_r(q)}\right)} \; \phi(q) \; dq \\
&= \int_{\mathbb{R}} \overline{\left(\frac{\hbar}{\mathrm{i}}\right)^r \sum_{s=0}^r \binom{r}{s} \frac{\partial^s \overline{f_r}}{\partial q^s}(q) \; \frac{\partial^{r-s}\psi}{\partial q^{r-s}}(q)} \; \phi(q) \; dq \\
&= \int_{\mathbb{R}} \overline{\left(\varrho_{\mathrm{S}} \left(\sum_{s=0}^r \binom{r}{s} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s \overline{f_r}}{\partial q^s} \, p^{r-s} \right) \psi \right)(q)} \; \phi(q) \; dq.
\end{aligned} \tag{3.9}$$

Thus the adjoint of $\varrho_{\mathrm{S}}(f_r p^r)$ is given by

$$\varrho_{\mathrm{S}}(f_r(q)p^r)^\dagger = \varrho_{\mathrm{S}} \left(\sum_{s=0}^r \binom{r}{s} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s \overline{f_r}}{\partial q^s} p^{r-s} \right). \tag{3.10}$$

In order to simplify this expression we need the following operators

$$\Delta = \frac{\partial^2}{\partial q \partial p} \quad \text{and} \quad N = \mathrm{e}^{\frac{\hbar}{2\mathrm{i}}\Delta} = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\hbar}{2\mathrm{i}}\right)^s \Delta^s, \tag{3.11}$$

which we view as operators acting on $\mathrm{Pol}(T^*\mathbb{R})$. Note that N is well-defined, i.e. for $f \in \mathrm{Pol}(T^*\mathbb{R})$ the series Nf only contains finitely many terms as $\Delta^s f = 0$ if s is larger than the polynomial degree in p of f. So consider again $f(q,p) = f_r(q)p^r$ as before then

$$\begin{aligned} N^2\overline{f} &= \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\hbar}{\mathrm{i}}\right)^s \Delta^s \overline{f} \\ &= \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s \overline{f_r}}{\partial q^s} \frac{\partial^s p^r}{\partial p^s} \\ &= \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s \overline{f_r}}{\partial q^s} \frac{r!}{(r-s)!} p^{r-s}, \end{aligned} \tag{3.12}$$

whence we obtain the important equation

$$\varrho_{\mathrm{S}}(f)^\dagger = \varrho_{\mathrm{S}}(N^2\overline{f}). \tag{3.13}$$

Since (3.13) is linear we conclude that (3.13) is valid for all $f \in \mathrm{Pol}(T^*\mathbb{R})$. In particular, for a real-valued function $f = \overline{f}$ the corresponding operator $\varrho_{\mathrm{S}}(f)$ needs not to be symmetric as in general $N^2 f \neq f$. Thus the classical and the quantum *-involution are *not compatible* and classical observables are not mapped to quantum observables.

3.1.2 *The Weyl ordering*

The above unphysical property of the standard ordering can be cured very easily using the operator N. We define the *Weyl representation* by

$$\varrho_{\mathrm{Weyl}}(f) = \varrho_{\mathrm{S}}(Nf) \tag{3.14}$$

for $f \in \mathrm{Pol}(T^*\mathbb{R})$. More explicitly we have

$$\varrho_{\mathrm{Weyl}}(f) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \left.\frac{\partial^r (Nf)}{\partial p^r}\right|_{p=0} \frac{\partial^r}{\partial q^r}, \tag{3.15}$$

viewed as differential operator acting on $C_0^\infty(\mathbb{R})$. Since $N = \mathrm{e}^{\frac{\hbar}{2\mathrm{i}}\Delta}$ is invertible on $\mathrm{Pol}(T^*\mathbb{R})$ it follows that ϱ_{Weyl} is again a bijection between $\mathrm{Pol}(T^*\mathbb{R})$ and the differential operators $\mathrm{DiffOp}(C_0^\infty(\mathbb{R}))$. After some combinatorical considerations one can show that for a polynomial $f(q,p) = q^k p^\ell$ the Weyl representation $\varrho_{\mathrm{Weyl}}(f)$ is the corresponding *totally symmetrized* polynomial in the operators $\hat{Q}$ and $\hat{P}$, e.g.

$$\varrho_{\mathrm{Weyl}}(q^2 p) = \frac{1}{3}(\hat{Q}^2\hat{P} + \hat{Q}\hat{P}\hat{Q} + \hat{P}\hat{Q}^2) = -\mathrm{i}\hbar q^2 \frac{\partial}{\partial q} - \mathrm{i}\hbar q. \tag{3.16}$$

This ordering is also called the *Weyl ordering*. Using (3.13) and the definition (3.14) one has

$$\varrho_{\mathrm{Weyl}}(f)^\dagger = \varrho_{\mathrm{S}}(Nf)^\dagger = \varrho_{\mathrm{S}}(N^2\overline{Nf}) = \varrho_{\mathrm{S}}(N^2N^{-1}\overline{f}) = \varrho_{\mathrm{S}}(N\overline{f}) = \varrho_{\mathrm{Weyl}}(\overline{f}), \tag{3.17}$$

whence the classical *-involution and the quantum *-involution are compatible for the Weyl ordering.

In order to investigate the other requirements of a quantization one has to discuss in particular the correspondence principle (2.29). I will not do this at the present stage but invite the reader to perform some simple computations:

Exercise 3.2 *Let* $f, g \in \mathrm{Pol}(T^*\mathbb{R})$. *Compare* $\varrho_{\mathrm{S}}(fg)$ *and* $\varrho_{\mathrm{S}}(f)\varrho_{\mathrm{S}}(g)$. *Also compare* $\varrho_{\mathrm{S}}(\{f,g\})$ *and* $\frac{1}{\mathrm{i}\hbar}[\varrho_{\mathrm{S}}(f), \varrho_{\mathrm{S}}(g)]$. *How can one interprete the result? Do the same computations for the Weyl ordered case.*

3.2 The first star products

3.2.1 *The standard ordered and Weyl ordered star product*

The following idea is very simply but enables us to formulate the notion of quantization in greater detail. Since both ordering prescriptions ϱ_{S} and ϱ_{Weyl} are linear bijections between $\mathcal{A}_{\mathrm{class}} = \mathrm{Pol}(T^*\mathbb{R})$ and $\mathcal{A}_{\mathrm{QM}} = \mathrm{DiffOp}(C_0^\infty(\mathbb{R}))$ we see that for these choices of $\mathcal{A}_{\mathrm{class}}$ and $\mathcal{A}_{\mathrm{QM}}$ the underlying *vector spaces* of observables are isomorphic. Thus we can pull-back the non-commutative product of $\mathcal{A}_{\mathrm{QM}}$ to $\mathcal{A}_{\mathrm{class}}$ in order to obtain a *new product* for $\mathcal{A}_{\mathrm{class}}$. One defines

$$f \star_{\mathrm{S}} g = \varrho_{\mathrm{S}}^{-1}\big(\varrho_{\mathrm{S}}(f)\varrho_{\mathrm{S}}(g)\big) \tag{3.18}$$

and

$$f \star_{\mathrm{Weyl}} g = \varrho_{\mathrm{Weyl}}^{-1}\big(\varrho_{\mathrm{Weyl}}(f)\varrho_{\mathrm{Weyl}}(g)\big) \tag{3.19}$$

for $f, g \in \mathrm{Pol}(T^*\mathbb{R})$. These new products are called *standard ordered star product* and *Weyl ordered star product* or *Weyl product* for short, respectively. Though we already expect the Weyl product to have physically more reasonable properties I will first discuss the more simple standard ordered product. First I shall derive a more explicit formula for $\star_{\mathrm{S}}$. To this end one considers functions with a fixed polynomial degree in the momentum variables $f(q,p) = f_k(q)p^k$ and

$g(q,p) = g_\ell(q)p^\ell$ with $k, \ell \in \mathbb{N}$ and $f_k, g_\ell \in C^\infty(\mathbb{R})$. Then one computes

$$\begin{aligned}
\varrho_{\mathrm{S}}(f)\varrho_{\mathrm{S}}(g) &= \left(\frac{\hbar}{\mathrm{i}}\right)^{k+\ell} f_k \frac{\partial^k}{\partial q^k} g_\ell \frac{\partial^l}{\partial q^\ell} \\
&= \left(\frac{\hbar}{\mathrm{i}}\right)^{k+\ell} f_k \sum_{s=0}^{k} \binom{k}{s} \frac{\partial^s g_\ell}{\partial q^s} \frac{\partial^{\ell+k-s}}{\partial q^{\ell+k-s}} \\
&= \varrho_{\mathrm{S}}\left(\sum_{s=0}^{k} \binom{k}{s} \left(\frac{\hbar}{\mathrm{i}}\right)^s f_k p^{k-s} \frac{\partial^s g_\ell}{\partial q^s} p^\ell\right) \\
&= \varrho_{\mathrm{S}}\left(\sum_{s=0}^{k} \frac{1}{s!} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s}{\partial p^s}(f_k p^k) \frac{\partial^s}{\partial q^s}(g_\ell p^\ell)\right) \\
&= \varrho_{\mathrm{S}}\left(\sum_{s=0}^{k} \frac{1}{s!} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s f}{\partial p^s} \frac{\partial^s g}{\partial q^s}\right),
\end{aligned} \tag{3.20}$$

whence one concludes that in general

$$f \star_{\mathrm{S}} g = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\hbar}{\mathrm{i}}\right)^s \frac{\partial^s f}{\partial p^s} \frac{\partial^s g}{\partial q^s} \tag{3.21}$$

for $f, g \in \mathrm{Pol}(T^*\mathbb{R})$. Again note that the infinite series actually contains only a finite number of non-zero terms as long as the function f is only a polynomial in p. Moreover, the result is again a function which is only polynomial in p.

Before I discuss the properties of $\star_{\mathrm{S}}$ let me mention the corresponding formulas for $\star_{\mathrm{Weyl}}$. By definition (3.19) and by (3.14) one has

$$f \star_{\mathrm{Weyl}} g = \varrho_{\mathrm{Weyl}}^{-1}(\varrho_{\mathrm{Weyl}}(f)\varrho_{\mathrm{Weyl}}(g)) = N^{-1}\varrho_{\mathrm{S}}^{-1}(\varrho_{\mathrm{S}}(Nf)\varrho_{\mathrm{S}}(Ng)) = N^{-1}(Nf \star_{\mathrm{S}} Ng), \tag{3.22}$$

whence the operator N intertwines between the products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$. Explicitly, one obtains after some simple computation

$$f \star_{\mathrm{Weyl}} g = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{2\mathrm{i}}\right)^r \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} \frac{\partial^r f}{\partial q^s \partial p^{r-s}} \frac{\partial^r g}{\partial q^{r-s} \partial p^s}. \tag{3.23}$$

Again, the infinite series terminates after finitely many terms if $f, g \in \mathrm{Pol}(T^*\mathbb{R})$. Using these explicit expressions one finds the following theorem:

Theorem 3.3 *The star products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ have the following properties:*

(1) *For $f, g \in \mathrm{Pol}(T^*\mathbb{R})$ one has $f \star_{\mathrm{S}} g$, $f \star_{\mathrm{Weyl}} g \in \mathrm{Pol}(T^*\mathbb{R})$.*

(2) *$\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ are associative multiplications*

$$f \star (g \star h) = (f \star g) \star h. \tag{3.24}$$

(3) $\star_{\mathrm{S}}$ *and* $\star_{\mathrm{Weyl}}$ *can be written as a series of bidifferential operators* C_r

$$f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g). \tag{3.25}$$

(4) *For both products one has*

$$f \star g = fg + \cdots \quad \textit{whence} \quad C_0(f, g) = fg \tag{3.26}$$

and

$$f \star g - g \star f = \mathrm{i}\hbar\{f, g\} + \cdots \quad \textit{whence} \quad C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}, \tag{3.27}$$

as well as

$$1 \star f = f = f \star 1, \tag{3.28}$$

where $\{f, g\}$ *is the canonical Poisson bracket* (2.21).

(5) *For the Weyl product one has in addition*

$$\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f}. \tag{3.29}$$

Exercise 3.4 *Prove the explicit formula (3.23) for the Weyl product and use it to show the theorem. Prove the associativity only using (3.23) and (3.21). Give an alternative proof of associativity using* ϱ_{S} *and* ϱ_{Weyl}*, respectively.*

Rephrasing the statement of the above theorem we have the following result: The new products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ are *associative* products for $\mathcal{A}_{\mathrm{class}}$. They *deform* the classical, original products in direction of the Poisson bracket. Thus the correspondence principle is manifested in the equations (3.26) and (3.27). For the Weyl product $\star_{\mathrm{Weyl}}$ the classical *-involution is still a *-involution for the new product $\star_{\mathrm{Weyl}}$. This is not true for $\star_{\mathrm{S}}$.

The maps ϱ_{S} and ϱ_{Weyl}, respectively, show that $\mathcal{A}_{\mathrm{class}}$ with the new products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$, respectively, are *isomorphic to* $\mathcal{A}_{\mathrm{QM}}$ *as algebras*. In case of $\star_{\mathrm{Weyl}}$ it is even a *-isomorphism.

This motivates the following *deformation problem* as a more concrete formulation of the quantization problem: Instead of constructing a completely new algebra $\mathcal{A}_{\mathrm{QM}}$ out of $\mathcal{A}_{\mathrm{class}}$ one keeps $\mathcal{A}_{\mathrm{class}}$ as a vector space and changes only the product structure from the commutative product into a non-commutative star product by deforming the original one.

3.2.2 *A first generalization: cotangent bundles*

The most important generalization of $\mathbb{R}^{2n}$ as phase space for classical mechanics is the class of cotangent bundles. I do not want to go into much details on the differential geometry here but just motivate why cotangent bundles are so interesting.

Let us consider a physical system of N particles moving in $\mathbb{R}^3$. Then the *configuration space* is $\mathbb{R}^{3N}$, probably with removed coincidence points. The phase space is $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$ where the second $\mathbb{R}^{3N}$ corresponds to the canonically conjugate momenta. It is clear that the star products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ have an immediate generalization to this situation.

Now consider the sitation where the *positions* of the particles have to fulfill some *constraints*. If there are f independent constraints then the configuration space will be a $n = 3N - f$ dimensional submanifold Q of the original configuration space $\mathbb{R}^{3N}$. To guarantee that the particles do not leave the constraint surface Q their possible velocities have to be tangent to the submanifold Q. Thus for the description of the allowed positions and velocities one uses the *tangent bundle* TQ of Q. The differential geometric definition of TQ can be understood heuristically as follows: for any point $q \in Q$ one attaches the tangent space T_qQ at q. Then the dijoint union of all these T_qQ gives TQ. This *velocity phase space* turns out to be a manifold of dimension $2n$ and is the basis for the Lagrangean formulation of mechanics.

In order to formulate mechanics on Q in the Hamiltonian way one has to pass to the *(momentum) phase space* by a *Legendre transform*. Geometrically this corresponds to the passage to the *cotangent bundle* T^*Q of Q. This cotangent bundle is constructed analogously to TQ where now the dual space T_q^*Q to the tangent space is attached at any point q. It turns out that on T^*Q one has a *canonical Poisson structure* whence $C^\infty(T^*Q)$ becomes a Poisson *-algebra.

Different from $\mathbb{R}^m$ on a general manifold M we do not have a global and preferred coordinate system. Hence a meaningful definition of polynomial functions is no longer available. In case of a cotangent bundle the situation is slightly nicer: Since the *momentum directions* are along the vector spaces T_q^*Q it is still meaningful to speak of functions which are *polynomial in the momenta*. In spacial directions however, this is no longer possible in a coordinate independent way. The smooth functions on T^*Q which are polynomial in momentum directions are denoted by $\mathrm{Pol}(T^*Q)$. They are a Poisson subalgebra of all smooth functions $C^\infty(T^*Q)$ for the canonical Poisson bracket. Moreover, for a typical physical system the kinetic energy is a quadratic function in the momenta while the potential is even constant in momentum directions. Thus $\mathcal{A}_{\mathrm{class}} = \mathrm{Pol}(T^*Q)$ is a good candidate for the classical observables.

For such a classical system with $\mathcal{A}_{\mathrm{class}} = \mathrm{Pol}(T^*Q)$ one can show now the following: It is possible to repeat all important steps in the construction of ϱ_{S}, N, ϱ_{Weyl} $\star_{\mathrm{S}}$, and $\star_{\mathrm{Weyl}}$ almost literally. The only modification comes from the fact that one has to use a *covariant derivative* ∇. A function depending linearly on the momenta is quantized using ∇ by $p_k \mapsto -\mathrm{i}\hbar\nabla_k$ instead of $-\mathrm{i}\hbar\partial_k$. This way one obtains a coordinate independent formulation. Moreover, for a concrete physical system there is usually a preferred choice of a covariant derivative: the kinetic energy determines a Riemannian metric on Q which has a unique compatible covariant derivative: the *Levi Civita connection*. The technical details of this *very explicit* construction can be found in [5–7].

4 Deformation quantization

4.1 Star products

4.1.1 *General definitions*

Having seen that the *deformation of the classical observable algebra* promisses to give a successful concept of quantization we shall now try to put things on a solid mathematical ground. I will discuss on one hand a geometric version based on Poisson manifolds and, on the other hand, a more algebraic version.

For the geometric situation one observes that on a general Poisson manifold M there are no 'polynomial functions' within $C^\infty(M)$. As classical observable algebra we can only use $C^\infty(M)$ at least as long as we do not have any additional specific information. In concrete examples there may be some physically interesting subalgebras, as e.g. in the case of cotangent bundles.

Using $C^\infty(M)$ the formulas (3.21) and (3.23) have to be reconsidered: If f, g are arbitrary smooth functions then the series do *not* converge in general. Even worse, one can always find functions f, g such that the power series in $\hbar$ do have radius of convergence equal to 0.

One way out is to look for star products for $\mathcal{A}_{\mathrm{class}} = C^\infty(M)$ which are only *formal power series in* $\hbar$, but share all other properties of $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$. In a *second step* one tries to find suitable subalgebras where convergence of the formal series is guaranteed. The examples of $\mathbb{R}^{2n}$ and T^*Q show that this might be successful, as here one has the subalgebras of polynomial functions. The point is that an a priori choice of a subalgebra may be very difficult without further knowledge of M but a posteriori the star product itself may single out 'nice' functions where the series converge. Thus one should view the formal power series as consequence of some missing additional information on the classical situation which typically depends strongly on the example.

After this considerations the following general definition of a star product according to Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer should be well motivated [2], see also [13] for a recent review.

Definition 4.1 *A star product for a Poisson manifold M is an associative product $\star$ for the formal power series $C^\infty(M)[[\lambda]]$ of the form*

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g), \tag{4.1}$$

such that:

(1) $f \star g = fg + \cdots$, *i.e.* $C_0(f,g) = fg$.
(2) $f \star g - g \star f = \mathrm{i}\lambda\{f,g\} + \cdots$, *i.e.* $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$.
(3) $f \star 1 = f = 1 \star f$, *i.e.* $C_r(1,f) = 0 = C_r(f,1)$ *for* $r \geq 1$.
(4) C_r *is a bidifferential operator.*

The star product is called Hermitian if in addition

$$\overline{f \star g} = \overline{g} \star \overline{f}. \tag{4.2}$$

The formal parameter λ plays the role of $\hbar$ and can be substituted by $\hbar$ as soon as one has established convergence.

The *associativity* $f \star (g \star h) = (f \star g) \star h$ is checked order by order in λ. This yields the following equivalent conditions on the C_r's: for all $k \in \mathbb{N}$ and all functions $f, g, h \in C^\infty(M)$ one has

$$\sum_{r=0}^{k} C_r(f, C_{k-r}(g, f)) = \sum_{r=0}^{k} C_r(C_{k-r}(f, g), h). \tag{4.3}$$

The Hermiticity of $\star$ is equivalent to the condition $\overline{C_r(f,g)} = C_r(\overline{g}, \overline{f})$.

The algebraic content of deformation quantization is Gerstenhaber's *deformation theory of associative algebras* [17]. Sometimes we do not want to make reference to the underlying geometric situation but start with the classical observable algebra $\mathcal{A}_{\mathrm{class}}$ as a Poisson *-algebra directly. Then one uses the following notion of a formal deformation:

Definition 4.2

(1) *A formal deformation of an associative algebra $\mathcal{A}$ is an associative product for $\mathcal{A}[[\lambda]]$ of the form*

$$A \star B = \sum_{r=0}^{\infty} \lambda^r C_r(A, B) \tag{4.4}$$

with $C_0(A, B) = AB$.

(2) *A formal deformation quantization of a Poisson algebra $\mathcal{A}_{\mathrm{class}}$ is an associative deformation $\star$ of $\mathcal{A}_{\mathrm{class}}$ with $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$.*

(3) *A Hermitian deformation of a *-algebra is a deformation with $(A \star B)^* = B^* \star A^*$.*

This way we have defined the notion of a *Hermitian deformation quantization of a Poisson *-algebra* in a purely algebraic way. This definition is useful if the geometry of the classical phase space is difficult to describe but an observable algebra is still available. As example one can consider simple field theories [14, 15].

4.1.2 *Existence and classification*

Having the general definition of a star product one is faced with the question whether such deformations exist for general Poisson manifolds. Moreover, one should clarify how many possibilities are there in the construction of star products. This is non-trivial as we have already seen in the case of $\mathbb{R}^2$ that there are at least two, $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$. The existence is guaranteed by the following theorem:

Theorem 4.3 *On any Poisson manifold there are star products.*

The proof of this theorem is far beyond this introduction. I only shall mention that first the existence for symplectic manifolds was shown by DeWilde and

Lecomte [12], Fedosov [16] and Omori, Maeda and Yoshioka [25]. The general Poisson case was unclear for long time untill Kontsevich succeeded with a proof in his fundamental work [21].

The question about uniqueness is slightly more difficult to formulate as we have already seen that there are at least two star products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ on $\mathbb{R}^{2n}$. However, there is a relation between them using the operator N, namely

$$f \star_{\mathrm{Weyl}} g = N^{-1}(Nf \star_{\mathrm{S}} Ng), \tag{4.5}$$

where $N = \exp(\frac{\lambda}{2\mathrm{i}}\Delta)$ is now viewed as formal series of differential operators. Thus N is an *algebra isomorphism* between the standard and Weyl ordered star product algebras. This is of course to be expected as both are isomorphic to the algebra of differential operators, by their very construction. The important observation is that N starts with the identity in zeroth order of λ: the isomorphism is simply the identity in the classical limit.

Definition 4.4 *Two star products $\star$ and $\star'$ for $C^\infty(M)$ are called equivalent if there is a formal series*

$$S = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r S_r \tag{4.6}$$

of differential operators S_r such that

$$f \star' g = S^{-1}(Sf \star Sg) \quad \textit{and} \quad S1 = 1. \tag{4.7}$$

In this case S is called an equivalence transformation. In the case of Hermitian star products one requires in addition $\overline{Sf} = S\overline{f}$.

Exercise 4.5 *Prove that the above notion indeed is an equivalence relation. Moreover, given such an S, show that $\star'$ defined by (4.7) is again a (Hermitian) star product provided $\star$ is one.*

Thus having one star product one gets a whole bunch of them by applying some equivalence transformation. The example of $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ shows that passing to an equivalent star product is related to another choice of an ordering prescription. In fact, the notion of equivalence can be seen as an axiomatic and generalized notion of passing from one to another ordering prescription. This raises the following two questions:

(1) Is there a classification of star products up to equivalence, i.e. up to the choice of an ordering prescription?
(2) Is there a physically motivated choice of a star product $\star$ within a given equivalence class $[\star]$?

It turns out that the first question can be answered in full generality. I shall not give the detail which are technical and can be found in the literature [3, 21, 24].

Theorem 4.6 *The equivalence classes $[\star]$ of star products $\star$ are classified by certain geometric properties of the underlying Poisson manifold M. In particular, there is only one class of symplectic star products on $\mathbb{R}^{2n}$.*

The choice of an equivalence class is not sufficient from the physical point of view. In order to construct the quantum mechanical observables $\mathcal{A}_{\mathrm{QM}}$ one has to answer also the above second question. In general, this is much more involved as the choice of a particular star product usually depends strongly on the example. Guidelines can be the classical *symmetries* and the wish for convergence.

4.2 Time evolution in deformation quantization

Let me now discuss the time evolution in the formalism of deformation quantization. Consider a classical Hamiltonian $H \in C^\infty(M)$. Then the classical time evolution is given by Hamilton's equations (2.27). As the vector spaces of $\mathcal{A}_{\mathrm{class}}$ and $\mathcal{A}_{\mathrm{QM}}$ coincide in deformation quantization it is clear what the star product analog of Heisenberg's equation (2.12) should be, namely

$$\frac{d}{dt} f(t) = \frac{\mathrm{i}}{\lambda}(H \star f(t) - f(t) \star H). \tag{4.8}$$

Here we encounter a little mathematical subtlety: in the framework of formal power series we can not simply divide by λ. This would require *formal Laurent series*. However, since

$$H \star f(t) - f(t) \star H = \mathrm{i}\lambda\{H, f(t)\} + \cdots \tag{4.9}$$

this is not necessary. Heisenberg's equation of motion reads

$$\frac{d}{dt} f(t) = \frac{\mathrm{i}}{\lambda}(H \star f(t) - f(t) \star H) = -\{H, f(t)\} + \cdots . \tag{4.10}$$

Thus the equation of motion in deformation quantization is a *deformation* of Hamilton's equation of motion where the *quantum corrections* in (4.10) result from higher order contributions of the star product commutator. Also here the classical limit is evidently build into the construction.

4.3 Star products and symmetries

4.3.1 *Classical symmetries*

Let me first recall the concept of *infinitesimal symmetries* in classical mechanics, see e.g. [23]. Take the angular momentum $\vec{L}(\vec{q}, \vec{p}) = \vec{q} \times \vec{p}$ as example. The functions L_1, L_2, L_3 generate the rotations around the corresponding axis in the sense that their Hamilton equations of motion yield flow maps which are exactly the rotations. The Poisson brackets of the three generators can be summarized by

$$\{L_i, L_j\} = \sum_k \epsilon_{ijk} L_k. \tag{4.11}$$

Thus the L_i build a subalgebra for the Poisson bracket. This can be seen as the infinitesimal version of the statement that the composition of two rotations is again a rotation.

More general symmetries are described by N generators $J_1, \ldots, J_n \in C^\infty(M)$, which form a subalgebra with respect to the Poisson bracket. Hence there are constants, the so-called *structure constants* $c_{ij}^k \in \mathbb{R}$ *of the symmetry* such that

$$\{J_i, J_j\} = \sum_k c_{ij}^k J_k. \tag{4.12}$$

Note that the J_i only form a subalgebra for the Poisson bracket, not for the pointwise product. Since the Poisson bracket satisfies the Jacobi identity the constants c_{ij}^k can not be completely arbitrary but have to satisfy the two conditions

$$c_{ij}^k = -c_{ji}^k \quad \text{and} \quad \sum_r (c_{ij}^r c_{rk}^\ell + c_{ki}^r c_{rj}^\ell + c_{jk}^r c_{ri}^\ell) = 0. \tag{4.13}$$

One obtains a finite dimensional vector space $\mathfrak{g}$ spanned by the $J_1, \ldots, J_N$ which has the structure of a *Lie algebra* with *Lie bracket*

$$[J_i, J_j] = \sum_k c_{ij}^k J_k. \tag{4.14}$$

Thus, abstractly speaking, an infinitesimal symmetry in classical mechanics is given by a Lie algebra $\mathfrak{g}$ which is realised by functions on the phase space, meaning that (4.12) is fullfilled. In the case of rotations the L_i give the Lie algebra $\mathfrak{so}(3)$.

4.3.2 *Quantum mechanical symmetries*

Quantum mechanically symmetries are described in a similar way. Infinitesimal symmetries are also encoded in a Lie algebra $\mathfrak{g}$ which is now realised by operators on the Hilbert space $\mathfrak{H}$. As this should be done by linear operators one calls this a *representation* of $\mathfrak{g}$ on $\mathfrak{H}$. More concrete this means that we have self-adjoint operators $\hat{J}_1, \ldots, \hat{J}_N$ on $\mathfrak{H}$ such that the commutation relations

$$[\hat{J}_i, \hat{J}_j] = \mathrm{i}\hbar \sum_k c_{ij}^k \hat{J}_k \tag{4.15}$$

hold. Note however that usually there are some technical subtleties since typically the operators $\hat{J}_i$ are only densely defined.

4.3.3 *Symmetries in deformation quantization*

If one has a classical symmetry described by a Lie algebra $\mathfrak{g}$ and corresponding functions $J_1, \ldots, J_n$ on M then one is interested in the question whether in the quantum mechanical description the same symmetry is realised or if *anomalies* occure. I only shalldiscuss some of the basic definitions, for a more detailed discussion see e.g. [1,4].

Definition 4.7 *A star product is called* $\mathfrak{g}$*-covariant if*

$$J_i \star J_j - J_j \star J_i = \mathrm{i}\lambda \sum_k c_{ij}^k J_k. \tag{4.16}$$

Note that from the definition of a star product, we always have $f \star g - g \star f = \mathrm{i}\lambda\{f, g\} + \cdots$. So the crucial point in the above definition is that there are *no* higher order corrections to the commutators of the J_i. Typically this is a strong requirement and limits the number of $\mathfrak{g}$-covariant star products quite drastically.

Exercise 4.8 *Prove that the Weyl star product $\star_{\mathrm{Weyl}}$ is $\mathfrak{so}(3)$-covariant.*

The general question whether such covariant star products always exist is still unsolved and very difficult. However, there is an important case where existence is guaranteed:

Theorem 4.9 *Let M be a symplectic manifold and $\mathfrak{g}$ the Lie algebra of a compact Lie group G acting on M in a Hamiltonian way. Then there exists a $\mathfrak{g}$-covariant star product.*

4.4 States in deformation quantization

4.4.1 *Positive functionals and positive deformations*

In the following I shall assume to have a Hermitian star product $\star$. Then the concept of states as discussed in Sect. 2.2.1 shall be applied to deformation quantization as well. Here one encounters the following problem: $\mathbb{C}$-linear functionals $\omega : C^\infty(M)[[\lambda]] \to \mathbb{C}$ are not of much use and not general enough to yield an interesting notion of states. Either they cut off higher orders in λ or one is faced with convergence problems immediately. It turns out that it is better to take the formal power series serious. Thus we consider $\mathbb{C}[[\lambda]]$-linear functionals

$$\omega : C^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]. \tag{4.17}$$

The values are also formal power series. This seems to be justified as λ should correspond to $\hbar$ and expecation values may depend on $\hbar$ as well. However, how shall the positivity condition (2.7) be interpreted?

Definition 4.10 *A real formal power series $a = \sum_{r=r_0}^{\infty} \lambda^r a_r \in \mathbb{R}[[\lambda]]$ is called positive if the lowest order a_{r_0} is positive.*

Hence a formal series in $\mathbb{R}[[\lambda]]$ is either negative, 0, or positive. Furthermore, the sum and the product of two positive series is again positive. This shows that the usual rules hold. In more mathematical terms the ring $\mathbb{R}[[\lambda]]$ becomes an *ordered ring*. Note however that this ordering is no longer *Archimedean* as e.g. $n\lambda < 1$ for all $n \in \mathbb{N}$. Physically interpreted this corresponds to the fact that a formal deformation has the right 'direction' but not yet the right 'size', as we have not yet imposed convergence conditions. Thus it is an *asymptotic* notion of positivity. Nevertheless this notion can be used to give a definition of a state:

Definition 4.11 *Let $(M, \star)$ be a Poisson manifold with a Hermitian star product. A state of $\mathcal{A}_{\mathrm{QM}} = C^\infty(M)[[\lambda]]$ is a $\mathbb{C}[[\lambda]]$-linear functional $\omega : C^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ such that for all f*

$$\omega(\overline{f} \star f) \geq 0 \quad \textit{and} \quad \omega(1) = 1. \tag{4.18}$$

Having a classically positive linear functional $\omega_0 : C^\infty(M) \to \mathbb{C}$ one can extend it $\mathbb{C}[[\lambda]]$-linearily to all of $C^\infty(M)[[\lambda]]$ by setting

$$\omega_0 : C^\infty(M)[[\lambda]] \ni f = \sum_{r=0}^{\infty} \lambda^r f_r \;\mapsto\; \sum_{r=0}^{\infty} \lambda^r \omega_0(f_r) \in \mathbb{C}[[\lambda]]. \tag{4.19}$$

Thus the question is whether ω_0 is not only positive for the pointwise product, i.e. $\omega_0(\overline{f}f) \geq 0$ but also for $\star$. The condition for positivity is in the lowest orders given by

$$0 \leq \omega_0(\overline{f} \star f) = \omega_0(\overline{f}f) + \lambda\omega_0(C_1(\overline{f}, f)) + \cdots . \tag{4.20}$$

Now it may happen that $\omega_0(\overline{f}f) = 0$. Then the positivity requires that in the next order we have something non-negativ. But the next term $\omega_0(C_1(\overline{f}, f))$ (and similarly all higher order terms) usually contains derivatives of f whence a definiteness can not be guaranteed so easily. This can be seen in the following example.

Let $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$ be the Hamiltonian of the harmonic oscillator and let $\omega_0 = \delta_0$ be the δ-functional at 0. Then we consider the Weyl product $\star_{\mathrm{Weyl}}$ and compute

$$\delta_0(\overline{H} \star_{\mathrm{Weyl}} H) = \delta_0\left(H^2 - \frac{\lambda^2}{4}\right) = -\frac{\lambda^2}{4} < 0. \tag{4.21}$$

Thus the δ-functional is no longer positive for $\star_{\mathrm{Weyl}}$. Physically this can be interpreted as the well-known fact that *points in phase space* are no longer states in quantum theory because of the uncertainty relations.

How can we obtain now every classical state as a classical limit of a quantum state as we have demanded this in Sect. 2.3? The solution to the above problem is that we can also allow for higher order terms in the positive functional. One considers functionals of the form

$$\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r \tag{4.22}$$

where each $\omega_r : C^\infty(M) \to \mathbb{C}$. Then such a functional clearly extends to a $\mathbb{C}[[\lambda]]$-linear functional $\omega : C^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$. Now the positivity condition reads

$$0 \leq \omega(\overline{f} \star f) = \omega_0(\overline{f}f) + \lambda\left(\omega_0(C_1(\overline{f}, f)) + \omega_1(\overline{f}f)\right) + \cdots . \tag{4.23}$$

For a given classically positive functional ω_0 we have to chose ω_1 in such a way that the term $\omega_0(C_1(\overline{f}, f))$ is compensated whenever $\omega_0(\overline{f}f) = 0$. Similarly the higher orders ω_r are used to compensate possibly negative terms occuring in the star product. A priori it is not clear whether one can find such *quantum corrections* whence I state the following definition:

Definition 4.12 *Let $\star$ be a Hermitian star product.*

(1) *A* $\mathbb{C}[[\lambda]]$*-linear functional* $\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r : C^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ *is called a deformation of* ω_0 *if* ω *is positive with respect to* $\star$*.*

(2) *A star product* $\star$ *is called a positive deformation if any classically positive functional can be deformed.*

The following theorem ensures that we indeed can deform always the classically positive functionals as this was expected from the discussion in Sect. 2.3, see [10].

Theorem 4.13 *On a symplectic manifold all Hermitian star products are positive deformations.*

In this sense any classical state arises as classical limit of a quantum state. Note that the above formulation is only meaningful because we have identified the underlying vector spaces of $\mathcal{A}_{\mathrm{class}}$ and $\mathcal{A}_{\mathrm{QM}}$.

4.4.2 *The GNS construction*

Let me now explain the GNS construction of a representation out of a given state. The construction was developed for C^*-algebras, see e.g. [8], but it works in much more general situations.

In the following I shall assume that a quantum observable algebra $\mathcal{A}_{\mathrm{QM}}$ has been constructed, e.g. by means of a Hermitian star product $(C^\infty(M)[[\lambda]], \star)$. The only important point is that $\mathcal{A}_{\mathrm{QM}}$ is a *-algebra over $\mathsf{C} = \mathbb{C}$ or $= \mathbb{C}[[\lambda]]$ in order to have a notion of positive functionals. In fact, it will be sufficient to have an ordered ring R and $\mathsf{C} = \mathsf{R} \oplus \mathsf{iR}$.

A *pre Hilbert space* over C is then defined to be a C-module $\mathfrak{H}$ with a positive definite inner product. To make this definition meaningful, one needs the notion of positive elements in R. Then a linear map $A : \mathfrak{H} \to \mathfrak{H}$ is called *adjointable* if there exists a $A^* : \mathfrak{H} \to \mathfrak{H}$ such that

$$\langle A^*\phi, \psi \rangle = \langle \phi, A\psi \rangle \tag{4.24}$$

for all $\phi, \psi \in \mathfrak{H}$. The set of all adjointable operators no $\mathfrak{H}$ is denoted by $\mathfrak{B}(\mathfrak{H})$. If $\mathfrak{H}$ is actually a Hilbert space over $\mathbb{C}$ then the above definition of $\mathfrak{B}(\mathfrak{H})$ coincides with the usual definition of bounded operators: this is the Hellinger-Toeplitz theorem, see e.g. [26, S. 84]. In general, the reader may easily show that $\mathfrak{B}(\mathfrak{H})$ is a *-algebra.

A *-representation is a notion which should make precise the idea of realizing the observables in $\mathcal{A}_{\mathrm{QM}}$ by operators on $\mathfrak{H}$.

Definition 4.14 *A* **-representation* π *of* $\mathcal{A}_{\mathrm{QM}}$ *on* $\mathfrak{H}$ *is a* **-homomorphism*

$$\pi : \mathcal{A}_{\mathrm{QM}} \to \mathfrak{B}(\mathfrak{H}). \tag{4.25}$$

With other words, π is linear and satisfies the identities

$$\pi(AB) = \pi(A)\pi(B) \quad \text{and} \quad \pi(A^*) = \pi(A)^* \tag{4.26}$$

for all $A, B \in \mathcal{A}_{\mathrm{QM}}$. Thus one can realise the abstractly given observable algebra $\mathcal{A}_{\mathrm{QM}}$ by a particular *-representation on a pre Hilbert space. In fact, the various 'representations' in quantum mechanics as the spacial, the momentum, and the energy representation, respectively, are indeed *-representation of the observable algebra. They are not really different as they turn out to be *unitarily equivalent*. In general two *-representations π and ρ of $\mathcal{A}_{\mathrm{QM}}$ on $\mathfrak{H}$ and $\mathfrak{K}$ are called unitarily equivalent if there is a unitary operator $U : \mathfrak{H} \to \mathfrak{K}$ such that

$$U\pi(A) = \rho(A)U \tag{4.27}$$

for all $A \in \mathcal{A}_{\mathrm{QM}}$. In this case U is called a unitary *intertwiner*.

For a given $\mathcal{A}_{\mathrm{QM}}$ one wants to understand how many *-representations $\mathcal{A}_{\mathrm{QM}}$ has, up to unitary equivalence. The GNS construction (after Gel'fand, Naimark and Segal) allows to *construct* a *-representation out of a state ω of $\mathcal{A}_{\mathrm{QM}}$.

Hence let $\omega : \mathcal{A}_{\mathrm{QM}} \to \mathsf{C}$ be a state. Since ω fulfilles the Cauchy Schwarz inequality (2.10) one shows that

$$\mathfrak{J}_\omega = \{A \in \mathcal{A} \mid \omega(A^*A) = 0\} \tag{4.28}$$

is a left ideal of $\mathcal{A}$, the so-called *Gel'fand ideal of* ω. One defines the quotient space

$$\mathfrak{H}_\omega = \mathcal{A}/\mathfrak{J}_\omega. \tag{4.29}$$

The equivalence class of B in $\mathfrak{H}_\omega$ is denoted by ψ_B. On this quotient space one has two additional structures: a representation of $\mathcal{A}$ and an inner product. The representation π_ω of $\mathcal{A}$ is defined by

$$\pi_\omega(A)\psi_B = \psi_{AB}, \tag{4.30}$$

i.e. the usual left representation of an algebra on itself modulo a left ideal. The inner product is defined by

$$\langle \psi_A, \psi_B \rangle_\omega = \omega(A^*B). \tag{4.31}$$

Since $\mathfrak{J}_\omega$ is quotiened out it follows that $\langle \cdot, \cdot \rangle_\omega$ is indeed positive definite. Finally, one shows that π_ω is a *-representation since

$$\langle \psi_C, \pi_\omega(A)\psi_B \rangle = \omega(C^*AB) = \omega((A^*C)^*B) = \langle \pi_\omega(A)\psi_C, \psi_B \rangle. \tag{4.32}$$

Thus one has constructed a *-representation π_ω of $\mathcal{A}_{\mathrm{QM}}$ on the pre Hilbert space $\mathfrak{H}_\omega$ out of a state $\omega : \mathcal{A}_{\mathrm{QM}} \to \mathsf{C}$. The representation is called the *GNS representation* for ω.

A remarkable and also characterizing property is that all vectors ψ_A can be obtained by applying the 'creation' operator $\pi_\omega(A)$ to the *vacuum vector* $\psi_{\mathbb{1}}$, since

$$\psi_A = \psi_{A\mathbb{1}} = \pi_\omega(A)\psi_{\mathbb{1}}. \tag{4.33}$$

A representation with this property is called *cyclic* with *cyclic vector* $\psi_{\mathbb{1}}$. Finally, the functional ω can be reconstructed as the *vacuum expectation value*

$$\omega(A) = \langle \psi_{\mathbb{1}}, \pi_\omega(A)\psi_{\mathbb{1}} \rangle, \tag{4.34}$$

whence the general notion of a state becomes again related to the naive notion of a state as an expectation value (2.3). The terminology comes from quantum field theory where also the GNS construction first was applied, see e.g. [18].

Exercise 4.15 *Prove the above properties of the GNS construction. Show that all operations are indeed well-defined, i.e. independent of the particular representatives of the equivalence classes.*

As a remark I would like to mention that in the case where $\mathcal{A}_{\mathrm{QM}}$ is a C^*-algebra the GNS representation π_ω always extends to the Hilbert space completion $\hat{\mathfrak{H}}_\omega$. Thus in this case one obtains a *-representation by bounded operators on a Hilbert space.

The importance of the GNS construction is that it works also for deformation quantization. To illustrate this consider the following example: The *Schrödinger functional* $\omega : C_0^\infty(\mathbb{R}^{2n})[[\lambda]] \to \mathbb{C}[[\lambda]]$ is defined by the integration over the configuration space coordinates for momentum $p = 0$, i.e.

$$\omega(f) = \int_{\mathbb{R}^n} f(q, 0)\, d^n q. \tag{4.35}$$

With some elementary partial integrations one shows that ω is indeed a positive linear functional with respect to the Weyl product

$$\omega(\overline{f} \star_{\mathrm{Weyl}} f) = \int_{\mathbb{R}^n} \overline{(Nf)(q, 0)} (Nf)(q, 0)\, d^n q \geq 0. \tag{4.36}$$

Thus ω defines a corresponding GNS representation. It turns out that this GNS representation is unitarily equivalent to the Weyl representation ϱ_{Weyl} on the pre Hilbert space $C_0^\infty(\mathbb{R}^n)[[\lambda]]$ with its usual L^2-inner product. A completely analogous construction is possible for cotangent bundles [5–7].

5 Star products beyond quantization

In this section I shall briefly discuss two applications of the methods of deformation quantization. This is meant to be an outlook not directly related the to quantization problem.

5.1 The quantum plane

Consider the plane $\mathbb{R}^2$ with the following bracket

$$\{f, g\}(x, y) = xy \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right). \tag{5.1}$$

A simple computation shows that this is indeed a Poisson bracket which vanishes along the x- and y-axis. Thus it can not be symplectic. Nevertheless there is a simple formula for a star product quantizing this Poisson bracket. First we define the two differential operators

$$D_x := x \frac{\partial}{\partial x} \quad \text{and} \quad D_y := y \frac{\partial}{\partial y} \tag{5.2}$$

acting on functions in $C^\infty(\mathbb{R}^2)$. A star product similar to the standard ordered one is then defined by

$$f \star g = \sum_{r=0}^{\infty} \frac{(\mathrm{i}\lambda)^r}{r!} D_x^r f D_y^r g. \tag{5.3}$$

It is easy to see that the first order commutator with respect to $\star$ gives indeed the Poisson bracket (5.1). It is also clear that $1 \star f = f = f \star 1$. So it remains to show the associativity of $\star$. This can be done by a lenghty but elementary computation using the crucial commutation relation

$$[D_x, D_y] = 0. \tag{5.4}$$

Exercise 5.1 *Prove that $\star$ is associative.*

To see the relations to the quantum plane we compute the commutator of the coordinate functions x and y. We have

$$x \star y = \sum_{r=0}^{\infty} \frac{(\mathrm{i}\lambda)^r}{r!} D_x^r x D_y^r y = \sum_{r=0}^{\infty} \frac{(\mathrm{i}\lambda)^r}{r!} xy = \mathrm{e}^{\mathrm{i}\lambda} xy, \tag{5.5}$$

since clearly $D_x^r x = x$ and $D_y^r y = y$ for all $r \in \mathbb{N}$. For the other direction we use $D_x^r y = 0$ and $D_y^r x = 0$ for $r \geq 1$ to obtain

$$y \star x = \sum_{r=0}^{\infty} \frac{(\mathrm{i}\lambda)^r}{r!} D_x^r y D_y^r x = yx. \tag{5.6}$$

Hence the coordinate functions x and y obey the commutation relation of the *quantum plane*

$$x \star y = q y \star x \quad \text{where} \quad q := \mathrm{e}^{\mathrm{i}\lambda}. \tag{5.7}$$

For a detailed discussion of the quantum plane and its relations to quantum groups see e.g. [20].

5.2 Noncommutative field theory

Another application of the methods of deformation quantization comes from generalizations of field theories. Let M be a space-time manifold, say Minkowski space for simplicity. Furthermore, let $x_1, \ldots, x_n$ be (global) coordinates for M. The one considers a *non-commutative* analog $\hat{M}$ of M in the sense, that one considers an algebra of operators $\hat{x}_1, \ldots, \hat{x}_n$ with commutation relations

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\lambda\theta_{ij}, \tag{5.8}$$

where the θ_{ij} are numerical constants and λ has the physical dimension of an area. Thus one enters the realm of Connes' *non-commutative geometry* [11]. The idea is that for very small distances the description of geometry as we are used to is no longer valid: here quantum and general relativistic effects have to be considered in a unified way. One (probably very naive) way to think of this *quantum* *gravity* is to impose non-commutativity also for the coordinate functions. Then the typical length scale where these effects are no longer negligible is the Planck scale whence λ should be of the size of the Planck area. Here λ plays in some sense the same role as $\hbar$ in quantization. In a limit $\lambda \to 0$ one should get back the coordinates of the 'commutative' space-time.

Having the definition of a star product it is clear how one can model these commutation relations directly on the classical space-time. One interprets the constants θ as the coefficients of a Poisson structure

$$\{x_i, x_j\} = \theta_{ij} \tag{5.9}$$

and looks for a star product $\star$ quantizing this Poisson bracket whose existence is guaranteed by Kontsevich's theorem. Here θ_{ij} can even be arbitrary functions such that the Jacobi identity (2.19) holds.

In a second step one is interested in field theories on such non-commutative space-time manifolds. Here one proceeds analogously to the commutative case and considers an action principle starting with a Lagrangean, say for a scalar field ϕ

$$\mathcal{L}_{\mathrm{comm}} = \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 + \cdots, \tag{5.10}$$

where $\cdots$ indicate some possible interaction terms. In the non-commutative field theory all the pointwise products have to be replaced by the non-commutative star product. This yields a Lagrangean of the form

$$\mathcal{L}_{\mathrm{n.c.}} = \partial_\mu \phi \star \partial^\mu \phi - m^2 \phi \star \phi + \cdots. \tag{5.11}$$

In non-commutative field theories one is interested in the solutions of the correspondig action principle. Note that one first treats this as a *classical* field theory with some strange Lagrangean on a non-commutative space-time. Only in a second step one tries to quantize this into a quantum field theory on the non-commutative space-time. We shall refer to [27] for a review on these topics with many additional references and finish this discussion with the remark that there is also a geometric interpretation of the non-commutative fields in terms of deformation quantization of vector bundles [9].

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